Introduction to Lattices and Order

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 400

1. Representation: The General Case
   - Stone's Representation Theorem for Boolean Algebras
   - Priestley's Representation Theorem for Distributive Lattices
   - Duality: Distributive Lattices and Priestley Spaces

Subsection 1

Stone's Representation Theorem for Boolean Algebras

The Prime Ideal Space of a Boolean Algebra

- We showed every finite Boolean algebra is isomorphic to some powerset algebra.
- Finiteness is essential, since we saw that the finite-cofinite algebra $\mathrm{FC}(\mathbb{N})$ is not isomorphic to a powerset algebra.
- However, it is true that any Boolean algebra B is isomorphic to a subalgebra of a powerset algebra.
- We refine this result by describing precisely which subalgebra this is.
- Let B be a Boolean algebra. The map $\eta : a \mapsto X_a := \{I \in \mathcal{I}_p(B) : a \notin I\}$ is a Boolean algebra embedding of B into $\mathcal{P}(\mathcal{I}_p(B))$.
- We seek a characterization of the image $\mathrm{im}\eta$ of the embedding η in terms of additional structure on the set of prime ideals.
- A **topology** on a set X is a family of subsets of X containing X and $\varnothing$ and closed under arbitrary unions and finite intersections.
- We assume familiarity with topological concepts (see preceding set).

The Prime Ideal Space

- The family of clopen subsets of a topological space $\langle X; \mathcal{T} \rangle$ forms a Boolean algebra.
- This suggests that we might try to impose a topology $\mathcal{T}$ on $\mathcal{I}_p(B)$ so that $\text{im}\eta$ is characterized as the family of clopen subsets of the topological space $\langle \mathcal{I}_p(B); \mathcal{T} \rangle$.
- Of course, $X_a := \{I \in \mathcal{I}_p(B) : a \notin I\}$ must be in $\mathcal{T}$, for each $a \in B$.
- The family $B := \{X_a : a \in B\}$ is not a topology because it is not closed under the formation of arbitrary unions.
- We have to define $\mathcal{T}$ on $\mathcal{I}_p(B)$ as follows:

$$\mathcal{T} := \{U \subseteq \mathcal{I}_p(B) : U \text{ is a union of members of } B\}.$$

- The family B is a basis for $\mathcal{T}$ (which is indeed a topology).
- The topological space $\langle \mathcal{I}_p(B); \mathcal{T} \rangle$ is called the **prime ideal space** or **dual space** of B.
- Let $X := \mathcal{I}_p(B)$. Each element of B is clopen in X, because $X \backslash X_a = X_{a'}$ and so $X \backslash X_a$ is open.

Compactness of Prime Ideal Space

- To prove that every clopen subset of $\langle X; \mathcal{T} \rangle$ is of the form X_a, we need further information about the prime ideal space.

Proposition

For B a Boolean algebra, the prime ideal space $\langle \mathcal{I}_p(B); \mathcal{T} \rangle$ is compact.

- Let $\mathcal{U}$ be an open cover of $X := \mathcal{I}_p(B)$. We have to show that there exist finitely many members of $\mathcal{U}$ whose union is X. Every open set is a union of sets X_a and we may therefore assume without loss of generality that $\mathcal{U} \subseteq \mathcal{B}$. Write $\mathcal{U} = \{X_a : a \in A\}$, where $A \subseteq B$. Let J be the smallest ideal containing A, that is $J = \{b \in B : b \leq a_1 \vee \cdots \vee a_n$, for some $a_1, \ldots, a_n \in A\}$.
  - If J is not proper, then $1 \in J$. So $a_1 \vee \cdots \vee a_n = 1$, for some finite subset $\{a_1, \ldots, a_n\}$ of A. Then $X = X_1 = X_{a_1 \vee \cdots \vee a_n} = X_{a_1} \cup \cdots \cup X_{a_n}$ and $\{X_{a_1}, \ldots, X_{a_n}\}$ provides the required finite subcover of $\mathcal{U}$.
  - If J is proper we can use (BPI) to obtain a prime ideal I containing J. But then I belongs to X but to no member of $\mathcal{U}$, a contradiction.

Clopen Subsets in the Prime Ideal Space

Proposition

Let $X := \mathcal{I}_p(B)$ and let $\langle X; \mathcal{T} \rangle$ be the prime ideal space of the Boolean algebra B. Then the clopen subsets of X are exactly the sets X_a for $a \in B$. Further, given distinct points $x, y \in X$, there exists a clopen subset V of X, such that $x \in V$ and $y \notin V$.

- As noted above, each set X_a is clopen. Also, given distinct I_1 and I_2 in $\mathcal{I}_p(B)$, there exists, without loss of generality, $a \in I_1 \backslash I_2$. Then X_a contains I_2 but not I_1. This proves the final assertion.

