Introduction to Lattices and Order

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LSSU Math 400

1. Representation: The Finite Case
   - Building Blocks for Lattices
   - Finite Boolean Algebras are Powerset Algebras
   - Finite Distributive Lattices are Down-Set Lattices
   - Finite Distributive Lattices and Finite Ordered Sets

Subsection 1

Building Blocks for Lattices

"Skeletal" Subset of a Lattice

- Recall that a non-zero element x of a lattice L is **join-irreducible** if $x = a \vee b$ implies $x = a$ or $x = b$, for all $a, b \in L$.
- We saw that, if L satisfies (DCC), and hence certainly if L is finite, the set $\mathcal{J}(L)$ of join-irreducible elements of L is join-dense: every element of L can be obtained as a join of elements from $\mathcal{J}(L)$.
- We would like to discover a way of building a lattice L from a suitable "skeletal" subset P of L, having the following properties:
  (i) P is "small" and readily identifiable;
  (ii) L is uniquely determined by the ordered set P;
  (iii) the process for obtaining L from P is simple to carry out.
- Conditions (i) and (ii) pull in opposite directions, since (ii) requires P to be, in some sense, large.
- Many important lattices are distributive. e.g., down-set lattices and, in the Boolean case, powerset lattices.

  We show that the join-irreducible elements of a finite distributive lattice form a good skeleton for it.

Atoms

- Our archetypal example of a Boolean algebra is a powerset algebra $\langle \mathcal{P}(X); \cup, \cap, ', \varnothing, X \rangle$.

  Any $A \in \mathcal{P}(X)$ is a union of singleton sets $\{x\}$ for $x \in A$:

  The singletons are precisely the join-irreducible elements.

  The singletons are exactly the elements in $\mathcal{P}(X)$ which cover 0.

- Let L be a lattice with least element 0.

  Then $a \in L$ is called an **atom** if $0 \lessdot a$. The set of atoms of L is denoted by $\mathcal{A}(L)$.

  The lattice L is called **atomic** if, given $a \neq 0$ in L, there exists $x \in \mathcal{A}(L)$, such that $x \leq a$.

  Example: Every finite lattice is atomic. By contrast, it may happen that an infinite lattice has no atoms at all. The chain of non-negative real numbers provides an example.

Atoms and Join-Irreducibles

- The following lemma compares atoms and join-irreducible elements. It shows that in any Boolean lattice, $\mathcal{J}(L)$ coincides with $\mathcal{A}(L)$.

Lemma

Let L be a lattice with least element 0. Then:

(i) $0 \lessdot x$ in L implies $x \in \mathcal{J}(L)$;

(ii) If L is a Boolean lattice, $x \in \mathcal{J}(L)$ implies $0 \lessdot x$.

(i) Suppose by way of contradiction that $0 \lessdot x$ and $x = a \vee b$ with $a < x$ and $b < x$. Since $0 \lessdot x$, we have $a = b = 0$. Thus, $x = 0$, a contradiction.

(ii) Now assume L is a Boolean lattice and that $x \in \mathcal{J}(L)$. Suppose $0 \leq y < x$. We want $y = 0$. We have $x = x \vee y = (x \vee y) \wedge (y' \vee y) = (x \wedge y') \vee y$. Since x is join-irreducible and $y < x$, we must have $x = x \wedge y'$, whence $x \leq y'$. But then $y = x \wedge y \leq y' \wedge y = 0$. So $y = 0$.