  It remains to prove that an arbitrary clopen subset U of X is of the form X_a, for some $a \in B$. Because U is open, $U = \bigcup_{a \in A} X_a$, for some subset A of B. But U is also a closed subset of X and so compact. Hence, there exists a finite subset A_1 of A, such that $U = \bigcup_{a \in A_1} X_a$. Then $U = X_a$, where $a = \bigvee A_1$.

Stone's Representation Theorem for Boolean Algebras

Stone's Representation Theorem for Boolean Algebras

Let B be a Boolean algebra. Then the map

$$\eta : a \mapsto X_a := \{I \in \mathcal{I}_p(B) : a \notin I\}$$

is a Boolean algebra isomorphism of B onto the Boolean algebra of clopen subsets of the dual space $\langle \mathcal{I}_p(B); \mathcal{T} \rangle$ of B.

- To exploit this representation to the full we need to know more about topological spaces with the properties possessed by $\mathcal{I}_p(B)$.
- The last part of the preceding proposition asserts that the prime ideal space of a Boolean algebra satisfies a separation condition guaranteeing that the space has "plenty" of clopen subsets.
- This result has some topological ramifications.

Totally Disconnected Spaces and Boolean Spaces

- A topological space $\langle X; \mathcal{T} \rangle$ is **totally disconnected** if, given distinct points $x, y \in X$, there exists a clopen subset V of X, such that $x \in V$ and $y \notin V$.
- If $\langle X; \mathcal{T} \rangle$ is both compact and totally disconnected, it is said to be a **Boolean space**.
- We have shown that $\langle \mathcal{I}_p(B); \mathcal{T} \rangle$ is a Boolean space for every Boolean algebra B.
- We denote by $\mathcal{P}^{\mathcal{T}}(X)$ the family of clopen subsets of a Boolean space $\langle X; \mathcal{T} \rangle$.
- Given distinct points x, y in a totally disconnected space X, there exist disjoint clopen sets V and $W := X \backslash V$, such that $x \in V$ and $y \in W$.
- This implies that a totally disconnected space is Hausdorff (exploited in the following slides).

Clopen Sets Satisfying Special Properties

Lemma

Let $\langle X; \mathcal{T} \rangle$ be a Boolean space.

(i) Let Y be a closed subset of X and $x \notin Y$. Then there exists a clopen set V, such that $Y \subseteq V$ and $x \notin V$.

(ii) Let Y and Z be disjoint closed subsets of X. Then there exists a clopen set U, such that $Y \subseteq U$ and $Z \cap U = \varnothing$.

(i) Since X is totally disconnected, for each $y \in Y$, there exists a clopen set V_y, with $y \in V_y$ and $x \notin V_y$. The open sets $\{V_y : y \in Y\}$ form an open cover of Y. Since Y is compact, there exist $y_1, \ldots, y_n \in Y$, such that $Y \subseteq V := V_{y_1} \cup \cdots \cup V_{y_n}$. As a finite union of clopen sets, V is clopen. By construction it does not contain x.

Clopen Sets Satisfying Special Properties (Cont'd)

(ii) Let Y and Z be disjoint closed subsets of X.

By Part (i), for all $z \in Z$, there exists a clopen set U_z, such that $Y \subseteq U_z$ and $z \notin U_z$. Let $V_z = X \setminus U_z$. The collection of clopen sets $\{V_z : z \in Z\}$ forms an open cover of Z. Since Z is compact, there exist $z_1, z_2, \ldots, z_n \in Z$, such that $Z \subseteq \bigcup_{i=1}^n V_{z_i}$. As a finite intersection of clopen sets, $U := \bigcap_{i=1}^n U_{z_i}$ is clopen. Moreover, we have $Y \subseteq U$ and:

$$\begin{array}{rcl} Z \cap U & = & Z \cap \bigcap_{i=1}^n U_{z_i} \\ & = & Z \cap \bigcap_{i=1}^n (X \setminus V_{z_i}) \\ & = & Z \cap (X \setminus \bigcup_{i=1}^n V_{z_i}) \\ & \stackrel{Z \subseteq \bigcup_{i=1}^n V_{z_i}}{=} & \varnothing. \end{array}$$

Obtaining Dual Spaces Indirectly

Theorem

(i) Let Y be a Boolean space, let B be the algebra $\mathcal{P}^{\mathcal{T}}(Y)$ of clopen subsets of Y and let X be the dual space of B. Then Y and X are homeomorphic.

(ii) Let C be a Boolean algebra and Y a Boolean space such that $C \cong \mathcal{P}^{\mathcal{T}}(Y)$. Then the dual space of C is (homeomorphic to) Y.

- We define $\varepsilon : Y \to X$ by $\varepsilon(y) := \{a \in B : y \notin a\}$. Certainly $\varepsilon(y)$ is a prime ideal in B. We shall show that ε is a continuous bijection from Y onto X. It then follows by a topological result that ε is a homeomorphism.
  - Because Y is totally disconnected, if $y \neq z$ in Y then there exists a clopen subset a of Y, such that $y \in a$ and $z \notin a$. Hence ε is injective.
  - To establish continuity of ε it suffices to show that $\varepsilon^{-1}(X_a)$ is clopen for each $a \in B$: By the definition of X_a and the definition of ε we have $\varepsilon^{-1}(X_a) = \{y \in Y : \varepsilon(y) \in X_a\} = \{y \in Y : a \notin \varepsilon(y)\} = a$.