Subsection 2

Finite Boolean Algebras are Powerset Algebras

Determining Elements via Atoms

- The set of atoms, $\mathcal{A}(L)$, of a finite Boolean lattice L meets the building block criteria:

Lemma

Let B be a finite Boolean lattice. Then, for all $a \in B$,

$$a = \bigvee\{x \in \mathcal{A}(B) : x \leq a\}.$$

- Fix $a \in B$. Let $S = \{x \in \mathcal{A}(B) : x \leq a\}$. Certainly a is an upper bound for S. Let b be any upper bound for S. We must show $a \leq b$. Suppose not. Then $0 < a \wedge b'$. Choose $x \in \mathcal{A}(B)$, such that $0 \lessdot x \leq a \wedge b'$. Then $x \in S$. So $x \leq b$. Since $x \leq b'$ also holds, we have $x \leq b \wedge b' = 0$, a contradiction.
- The lemma tells us how each individual element of L is determined by the atoms, but it does not by itself fulfill the aim of Criterion (ii).

Representation Theorem for Finite Boolean Algebras

The Representation Theorem for Finite Boolean Algebras

Let B be a finite Boolean algebra. Then $\eta : a \mapsto \{x \in \mathcal{A}(B) : x \leq a\}$ is an isomorphism of B onto $\mathcal{P}(X)$, where $X = \mathcal{A}(B)$, with the inverse of η given by $\eta^{-1}(S) = \bigvee S$ for $S \in \mathcal{P}(X)$.

- η maps B onto $\mathcal{P}(X)$: Clearly $\varnothing = \eta(0)$. Now let $S = \{a_1, \ldots, a_k\}$ be a non-empty set of atoms of B and define $a = \bigvee S$. We claim $S = \eta(a)$. Certainly $S \subseteq \eta(a)$. Now let x be any atom, such that $x \leq a = a_1 \vee \cdots \vee a_k$. For each i, we have $0 \leq x \wedge a_i \leq x$. Because x is an atom, either $x \wedge a_i = 0$, for all i, or there exists j, such $x \wedge a_j = x$. In the former case, $x = x \wedge a = (x \wedge a_1) \vee \cdots \vee (x \wedge a_k) = 0$, a contradiction. Therefore $x \leq a_j$, for some j, which forces $x = a_j$, as a_j and x are atoms. Hence $\eta(a) \subseteq S$, as we wished to show. Let $a, b \in B$. Then $\eta(a) \subseteq \eta(b)$ implies that $a = \bigvee \eta(a) \leq \bigvee \eta(b) = b$. It is trivial (by the transitivity of $\leq$) that $\eta(a) \subseteq \eta(b)$ whenever $a \leq b$. So η is an order-isomorphism. Thus, it is an isomorphism of Boolean algebras.

Shape of Finite Boolean Lattices

Corollary

Let B be a finite lattice. Then the following statements are equivalent:

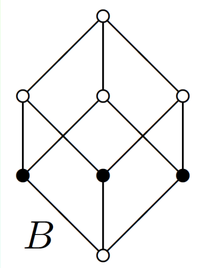
(i) B is a Boolean lattice;

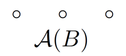
(ii) $B \cong \mathcal{P}(\mathcal{A}(B))$;

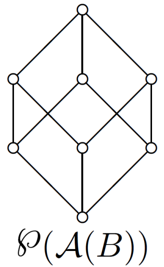
(iii) B is isomorphic to $\mathbf{2}^n$, for some $n \geq 0$.

Further, any finite Boolean lattice has $\mathbf{2}^n$ elements, for some $n \geq 0$.

Example:







Subsection 3

Finite Distributive Lattices are Down-Set Lattices

Join Irreducible Down-Sets

- Let P be an ordered set.

  Claim: Each set $\downarrow x$, for $x \in P$, is join-irreducible in $\mathcal{O}(P)$.

  Suppose that $\downarrow x = U \cup V$, where $U, V \in \mathcal{O}(P)$. Without loss of generality, $x \in U$. But then $\downarrow x \subseteq U$. Since $\downarrow x = U \cup V$ implies $U \subseteq \downarrow x$, we conclude that $\downarrow x = U$. This shows that $\downarrow x \in \mathcal{J}(\mathcal{O}(P))$.
- Now assume that P is finite. Any non-empty $U \in \mathcal{O}(P)$ is the union of sets $\downarrow x_i$, $i = 1, \ldots, k$, where $x_i \parallel x_j$, for $i \neq j$. Unless $k = 1$, the set U is not join-irreducible. Hence, $\mathcal{J}(\mathcal{O}(P)) = \{\downarrow x : x \in P\}$.
- In the previous paragraph P must be finite:

  $\{q \in \mathbb{Q} : q < 0\}$ is join-irreducible in $\mathcal{O}(\mathbb{Q})$, but is not of the form $\downarrow x$.