Obtaining Dual Spaces Indirectly (Cont'd)

- It remains to prove the surjectivity of ε:
  - We prove that ε is surjective. $\varepsilon(Y)$ is a closed subset of X. Suppose by way of contradiction that there exists $x \in X \backslash \varepsilon(Y)$. Then there is a subset X_a of X, such that $\varepsilon(Y) \cap X_a = \varnothing$ and $x \in X_a$. We have

$$\varnothing \stackrel{\varepsilon(Y) \cap X_a = \varnothing}{=} \{x : \varepsilon(x) \in X_a\} = \varepsilon^{-1}(X_a) = a.$$

    But this contradicts $x \in X_a$.

This proves (i), and (ii) follows from it.

Example: The Finite-Cofinite Algebra $\mathrm{FC}(\mathbb{N})$

- Denote by $\mathbb{N}_\infty$ the set of natural numbers with an additional point, ∞, adjoined. We define $\mathcal{T}$ as follows: A subset U of $\mathbb{N}_\infty$ belongs to $\mathcal{T}$ if:

$$\begin{array}{rl} \text{either (a)} & \infty \notin U \\ \text{or (b)} & \infty \in U \text{ and } \mathbb{N}_\infty \backslash U \text{ is finite.} \end{array}$$

- $\mathcal{T}$ is a topology.
- A subset V of $\mathbb{N}_\infty$ is clopen if and only if both V and $\mathbb{N}_\infty \backslash V$ are open. It follows that the clopen subsets of $\mathbb{N}_\infty$ are the finite sets not containing ∞ and their complements.

  Claim: $\mathbb{N}_\infty$ is totally disconnected.

  Given distinct points $x, y \in \mathbb{N}_\infty$, we may assume without loss of generality that $x \neq \infty$. Then $\{x\}$ is clopen and contains x but not y.

The Finite-Cofinite Algebra $\mathrm{FC}(\mathbb{N})$ (Cont'd)

Claim: $\mathbb{N}_\infty$ is compact.

Take an open cover U of $\mathbb{N}_\infty$. Some member of U must contain ∞; say U is such a set. Then $\mathbb{N}_\infty \backslash U$ is finite, by (b). Hence only finitely many members of U are needed to cover $\mathbb{N}_\infty \backslash U$. These, together with U, provide the required finite subcover of U.

- The algebra B of clopen sets of the Boolean space $\mathbb{N}_\infty$ consists of the finite sets not containing ∞ and their complements. Define

  $$f : \mathrm{FC}(\mathbb{N}) \to B \text{ by } f(a) = \begin{cases} a, & \text{if } a \text{ is finite} \\ a \cup \{\infty\}, & \text{if } a \text{ is cofinite} \end{cases}$$

  This map is easily seen to be an isomorphism. Therefore, the dual space of $\mathrm{FC}(\mathbb{N})$ can be identified with $\mathbb{N}_\infty$.
- We can now recognize the elements of $\mathcal{I}_p(B)$.

  The points of $\mathbb{N}$ are in one-to-one correspondence with the principal prime ideals of $\mathrm{FC}(\mathbb{N})$, via the map $n \mapsto \downarrow(\mathbb{N}\backslash\{n\})$. There is a single non-principal prime ideal, associated with ∞: it consists of all finite subsets of $\mathbb{N}$.

Subsection 2

Priestley's Representation Theorem for Distributive Lattices

Order and Topology: Boolean and Finite Distributive Case

- Let L be a distributive lattice and let $X = \mathcal{I}_p(L)$ be its set of prime ideals ordered by inclusion.
- We already have representations for L in two special cases:
  - When L is Boolean and X is topologized in the way described above, L is isomorphic to the algebra $\mathcal{P}^{\mathcal{T}}(X)$ of clopen subsets of X.
    Every prime ideal of a Boolean algebra is maximal. So the order on X is discrete (that is, $x \leq y$ in X if and only if $x = y$). Thus the order has no active role in this case.
  - When L is finite, L is isomorphic to the lattice $\mathcal{O}(X)$ of down-sets of X.
    Suppose X has a topology $\mathcal{T}$ making it a Boolean space. Then $\mathcal{T}$ is the discrete topology, in which every subset is clopen. In this case the topology contributes nothing.