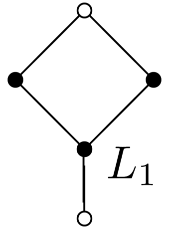
Join-Irreducible Lattice of Down-Set Lattice

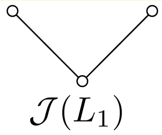
Theorem

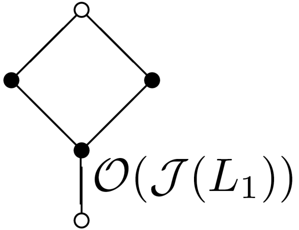
Let P be a finite ordered set. Then the map $\varepsilon : x \mapsto \downarrow x$ is an order-isomorphism from P onto $\mathcal{J}(\mathcal{O}(P))$.

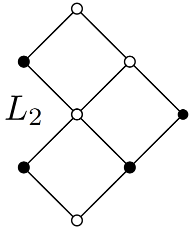
- We know that ε is an order-embedding of P into $\mathcal{O}(P)$. By the preceding claim, the image of ε is $\mathcal{J}(\mathcal{O}(P))$.
- For order-isomorphic ordered sets P and Q we have $\mathcal{O}(P) \cong \mathcal{O}(Q)$. Therefore, the theorem tells us that, when L is a finite down-set lattice $\mathcal{O}(P)$, we have $L \cong \mathcal{O}(\mathcal{J}(L))$.

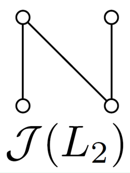
Examples I

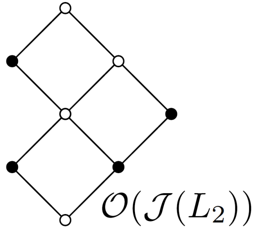






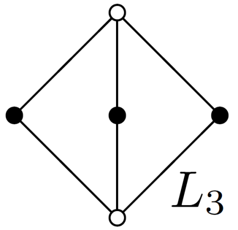


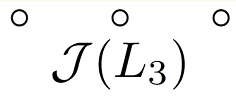


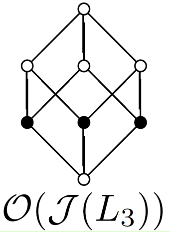


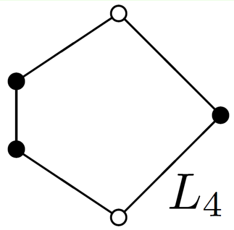
Observe that $L \cong \mathcal{O}(\mathcal{J}(L))$.

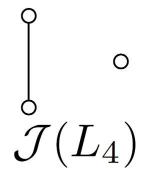
Examples II

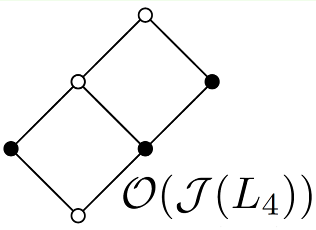












Since $\mathcal{O}(\mathcal{J}(L))$ is distributive, we cannot have $L \cong \mathcal{O}(\mathcal{J}(L))$ unless L is distributive.

Join-Irreducibles in Distributive Lattices

Lemma

Let L be a distributive lattice and let $x \in L$, with $x \neq 0$ in case L has a zero. Then the following are equivalent:

(i) x is join-irreducible;

(ii) if $a, b \in L$ and $x \leq a \vee b$, then $x \leq a$ or $x \leq b$;

(iii) for any $k \in \mathbb{N}$, if $a_1, \ldots, a_k \in L$ and $x \leq a_1 \vee \cdots \vee a_k$, then $x \leq a_i$, for some i $(1 \leq i \leq k)$.