Encompassing all Bounded Distributive Lattices

- To represent L in general we should equip X with
  - the inclusion order and
  - a suitable Boolean space topology.
- A prime candidate for a lattice isomorphic to L would then be the lattice of all clopen down-sets of X.
- This lattice coincides with:
  - $\mathcal{O}(X)$ when L is finite,
  - $\mathcal{P}^{\mathcal{T}}(X)$ when L is Boolean and
  - $\mathcal{P}(X)$ when L is both finite and Boolean.
- We prove that a bounded distributive lattice L is indeed isomorphic to the lattice of clopen down-sets of $\mathcal{I}_p(L)$, ordered by inclusion and appropriately topologized, and thereby obtain a natural common generalization of Birkhoff's and Stone's theorems.
  - The boundedness restriction is necessary because the lattice of clopen down-sets is bounded.
  - Extensions of the theorem to lattices lacking bounds do exist, but are not discussed here.

The Prime Ideal Space of a Bounded Distributive Lattice

- Let L be a distributive lattice with 0 and 1 and, for each $a \in L$, let $X_a := \{I \in \mathcal{I}_p(L) : a \notin I\}$.
- Let $X := \mathcal{I}_p(L)$.
- We want a topology $\mathcal{T}$ on X so that each X_a is clopen. Accordingly, we want all elements in $\mathcal{S} := \{X_b : b \in L\} \cup \{X \backslash X_c : c \in L\}$ to be in $\mathcal{T}$.
- Compared with the Boolean case, there is a double complication:
  - The family $\mathcal{S}$ contains sets of two types;
  - It is also not closed under finite intersections.
- We let
$$\mathcal{B} := \{X_b \cap (X \backslash X_c) : b, c \in L\}.$$
  - Since L has 0 and 1, the set $\mathcal{B}$ contains $\mathcal{S}$.
  - Also $\mathcal{B}$ is closed under finite intersections.
- Finally, we define $\mathcal{T}$ by $U \in \mathcal{T}$ if U is a union of members of $\mathcal{B}$.
- Then $\mathcal{T}$ is the smallest topology containing $\mathcal{S}$, i.e., $\mathcal{S}$ is a subbasis for $\mathcal{T}$ and $\mathcal{B}$ a basis.

Compactness of Prime Ideal Space

Theorem

Let L be a bounded distributive lattice. Then the prime ideal space $\langle \mathcal{I}_p(L); \mathcal{T} \rangle$ is compact.

- By Alexander's Subbasis Lemma, it is sufficient to prove that any open cover $\mathcal{U}$ of $X = \mathcal{I}_p(L)$ by sets in the subbasis $\mathcal{S}$ has a finite subcover. Let

$$\mathcal{U} = \{X_b : b \in A_0\} \cup \{X \backslash X_c : c \in A_1\}.$$

Let J be the ideal generated by A_0 (this is $\{0\}$ if A_0 is empty) and let G be the filter generated by A_1 (this is $\{1\}$ if A_1 is empty).

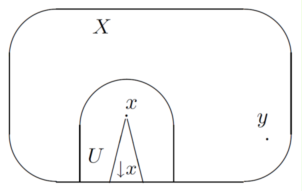
  - Assume first that $J \cap G = \varnothing$. Invoke (DPI) to find a prime ideal I, such that $J \subseteq I$ and $G \cap I = \varnothing$. Then, for all $b \in A_0$, $b \in J \subseteq I$, whence $I \notin X_b$. Moreover, for all $c \in A_1$, $c \in G$, whence, since $G \cap I = \varnothing$, $I \notin X \backslash X_c$. This means that $\mathcal{U}$ does not cover X, a contradiction.

Compactness of Prime Ideal Space (Cont'd)

- Hence $J \cap G \neq \varnothing$. Take $a \in J \cap G$.
  - If A_0 and A_1 are both non-empty, there exist $b_1, \ldots, b_j \in A_0$ and $c_1, \ldots, c_k \in A_1$, such that $c_1 \wedge \cdots \wedge c_k \leq a \leq b_1 \vee \cdots \vee b_j$, whence $X = X_1 = X_{b_1} \cup \cdots \cup X_{b_j} \cup (X \backslash X_{c_1}) \cup \cdots \cup (X \backslash X_{c_k})$. In this case, therefore, $\mathcal{U}$ has a finite subcover.
  - Suppose, now, that $A_1 = \varnothing$. Then $G = \{1\}$. Hence, since $J \cap G \neq \varnothing$, $1 \in J$ and J is improper. Thus, there exist $b_1, \ldots, b_j \in A_0$, such that $1 = b_1 \vee \cdots \vee b_j$. Hence, $X = X_1 = X_{b_1 \vee \cdots \vee b_j} = X_{b_1} \cup \cdots \cup X_{b_j}$. So $\mathcal{U}$ has a finite subcover.
  - Suppose, finally, that $A_0 = \varnothing$. Then $J = \{0\}$. Hence, since $J \cap G \neq \varnothing$, $0 \in G$ and G is improper. Thus, there exist $c_1, \ldots, c_k \in A_1$, such that $0 = c_1 \wedge \cdots \wedge c_k$. Hence,
  $X = X \backslash X_0 = X \backslash X_{c_1 \wedge \cdots \wedge c_k} = X \backslash (X_{c_1} \cap \cdots \cap X_{c_k}) = (X \backslash X_{c_1}) \cup \cdots \cup (X \backslash X_{c_k})$.
  So $\mathcal{U}$ has a finite subcover.

Totally Order-Disconnected Spaces

- A set X carrying an order relation $\leq$ and a topology $\mathcal{T}$ is called an **ordered (topological) space** and denoted $\langle X; \leq, \mathcal{T} \rangle$ (or by X when no ambiguity would result).
- It is said to be **totally order-disconnected** if, given $x, y \in X$, with $x \not\geq y$, there exists a clopen down-set U, such that $x \in U$ and $y \notin U$.



- A compact totally order-disconnected space is called a **Priestley space**, also known as an **ordered Stone space** or a **CTOD space**.
- We shall denote by $\mathcal{O}^{\mathcal{T}}(X)$ the family of clopen down-sets of a Priestley space X:
  - When the order on X is discrete, $\mathcal{O}^{\mathcal{T}}(X)$ coincides with $\mathcal{P}^{\mathcal{T}}(X)$;
  - When X is finite, $\mathcal{O}^{\mathcal{T}}(X)$ coincides with $\mathcal{O}(X)$.

Existence of Certain Clopen Down-Sets

- In many ways Priestley spaces behave like a cross between Boolean spaces and ordered sets.

Lemma

Let $\langle X; \leq, \mathcal{T}\rangle$ be a Priestley space.

(i) $x \leq y$ in X if and only if $y \in U$ implies $x \in U$, for every $U \in \mathcal{O}^{\mathcal{T}}(X)$.

(ii) (a) Let Y be a closed down-set in X and let $x \notin Y$. Then there exists a clopen down-set U such that $Y \subseteq U$ and $x \notin U$.

(i) Let Y and Z be disjoint closed subsets of X such that Y is a down-set and Z is an up-set. Then there exists a clopen down-set U such that $Y \subseteq U$ and $Z \cap U = \varnothing$.

(a) Suppose $x \leq y$ and $U \in \mathcal{O}^{\mathcal{T}}(X)$, such that $y \in U$. Since U is a down-set, $x \in U$.

Suppose, conversely, that $x \nleq y$. Since X is a Priestley space, there exists $U \in \mathcal{O}^{\mathcal{T}}(X)$, such that $y \in U$ and $x \notin U$.

Existence of Certain Clopen Down-Sets (Part (ii))

(a) Let Y be a closed down-set in X and let $x \notin Y$. Since Y is a down-set and $x \notin Y$, for all $y \in Y$, $x \nleq y$. Since X is a Priestley space, there exists a clopen down-set U_y, such that $y \in U_y$ and $x \notin U_y$. The collection $\{U_y : y \in Y\}$ covers Y. Since Y is closed, it is compact. Thus, there exists a finite subcover $\{U_{y_1}, \ldots, U_{y_n}\}$. Set $U = \bigcup_{i=1}^n U_{y_i}$. Since, for all $y \in Y$, U_y is a clopen downset, the same holds for U. Moreover, $Y \subseteq U$ and $x \notin U$, since $x \notin U_y$, for all $y \in Y$.

(b) By Part (a), for all $z \in Z$, there exists a clopen down-set U_z, such that $Y \subseteq U_z$ and $z \notin U_z$. Then $\{X \backslash U_z : z \in Z\}$ is a cover of the compact set Z. Thus, there exists a finite subcover $\{X \backslash U_{z_1}, \ldots, X \backslash U_{z_n}\}$. Set $U = \bigcap_{i=1}^n U_{z_i}$. Since each U_z is a clopen down-set, so is U. Moreover, since $Y \subseteq U_z$, for all $z \in Z$, $Y \subseteq U$. Finally, since $Z \subseteq \bigcup_{i=1}^n (X \backslash U_{z_i}) = X \backslash \bigcap_{i=1}^n U_{z_i}$, we get $Z \cap U = \varnothing$.

Clopen Sets and Down-Sets in $\mathcal{I}p(L)$

- We characterize clopen sets and clopen down-sets in the dual space $\langle \mathcal{I}_p(L); \subseteq, \mathcal{T} \rangle$ of a bounded distributive lattice L.

Lemma

Let L be a bounded distributive lattice with dual space $\langle X; \subseteq, \mathcal{T} \rangle$, where $X = \mathcal{I}_p(L)$. Then:

(i) the clopen subsets of X are the finite unions of sets of the form $X_b \cap (X \backslash X_c)$, for $b, c \in L$;

(ii) the clopen down-sets of X are exactly the sets X_a, for $a \in L$.

(i) Suppose $Y = \bigcup_{i=1}^n (X_{b_i} \cap (X \backslash X_{c_i}))$. Since, for all i, $X_{b_i} \cap (X \backslash X_{c_i}) \in \mathcal{B}$, we get that Y is open. On the other hand, $X \backslash Y = X \backslash \bigcup_{i=1}^n (X_{b_i} \cap (X \backslash X_{c_i})) = \bigcap_{i=1}^n (X \backslash (X_{b_i} \cap (X \backslash X_{c_i}))) = \bigcap_{i=1}^n ((X \backslash X_{b_i}) \cup X_{c_i})$. Since $X \backslash X_{b_i}$ and X_{c_i} are open, we get that $X \backslash Y$ is open, whence Y is also closed. Thus, it is clopen.