(i)⇒(ii): Assume that $x \in \mathcal{J}(L)$ and that $a, b \in L$ are such that $x \leq a \vee b$. We have $x = x \wedge (a \vee b)$ (since $x \leq a \vee b$) $= (x \wedge a) \vee (x \wedge b)$ (since L is distributive). Because x is join-irreducible, $x = x \wedge a$ or $x = x \wedge b$. Hence $x \leq a$ or $x \leq b$, as required.

Join-Irreducibles in Distributive Lattices (Cont'd)

(ii)⇒(iii): This is proved by induction on k:

The case $k = 1$ is trivial;

The case $k = 2$ is by the hypothesis (ii).

Assume the conclusion holds for $k = n$.

Let $a_1, \ldots, a_n, a_{n+1} \in L$, such that $x \leq a_1 \vee \cdots \vee a_n \vee a_{n+1}$. Then, $x \leq (a_1 \vee \cdots \vee a_n) \vee a_{n+1}$. By (ii), $x \leq a_1 \vee \cdots \vee a_n$ or $x \leq a_{n+1}$. By the Induction Hypothesis, $x \leq a_1$ or $\cdots$ or $x \leq a_n$ or $x \leq a_{n+1}$.

Therefore, (iii) holds for all $k \in \mathbb{N}$.

(iii)⇒(ii): Trivial.

(ii)⇒(i): Suppose (ii) holds and that $x = a \vee b$. Then certainly $x \leq a \vee b$, so $x \leq a$ or $x \leq b$. But $x = a \vee b$ forces $x \geq a$ and $x \geq b$. Hence $x = a$ or $x = b$.

Representation Theorem for Finite Distributive Lattices

Birkhoff's Representation Theorem for Finite Distributive Lattices

Let L be a finite distributive lattice. Then the map $\eta : L \to \mathcal{O}(\mathcal{J}(L))$ defined by

$$\eta(a) = \{x \in \mathcal{J}(L) : x \leq a\} (= \mathcal{J}(L) \cap \downarrow a)$$

is an isomorphism of L onto $\mathcal{O}(\mathcal{J}(L))$.

- It is immediate that $\eta(a) \in \mathcal{O}(\mathcal{J}(L))$ (since $\leq$ is transitive). It remains only to show that η is an order-isomorphism:
  - We have seen $a \leq b$ implies $\eta(a) \subseteq \eta(b)$. Conversely, suppose $\eta(a) \subseteq \eta(b)$. Then $a = \bigvee \eta(a) \leq \eta(b) = b$.
  - Finally, we prove that η is onto. Certainly $\varnothing = \eta(0)$. Now let $\varnothing \neq U \in \mathcal{O}(\mathcal{J}(L))$ and write $U = \{a_1, \ldots, a_k\}$. Define a to be $a_1 \vee \cdots \vee a_k$. We claim $U = \eta(a)$. First, let $x \in U$, so $x = a_i$, for some i. Then x is join-irreducible and $x \leq a$, hence $x \in \eta(a)$. Next, suppose $x \in \eta(a)$. Then $x \leq a = a_1 \vee \cdots \vee a_k$. Thus, $x \leq a_i$, for some i. Since U is a down-set and $a_i \in U$, we have $x \in U$.

Characterizing Finite Distributive Lattices

Corollary

Let L be a finite lattice. Then the following statements are equivalent:

(i) L is distributive;

(ii) $L \cong \mathcal{O}(\mathcal{J}(L))$;

(iii) L is isomorphic to a down-set lattice;

(iv) L is isomorphic to a lattice of sets;

(v) L is isomorphic to a sublattice of $\mathbf{2}^n$ for some $n \geq 0$.