Clopen Sets and Down-Sets in $\mathcal{I}p(L)$ (Cont'd)

Suppose, conversely, that Y is clopen. Since it is open it is a union of sets in $\mathcal{B}$, i.e., $Y = \bigcup_{i \in I}(X_{b_i} \cap (X \backslash X_{c_i}))$. Thus, $\{X_{b_i} \cap (X \backslash X_{c_i}) : i \in I\}$ is an open cover of Y. But Y is also closed and, hence, compact. Thus, there exists a finite subcover $\{X_{b_i} \cap (X \backslash X_{c_i}) : i = 1, \ldots, n\}$ of Y. Thus, $Y = \bigcup_{i=1}^{n}(X_{b_i} \cap (X \backslash X_{c_i}))$.

(ii) Suppose, first, that $J \in X_a$ and $I \subseteq J$. Then, by definition, $a \notin J$. Since $I \subseteq J$, $a \notin I$. Hence, again by definition, $I \in X_a$. Thus, X_a is a down-set.

Suppose conversely, that Y is a clopen down-set in X. Since it is clopen, by Part (i), it is of the form $Y = \bigcup_{i=1}^{n}(X_{b_i} \cap (X \backslash X_{c_i}))$. Since, for all i, $X_{b_i} \cap (X \backslash X_{c_i}) \subseteq Y$ and Y is a down-set, $c_i = 0$. Hence $Y = \bigcup_{i=1}^{n} X_{b_i}$ Therefore, $Y = X_{b_1 \vee \cdots \vee b_n}$.

Priestley's Representation for Distributive Lattices

Priestley's Representation Theorem for Distributive Lattices

Let L be a bounded distributive lattice. Then the map

$$\eta : a \mapsto X_a := \{I \in \mathcal{I}_p(L) : a \notin I\}$$

is an isomorphism of L onto the lattice of clopen down-sets of the dual space $\langle \mathcal{I}_p(L); \subseteq, \mathcal{T} \rangle$ of L.

- Combine a preceding representation theorem with the preceding characterization of the clopen down-sets of X.

Priestley Spaces and Order-Homeomorphisms

- Ordered spaces X and Y are "essentially the same" if there exists a map φ from X onto Y which is simultaneously an order-isomorphism and a homeomorphism. We call such a map an **order-homeomorphism** and say X and Y are **order-homeomorphic**.

Theorem

(i) Let Y be a Priestley space, let L be the lattice $\mathcal{O}^{\mathcal{T}}(Y)$ of clopen down-sets of Y and let X be the dual space of L. Then Y and X are order-homeomorphic.

(ii) Let L be a bounded distributive lattice and Y a Priestley space such that $\mathcal{O}^{\mathcal{T}}(Y) \cong L$. Then the dual space of L is (order-homeomorphic to) Y.

- The second part follows from the first. The proof of (i) is similar to the proof given for the Boolean case: We define $\varepsilon : Y \to X$ by $\varepsilon(y) := \{a \in L : y \notin a\}$. Certainly $\varepsilon(y)$ is a prime ideal in L.

Priestley Spaces and Order-Homeomorphisms (Cont'd)

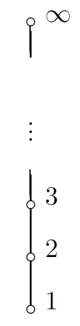
- We set $\varepsilon(y) := \{a \in L : y \notin a\}$. We must show that:
  - (a) ε is an order-embedding;
  - (b) ε is continuous;
  - (c) ε maps Y onto X.

  Combined with A.7 this will establish (i).
  - (a) Note that $y \le z$ in Y iff $(\forall a \in L)(z \in a \Rightarrow y \in a)$ iff $\varepsilon(y) \subseteq \varepsilon(z)$.
  - (b) We use A.4: (b) holds so long as $\varepsilon^{-1}(X_a)$ and $\varepsilon^{-1}(X \backslash X_a)$ are open, for each $a \in L$. But $\varepsilon^{-1}(X \backslash X_a) = \{y \in Y : \varepsilon(y) \notin X_a\} = Y \backslash \varepsilon^{-1}(X_a)$. Thus (b) holds provided $\varepsilon^{-1}(X_a)$ is clopen in Y, for each $a \in L$. By the definitions of X_a and ε, $\varepsilon^{-1}(X_a) = \{y \in Y : \varepsilon(y) \in X_a\} = \{y \in Y : a \notin \varepsilon(y)\} = a$, and this is clopen, by the definition of L.
  - (c) By Lemma A.7, $\varepsilon(Y)$ is a closed subset of X. Suppose by way of contradiction that there exists $x \in X \backslash \varepsilon(Y)$. Then there is a clopen subset V of X, such that $\varepsilon(Y) \cap V = \varnothing$ and $x \in V$. By the last lemma, we may assume that $V = X_b \cap (X \backslash X_c)$, for some $b, c \in L$. We have $\varnothing = \varepsilon^{-1}(V) = b \cap (Y \backslash c)$. Thus $b \subseteq c$, contradicting $x \in X_b \cap (X \backslash X_c)$.