- Of course, no non-distributive lattice could be isomorphic to a down-set lattice.
- Birkhoff's Representation Theorem provides an alternative to the $\mathbf{M}_3$-$\mathbf{N}_5$ Theorem for establishing nondistributivity of a finite lattice L: If $L \cong \mathcal{O}(\mathcal{J}(L))$ fails, then L cannot be distributive.

  Example: We saw that $\mathbf{M}_3$ and $\mathbf{N}_5$ form such examples.

Subsection 4

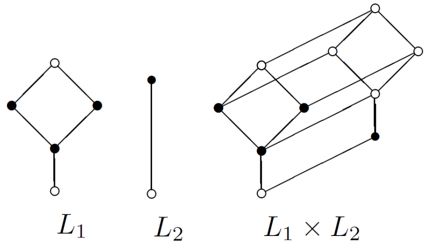
Finite Distributive Lattices and Finite Ordered Sets

The Join-Irreducible Elements of a Product Lattice

- Consider the product $L_1 \times L_2$ of lattices L_1 and L_2 each with a least element, but not necessarily distributive.
- Note that $(x_1, x_2) = (x_1, 0) \vee (0, x_2)$. Thus, (x_1, x_2) is not join-irreducible unless either x_1 or x_2 is zero. Further, $x_1 = a_1 \vee b_1$ in L_1 implies $(x_1, 0) = (a_1, 0) \vee (b_1, 0)$. It follows that $\mathcal{J}(L_1 \times L_2) \subseteq (\mathcal{J}(L_1) \times \{0\}) \cup (\{0\} \times \mathcal{J}(L_2))$.

  It is readily seen that the reverse inclusion also holds.

  We have an order-isomorphism $\mathcal{J}(L_1 \times L_2) \cong \mathcal{J}(L_1) \,\dot{\cup}\, \mathcal{J}(L_2)$.



Product of Finite Distributive Lattices

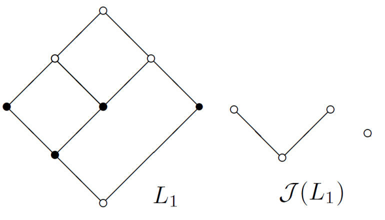
- Now assume that L_1 and L_2 are finite and distributive.
- In this case, the result of the previous paragraph can be derived from
  (a) $P \cong \mathcal{J}(\mathcal{O}(P))$;
  (b) Birkhoff's Representation Theorem;
  (c) the fact that $\mathcal{O}(P_1 \dot{\cup} P_2)$ is isomorphic to $\mathcal{O}(P_1) \times \mathcal{O}(P_2)$.

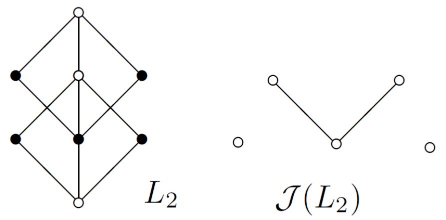
  Namely, we have:

$$\begin{array}{rcl} \mathcal{J}(L_1 \times L_2) & \overset{\text{(b)}}{\cong} & \mathcal{J}(\mathcal{O}(\mathcal{J}(L_1)) \times \mathcal{O}(\mathcal{J}(L_2))) \\ & \overset{\text{(c)}}{\cong} & \mathcal{J}(\mathcal{O}(\mathcal{J}(L_1) \dot{\cup} \mathcal{J}(L_2))) \\ & \overset{\text{(a)}}{\cong} & \mathcal{J}(L_1) \dot{\cup} \mathcal{J}(L_2). \end{array}$$

Some Examples

(1) Consider the lattice L_1.





The ordered set $\mathcal{J}(L_1)$ is also shown. Since $\mathcal{J}(L_1) \cong \mathbf{1} \cup (\mathbf{1} \oplus \overline{\mathbf{2}})$, we have $\mathcal{O}(\mathcal{J}(L_1)) \cong \mathbf{2} \times (\mathbf{1} \oplus \mathbf{2}^2)$, which has 10 elements. We deduce that L_1 is not isomorphic to $\mathcal{O}(\mathcal{J}(L_1))$. So L_1 is not distributive.

(2) Now consider L_2. We could compute $\mathcal{O}(\mathcal{J}(L_2))$ directly. Instead, we note that $L_2 \cong L_2^\partial$, but $\mathcal{J}(L_2)$ is not isomorphic to its order dual. Hence, L_2 cannot be isomorphic to $\mathcal{O}(\mathcal{J}(L_2))$. Consequently, L_2 is not distributive.

Finite Distributive Lattices and Finite Ordered Sets

- We denote by $\mathbf{D}_F$ the class of all finite distributive lattices and by $\mathbf{P}_F$ the class of all finite ordered sets.

  Then, we have

  $$L \cong \mathcal{O}(\mathcal{J}(L)) \quad \text{and} \quad P \cong \mathcal{J}(\mathcal{O}(P)),$$

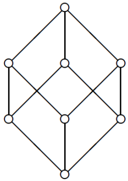
  for all $L \in \mathbf{D}_F$ and $P \in \mathbf{P}_F$.
- We call $\mathcal{J}(L)$ the **dual** of L and $\mathcal{O}(P)$ the **dual** of P.
- When we identify each finite distributive lattice L with the isomorphic lattice $\mathcal{O}(\mathcal{J}(L))$ of down-sets of $\mathcal{J}(L)$, we may regard $\mathbf{D}_F$ as consisting of the concrete lattices $\mathcal{O}(P)$, for $P \in \mathbf{P}_F$, rather than as abstract objects satisfying certain identities.
- Up to isomorphism, we have a one-to-one correspondence

  $$\mathcal{O}(P) = L \rightleftarrows P = \mathcal{J}(L),$$

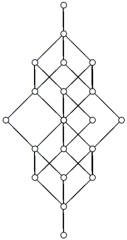
  for $L \in \mathbf{D}_F$ and $P \in \mathbf{P}_F$.

Example

- The figure shows $\mathcal{P}(X)$ and $\mathcal{O}(\mathcal{P}(X))$ for $|X| = 3$:



$\wp(\{1,2,3\})$



$\mathcal{O}(\wp(\{1,2,3\}))$

$\lvert X\rvert$	$\lvert\wp(X)\rvert$	$\lvert\mathcal{O}(\wp(X))\rvert$
1	2	3
2	4	6
3	8	20
4	16	168
5	32	7581
6	64	7828354
7	128	2414682040998
8	256	56130437228687557907788

- We also see the table with $|\mathcal{P}(X)|$ and $|\mathcal{O}(\mathcal{P}(X))|$ for $|X| \le 8$.
- The dual $\mathcal{J}(L)$ of a finite distributive lattice L is generally much smaller and less complex than L itself. So lattice problems concerning $\mathbf{D}_F$ are likely to become simpler when translated into problems about $\mathbf{P}_F$.

    In some sense the maps $L \mapsto \mathcal{J}(L)$ and $P \mapsto \mathcal{O}(P)$ play a role analogous to that of the logarithm and exponential functions.

Duality for Boolean Lattices and Chains

- Special properties of a finite distributive lattice are reflected in special properties of its dual.

Lemma

Let $L = \mathcal{O}(P)$ be a finite distributive lattice. Then:

(i) L is a Boolean lattice if and only if P is an antichain; $\mathcal{O}(\overline{\boldsymbol{n}}) = \mathbf{2}^n$.

(ii) L is a chain if and only if P is a chain; $\mathcal{O}(\boldsymbol{n}) = \boldsymbol{n} + \mathbf{1}$.

(i) Recall that L is a finite Boolean lattice if and only if $L \cong \mathbf{2}^n$, for some n. Now it suffices to observe that $L \cong \mathbf{2}^n$ implies $\mathcal{J}(L) \cong \overline{\boldsymbol{n}}$ and that $P \cong \overline{\boldsymbol{n}}$ implies $\mathcal{O}(P) \cong \mathbf{2}^n$.

(ii) We have $L \cong \boldsymbol{n} + \mathbf{1}$ implies $\mathcal{J}(L) \cong \boldsymbol{n}$ and $P \cong \boldsymbol{n}$ implies $\mathcal{O}(P) \cong \boldsymbol{n} + \mathbf{1}$.

The Maps in Duality

- Setting up a correspondence between $\{0,1\}$-homomorphisms from $\mathcal{O}(P)$ to $\mathcal{O}(Q)$ and order-preserving maps from Q to P, for $P, Q \in \mathbf{P}_F$ is harder to formulate and to prove:

Theorem

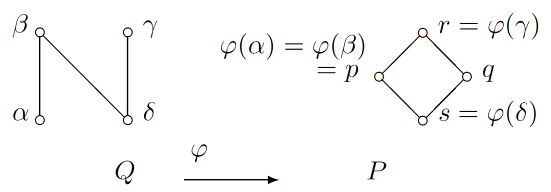
Let P and Q be finite ordered sets and let $L = \mathcal{O}(P)$ and $K = \mathcal{O}(Q)$. Given a $\{0,1\}$-homomorphism $f : L \to K$, there is an associated order-preserving map $\varphi_f : Q \to P$ defined by $\varphi_f(y) = \min\{x \in P : y \in f(\downarrow x)\}$, for all $y \in Q$. Given an order-preserving map $\varphi : Q \to P$, there is an associated $\{0,1\}$-homomorphism $f_\varphi : L \to K$ defined by $f_\varphi(a) = \varphi^{-1}(a)$, for all $a \in L$. Equivalently, $\varphi(y) \in a$ if and only if $y \in f_\varphi(a)$, for all $a \in L$, $y \in Q$. The maps $f \mapsto \varphi_f$ and $\varphi \mapsto f_\varphi$ establish a one-to-one correspondence between $\{0,1\}$-homomorphisms from L to K and order-preserving maps from Q to P. Further,

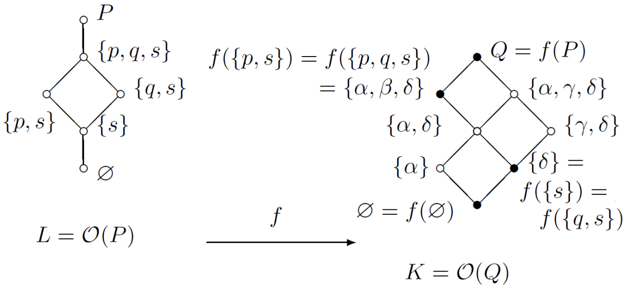
(i) f is one-to-one if and only if φ_f is onto,

(ii) f is onto if and only if φ_f is an order-embedding.

Example

- An order-preserving map $\varphi : Q \to P$ and the associated $\{0,1\}$-homomorphism $f : \mathcal{O}(P) \to \mathcal{O}(Q)$:





The Relation Between the Two "Dualities"

- We established a correspondence between $\mathbf{D}_F + \{0,1\}$-homomorphisms and $\mathbf{P}_F +$ order-preserving maps (a **duality** or a **dual equivalence of categories**).
- It follows that statements about finite distributive lattices can be translated into statements about finite ordered sets, and vice versa.
- We can now see that our two uses of the word "dual" have an underlying commonality:
  - If, in an ordered set P, we think of $x \leq y$ as representing an "arrow" from x to y, then P^∂ is obtained by reversing the arrows.
  - Similarly, for $L, K \in \mathbf{D}_F$, a $\{0,1\}$-homomorphism $f : L \to K$ provides an "arrow" from L to K, and, when we pass from $\mathbf{D}_F$ to $\mathbf{P}_F$, the arrows again reverse.