Example I

- A variety of Priestley spaces can be obtained by equipping the Boolean space $\mathbb{N}_\infty$ with an order.

  For a very simple example, order $\mathbb{N}_\infty$ as the chain $\mathbb{N}$ with ∞ adjoined as top element:

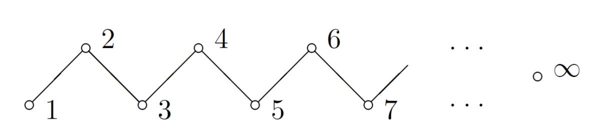


  Take $x \not\geq y$. Then $y > x$ and $\downarrow x$, which is clopen because it is finite and does not contain ∞, contains x but not y. Hence we have a Priestley space.

  Its lattice of clopen down-sets is isomorphic to the chain $\mathbb{N} \oplus \mathbf{1}$.

Example II

- Consider the ordered space Y obtained by equipping $\mathbb{N}_\infty$ with the order depicted below:



We have $n \succ n-1$ and $n \succ n+1$, for each even n.

For each $n \in \mathbb{N}$, the down-set $\downarrow n$ is finite and does not contain ∞ and so is clopen.

Let $x \not\geq y$ in Y.

Claim: There exists $U \in \mathcal{O}^{\mathcal{T}}(Y)$, such that $x \in U$ and $y \notin U$.

- If $x \neq \infty$, $y \notin \downarrow x$. Thus, we may take $U = \downarrow x$.
- If $x = \infty$, we may take $U = Y \backslash \{1, 2, \ldots, 2y\}$.

Hence Y is a Priestley space.

The lattice $\mathcal{O}^{\mathcal{T}}(Y)$, a sublattice of $\mathrm{FC}(\mathbb{N})$, is easily described.

Subsection 3

Duality: Distributive Lattices and Priestley Spaces

Duality

- Denote the class of bounded distributive lattices by **D**, and the class of Priestley spaces (compact totally order-disconnected spaces) by **P**.
- Define maps $D : \mathbf{D} \to \mathbf{P}$ and $E : \mathbf{P} \to \mathbf{D}$ by
$$D : L \mapsto \mathcal{I}_p(L) \quad (L \in \mathbf{D}) \qquad \text{and} \qquad E : X \mapsto \mathcal{O}^{\mathcal{T}}(X) \quad (X \in \mathbf{P}).$$
- Preceding theorems assert that, for all $L \in \mathbf{D}$ and $X \in \mathbf{P}$,
$$ED(L) \cong L \quad \text{and} \quad DE(X) \cong X,$$
the latter $\cong$ means "is order-homeomorphic to".
- We may use the isomorphism between L and $ED(L)$ to represent the members of **D** concretely as lattices of the form $\mathcal{O}^{\mathcal{T}}(X)$ for $X \in \mathbf{P}$.
- As an immediate application we note that the representation allows us to construct a "smallest" Boolean algebra B containing (an isomorphic copy of) a given lattice $L \in \mathbf{D}$:
  - Identify L with $\mathcal{O}^{\mathcal{T}}(X)$ and take $B = \mathcal{P}^{\mathcal{T}}(X)$.
  - We already saw how $\mathcal{O}^{\mathcal{T}}(X)$ and $\mathcal{P}^{\mathcal{T}}(X)$ are related.

Pseudocomplements

- There are many ways to weaken the condition that every element of a bounded lattice L have a complement.
- One possibility is to define the **pseudocomplement** of an element a in a lattice L with 0 to be

$$a^* = \max\{b \in L : b \wedge a = 0\},$$

  if this exists.
- Now consider $L = \mathcal{O}^{\mathcal{T}}(X)$, where X is a Priestley space.

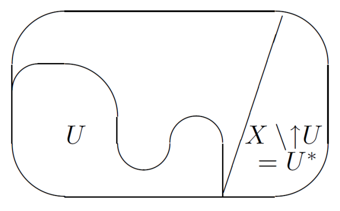
  Claim: $U \in L$ has a pseudocomplement if and only if $\uparrow U$ is clopen, and, in that case, $U^* = X \backslash \uparrow U$.

  Observe that a down-set W in X does not intersect U if and only if $W \subseteq X \backslash \uparrow U$. Hence $X \backslash \uparrow U$ is the largest down-set disjoint from U. If it is also clopen, it must be U^*.

Pseudocomplements (Cont'd)

- Conversely, assume U^* exists. Take $x \notin \uparrow U$. We show $x \in U^*$, from which it follows that $U^* = X \backslash \uparrow U$.

  U is clopen. Since, for all Y closed $\uparrow Y$ is closed, $\uparrow U$ is closed. By the dual of a preceding lemma, we can find a clopen up-set V, such that $x \notin V$ and $\uparrow U \subseteq V$.



  Then $(X \backslash V) \cap U = \varnothing$, so $X \backslash V \subseteq U^*$, by definition of U^*. This implies that $x \in U^*$.

Duality for ideals

- Let $L = \mathcal{O}^{\mathcal{T}}(X)$, where X is a Priestley space whose family of open down-sets we denote by $\mathcal{L}$.
- An ideal J of L is determined by its members, which are clopen down-sets of X.

  Define
  $$\Phi(J) = \bigcup\{U :| U \in J\} \quad (\text{for } J \in \mathcal{I}(L)).$$
  As a union of clopen sets, $\Phi(J)$ is an open set (but not in general clopen).
- In the other direction, define
  $$\Psi(W) = \{U \in \mathcal{O}^{\mathcal{T}}(X) : U \subseteq W\} \quad (\text{for } W \in \mathcal{L}).$$
  It is easily checked that $\Psi(W)$ is an ideal of L.

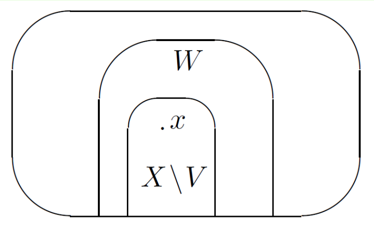
Duality for ideals (Cont'd)

Claim:

- $\Phi(\Psi(W)) = W$, for all $W \in \mathcal{L}$;
- $\Psi(\Phi(J)) = J$, for all $J \in \mathcal{I}(L)$.

The first equation, $W = \bigcup\{U \in \mathcal{O}^{\mathcal{T}}(X) : U \subseteq W\}$, asserts that an open down-set W is the union of the clopen down-sets contained in it.

Let $x \in W$. Use the dual of a preceding lemma to find a clopen up-set V containing the closed up-set $X \backslash W$ but with $x \notin V$.



Then $X \backslash V$ is a clopen down-set, and $x \in X \backslash V \subseteq W$.

Duality for ideals (Cont'd)

Claim:

- $\Phi(\Psi(W)) = W$, for all $W \in \mathcal{L}$;
- $\Psi(\Phi(J)) = J$, for all $J \in \mathcal{I}(L)$.

For the second equation, $J = \{U \in \mathcal{O}^{\mathcal{T}}(X) : U \subseteq \bigcup\{W : W \in J\}\}$, $J \subseteq \Psi(\Phi(J))$.

Let $V \in \Psi(\Phi(J))$. This means that J, regarded as a family of open subsets of X, is an open cover of the clopen set V. Since V is closed, it is compact. Thus, only finitely many elements of J are needed to cover V, say $U_1, \ldots, U_n$. But $V \subseteq U_1 \cup \cdots \cup U_n$ implies $V \subseteq J$, since J is an ideal. This establishes the second equation.

- The bijective correspondence we have set up between $\mathcal{I}(L)$ and $\mathcal{L}$ is in fact a lattice isomorphism.

  In addition, special types of ideal correspond to special types of open set.
- Similarly, for filters $\mathcal{F}(L) \cong \mathcal{F}$, the lattice of open up-sets of X.

Extending **D** and **P** to Encompass Morphisms

- There exists what is known as a (**full**) **duality** between **D** (bounded distributive lattices $+\{0,1\}$-homomorphisms) and **P** (Priestley spaces + continuous order-preserving maps).
- Here, the symbols **D** and **P** encompass structure-preserving maps as well as objects.
- For $L, K \in \mathbf{D}$, we denote the set of $\{0,1\}$-homomorphisms from L to K by $\mathbf{D}(L, K)$.
- For $X, Y \in \mathbf{P}$, we denote the set of continuous order-preserving maps from Y to X by $\mathbf{P}(Y, X)$.

Duality

- The way the duality is required to work is given by:
  - (O) There exist maps $D : \mathbf{D} \to \mathbf{P}$ and $E : \mathbf{P} \to \mathbf{D}$, such that:
    - (i) for each $L \in \mathbf{D}$, there exists $\eta_L : L \to ED(L)$, such that η_L is an isomorphism;
    - (ii) for each $X \in \mathbf{P}$, there exists $\varepsilon_X : X \to DE(X)$, such that ε_X is an order-homeomorphism.
  - (M) For any $L, K \in \mathbf{D}$, there exists, for each $f \in \mathbf{D}(L, K)$, a map $D(f) \in \mathbf{P}(D(K), D(L))$. For each $X, Y \in \mathbf{P}$, there exists, for each $\varphi \in \mathbf{P}(Y, X)$, a map $E(\varphi) \in \mathbf{D}(E(X), E(Y))$. The maps $D : \mathbf{D}(L, K) \to \mathbf{P}(D(K), D(L))$ and $E : \mathbf{P}(Y, X) \to \mathbf{D}(E(X), E(Y))$ are bijections and the diagrams below commute: