Introduction to Lattices and Order

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Subsection 1

Closure Operators

Closure Operators

- Let P be an ordered set. A map $c : P \to P$ is called a **closure operator** (on P) if, for all $x, y \in P$:

(clo1) $x \leq c(x)$;
(clo2) $x \leq y \Rightarrow c(x) \leq c(y)$;
(clo3) $c(c(x)) = c(x)$.

- An element $x \in P$ is called **closed** if $c(x) = x$.
- The set of all closed elements of P is denoted by P_c.
- If $P = \langle \mathcal{P}(X); \subseteq \rangle$, for some set X, we customarily use the symbol C rather than c and shall refer to a closure operator $C : \mathcal{P}(X) \to \mathcal{P}(X)$ on X.

Complete Lattice of Closed Sets

Proposition

Let c be a closure operator on an ordered set P.

(i) $P_c = \{c(x) : x \in P\}$ and P_c contains the top element of P when it exists.

(ii) Assume P is a complete lattice.

(a) For any $x \in P$, $c(x) = \bigwedge_P \{y \in P_c : x \leq y\}$.

(b) P_c is a complete lattice, under the order inherited from P, such that, for every subset S of P_c, $\bigwedge_{P_c} S = \bigwedge_P S$ and $\bigvee_{P_c} S = c(\bigvee_P S)$.

(i) Let $y \in P$. If $y \in P_c$, then $y = c(y)$. If $y = c(x)$, for some $x \in P$, then $c(y) = c(c(x)) = c(x) = y$. Hence, $y \in P_c$.

If $\top$ exists in P, then $\top = c(\top)$.

(ii) (a) Note $c(x) \in \{y \in P_c : x \leq y\}^\ell$. Since $c(x)$ belongs to $\{y \in P_c : x \leq y\}$, it is the greatest lower bound.

Meets and Joins of Closed Sets

(ii) (b). To show P_c is a complete lattice it suffices to show that $\bigwedge_{P_c} S$ exists for every $S \subseteq P_c$. This happens provided $\bigwedge_P S \in P_c$, and in that case $\bigwedge_P S$ serves as $\bigwedge_{P_c} S$. But, for all $s \in S$, $c(\bigwedge_P S) \leq c(s) = s$. So $c(\bigwedge_P S) \leq \bigwedge_P S$. Finally, note that

$$\begin{array}{rcll} \bigvee_{P_c} S & = & \bigwedge_{P_c}\{y \in P_c : (\forall s \in S)\ s \leq y\} & \text{(by join in } P_c\text{)} \\ & = & \bigwedge_P\{y \in P_c : (\forall s \in S)\ s \leq y\} & \text{(from above)} \\ & = & \bigwedge_P\{y \in P_c : \bigvee_P S \leq y\} & \\ & = & c(\bigvee_P S). & \text{(by (ii)(a))} \end{array}$$

Topped $\bigcap$-Structures and Closure Operators

- The next result says that every topped $\bigcap$-structure gives rise to a closure operator and conversely.

Theorem

Let C be a closure operator on a set X. Then the family

$$\mathcal{L}_C := \{A \subseteq X : C(A) = A\}$$

of closed subsets of X is a topped $\bigcap$-structure and so forms a complete lattice, when ordered by inclusion, in which

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i, \quad \bigvee_{i \in I} A_i = C(\bigcup_{i \in I} A_i).$$

Conversely, given a topped $\bigcap$-structure $\mathcal{L}$ on X, the formula

$$C_{\mathcal{L}}(A) := \bigcap \{B \in \mathcal{L} : A \subseteq B\}$$

defines a closure operator $C_{\mathcal{L}}$ on X.

Proof of the Theorem

- Suppose $C:\mathcal{P}(X)\to\mathcal{P}(X)$ is a closure operator.
  - Clearly, $C(X)=X$. Hence, $X\in\mathcal{L}_C$.
  - Suppose $A_i\in\mathcal{L}_C$, $i\in I$. Then $C(\bigcap_{i\in I}A_i)\subseteq\bigcap_{i\in I}C(A_i)=\bigcap_{i\in I}A_i$. Hence, $\bigcap_{i\in I}A_i\in\mathcal{L}_C$.

  Thus, $\mathcal{L}_C$ is a topped $\bigcap$-structure on X.
- Suppose, conversely, that $\mathcal{L}$ is a topped $\bigcap$-structure on X.
  - $A\subseteq\bigcap\{B\in\mathcal{L}:A\subseteq B\}=C_{\mathcal{L}}(A)$;
  - Suppose $A\subseteq A'$. Then $\{B\in\mathcal{L}:A'\subseteq B\}\subseteq\{B\in\mathcal{L}:A\subseteq B\}$. Now, we have
$$\begin{array}{rcl}C_{\mathcal{L}}(A)&=&\bigcap\{B\in\mathcal{L}:A\subseteq B\}\\&\subseteq&\bigcap\{B\in\mathcal{L}:A'\subseteq B\}\\&=&C_{\mathcal{L}}(A').\end{array}$$
  - Taking into account that $C_{\mathcal{L}}(A)\in\mathcal{L}$, for every set $A\subseteq X$, we get
$$C_{\mathcal{L}}(C_{\mathcal{L}}(A))=\bigcap\{B\in\mathcal{L}:C_{\mathcal{L}}(A)\subseteq B\}=C_{\mathcal{L}}(A).$$

  Thus, $C_{\mathcal{L}}:\mathcal{P}(X)\to\mathcal{P}(X)$ is a closure operator on X.

Bijection: Topped $\bigcap$-Structures and Closure Operators

- The relationship between closure operators and topped $\bigcap$-structures on a given set is a bijective one:
  - The closure operator induced by the topped $\bigcap$-structure $\mathcal{L}_C$ is C itself;
  - The topped $\bigcap$-structure induced by the closure operator $C_{\mathcal{L}}$ is $\mathcal{L}$.

  Summarizing in symbols,

$$C_{(\mathcal{L}_C)} = C \quad \text{and} \quad \mathcal{L}_{(C_{\mathcal{L}})} = \mathcal{L}.$$

- In practice this means that whether we work with a topped $\bigcap$-structure or the corresponding closure operator is a matter of convenience.
- Every complete lattice arises (up to order isomorphism) as a topped $\bigcap$-structure on some set. Thus, equivalently, every complete lattice is isomorphic to the lattice of closed sets with respect to some closure operator.

Examples

(1) Let G be a group. Then the closure operator corresponding to the topped $\bigcap$-structure $\mathrm{Sub}G$ maps a subset A of G to the subgroup $\langle A \rangle$ generated by A.

(2) Let V be a vector space over a field F and let $\mathrm{Sub}V$ be the complete lattice of linear subspaces of V. The corresponding closure operator on V maps a subset A of V to its linear span.

(3) Let L be a lattice and, for all $X \subseteq L$, $[X] := \bigcap\{K \in \mathrm{Sub}_0 L : X \subseteq K\}$. Then $[-] : \mathcal{P}(L) \to \mathcal{P}(L)$ is the closure operator corresponding to the topped $\bigcap$-structure $\mathrm{Sub}_0 L$.

(4) Let L be a lattice with 0. Then the closure operator corresponding to the topped $\bigcap$-structure $\mathcal{I}(L)$ consisting of all ideals of L is $(-] : \mathcal{P}(L) \to \mathcal{P}(L)$.

(5) Let P be an ordered set. The map $\downarrow : \mathcal{P}(P) \to \mathcal{P}(P)$ is easily seen to be a closure operator. The corresponding topped $\bigcap$-structure is the down-set lattice $\mathcal{O}(P)$.

Subsection 2

Complete Lattices From Algebra: Algebraic Lattices

An Example from Groups

- We explore the circumstances under which joins are given by union.

  Example: Let G be a group and $\mathcal{H} := \{H_i\}_{i\in I}$ be a non-empty family of subgroups of G with the property that, for each $i_1, i_2 \in I$, there exists $k \in I$, such that $H_{i_1} \cup H_{i_2} \subseteq H_k$.

  Claim: $H := \bigcup_{i\in I} H_i$ is a subgroup.

  Choose $g_1, g_2 \in H$. It suffices to show that $g_1 g_2^{-1} \in H$. For $j = 1, 2$, there exists $i_j \in I$, such that $g_j \in H_{i_j}$. By hypothesis we can find $H_k \in \mathcal{H}$ so that $H_{i_1} \subseteq H_k$ and $H_{i_2} \subseteq H_k$. Then g_1, g_2 both belong to a common subgroup H_k, so $g_1 g_2^{-1} \in H_k$. Hence $g_1 g_2^{-1} \in H$, as required.

  As a special case, note that if $H_1 \subseteq H_2 \subseteq \cdots$ is a non-empty chain of subgroups, then $\bigcup_{n\geq 1} H_n$ is a subgroup.

- Crucial to the argument above is not that it concerns groups, but the existence, for a given pair H_1, H_2 of members of $\mathcal{H}$, of a member H of $\mathcal{H}$ which contains both H_1 and H_2, so that we can exploit the closure properties of the group operations in H.

Directed Sets and CPOs

- Let S be a non-empty subset of an ordered set P. Then S is said to be **directed** if, for every pair of elements $x, y \in S$, there exists $z \in S$, such that $z \in \{x, y\}^u$.
- An easy induction shows that S is directed if and only if, for every finite subset F of S, there exists $z \in S$, such that $z \in F^u$.
- When D is a directed set for which $\bigvee D$ exists, then we often write $\bigsqcup D$ in place of $\bigvee D$ as a reminder that D is directed.
- Directed joins arise very naturally in computer science in the context of CPOs:

  A **CPO** is an ordered set P with $\perp$ in which $\bigsqcup D$ exists, for every directed subset D of P.

Examples

(1) In any ordered set P, any non-empty chain is directed and any subset of P with a greatest element is directed.

(2) The only directed subsets of an antichain are the singletons.
More generally, in an ordered set with (ACC) a set is directed if and only if it has a greatest element.

(3) Let X be a set. Then any non-empty family $\mathcal{D}$ of subsets of X which is closed under finite unions is directed: for $A, B \subseteq \mathcal{D}$, we have $A \cup B \in \{A, B\}^u$ in $\mathcal{D}$.
Hence, for example, the family of finite subsets of $\mathbb{N}$ is directed.

(4) The finitely generated subgroups of a group G form a directed subset $\mathcal{L}$ of Sub$G$. To check this claim, let H and K be subgroups of G generated, respectively, by $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$. Let M be the subgroup generated by $\{a_1, \ldots, a_m, b_1, \ldots, b_n\}$. Then $M \in \{H, K\}^u$ in $\mathcal{L}$.
By contrast with the preceding example, the exhibited upper bound is not given by set union: in general $H \cup K$ is not a subgroup.

Directed Families of Sets

- The union of a directed family of sets will be called a **directed union**.
- Recall that if $\{A_i\}_{i\in I}$ is a subset of a family $\mathcal{L}$ of subsets of a set X, then
$$\bigcup_{i\in I} A_i \in \mathcal{L} \;\Rightarrow\; \bigvee_{\mathcal{L}}\{A_i : i \in I\} \text{ exists and equals } \bigcup_{i\in I} A_i.$$
We deduce that if the family $\mathcal{L}$ is closed under directed unions, we have
$$\bigsqcup_{i\in I} A_i = \bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i,$$
whenever $\{A_i\}_{i\in I} \subseteq \mathcal{L}$ is directed.
- A subset $\mathcal{D} = \{A_i\}_{i\in I}$ of $\mathcal{P}(X)$ is directed if and only if, given $A_{i_1}, \ldots, A_{i_n}$ in $\mathcal{D}$, there exists $k \in I$, such that $A_{i_j} \subseteq A_k$, for $i = 1, \ldots, n$ (equivalently, $\bigcup\{A_{i_j} : j = 1, \ldots, n\} \subseteq A_k$). It follows that if $\mathcal{D}$ is directed and $Y = \{y_1, \ldots, y_n\}$ is a finite subset of $\bigcup A_i$, then there exists $A_k \in \mathcal{D}$, such that $Y \subseteq A_k$.

Algebraic $\bigcap$-Intersection Structures

- A non-empty family $\mathcal{L}$ in $\mathcal{P}(X)$ is said to be **closed under directed unions** if $\bigcup_{i \in I} A_i \in \mathcal{L}$, for any directed family $\mathcal{D} = \{A_i\}_{i \in I}$ in $\mathcal{L}$.
- A non-empty family $\mathcal{L}$ of subsets of a set X is said to be an **algebraic** $\bigcap$**-structure** if
  (i) $\bigcap_{i \in I} A_i \in \mathcal{L}$, for any non-empty family $\{A_i\}_{i \in I}$ in $\mathcal{L}$;
  (ii) $\bigcup_{i \in I} A_i \in \mathcal{L}$, for any directed family $\{A_i\}_{i \in I}$ in $\mathcal{L}$.
- Thus an algebraic $\bigcap$-structure is an $\bigcap$-structure which is closed under directed unions.

  In such a structure the join of any directed family is given by set union.

  Example: The $\bigcap$-structure Sub$G$ is algebraic.

  Each of the $\bigcap$-structures presented previously can be shown to be algebraic.

  The congruence lattice Con$L$, for any lattice L, is another example.

Algebraic Closure Operators

- We extend the correspondence between topped $\bigcap$-structures and closure operators to the algebraic case.
- A closure operator C on a set X is called **algebraic** if, for all $A \subseteq X$,

$$C(A) = \bigcup \{ C(B) : B \subseteq A \text{ and } B \text{ is finite} \}.$$

- For any closure operator C, $C(A) \supseteq \bigcup \{ C(B) : B \subseteq A \text{ and } B \text{ is finite} \}$, so to prove that a closure operator C is algebraic it is only necessary to prove the reverse inclusion.

  Example: The closure operator corresponding to the $\bigcap$-structure $\text{Sub}G$ maps a subset A of G to the subgroup $\langle A \rangle$ generated by A.

  Claim: This closure operator is algebraic.

  It suffices to show that $\langle A \rangle \subseteq \{ \langle B \rangle : B \subseteq A \text{ and } B \text{ is finite} \}$. Let $g \in \langle A \rangle$. Then, there exist $a_1, a_2, \ldots, a_n \in A$, such that $g = a_1' a_2' \cdots a_n'$, where $a_i' \in \{ a_i, a_i^{-1} \}$, for each i. Thus $g \in \langle \{ a_1, \ldots, a_n \} \rangle$, and this gives the required containment.

Algebraic Closure Operators and Algebraic $\bigcap$-Structures

Theorem

Let C be a closure operator on a set X and let $\mathcal{L}_C$ be the associated topped $\bigcap$-structure. Then the following are equivalent:

(i) C is an algebraic closure operator;

(ii) for every directed family $\{A_i\}_{i \in I}$ of subsets of X, $C(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} C(A_i)$;

(iii) $\mathcal{L}_C$ is an algebraic $\bigcap$-structure.

(i)⇒(ii) Let $\{A_i\}_{i \in I}$ be a directed family of subsets of X. First observe that if B is finite and $B \subseteq \bigcup_{i \in I} A_i$, then $B \subseteq A_k$, for some $k \in I$. Consequently, $C(\bigcup_{i \in I} A_i) = \bigcup\{C(B) : B \subseteq \bigcup_{i \in I} A_i \text{ and } B \text{ is finite}\} = \bigcup\{C(B) : B \subseteq A_k, \text{ for some } k \in I \text{ and } B \text{ is finite}\} \subseteq \bigcup_{i \in I} C(A_i)$. The reverse inclusion is always valid.

(ii)⇒(iii) Trivial, since $\mathcal{L}_C = \{C(A) : A \subseteq X\}$.

Algebraic Closure Operators and $\bigcap$-Structures (Cont'd)

(iii)⇒(i) Let $A \subseteq X$. The family $\mathcal{D} := \{C(B) : B \subseteq A \text{ and } B \text{ is finite}\}$ is directed. Hence $\bigcup \mathcal{D} \in \mathcal{L}_C$. For each $x \in A$, we have

$$x \in \{x\} \subseteq C(\{x\}) \subseteq \bigcup \mathcal{D}.$$

So, $A \subseteq \bigcup \mathcal{D}$. Hence

$$C(A) \subseteq C(\bigcup \mathcal{D}) = \bigcup \mathcal{D} = \bigcup \{C(B) : B \subseteq A \text{ and } B \text{ is finite}\}.$$

Since the reverse inclusion always holds, C is algebraic.

Finite and Compact Elements

- We aim to characterize, in a lattice-theoretic way, the closures, with respect to an algebraic closure operator on X, of the finite subsets of X.
- Let L be a complete lattice and let $k \in L$.
  (i) k is called **finite** (in L) if, for every directed set D in L,
  $$k \leq \bigsqcup D \Rightarrow k \leq d, \text{for some } d \in D.$$
  The set of finite elements of L is denoted $F(L)$.
  (ii) k is said to be **compact** if, for every subset S of L,
  $$k \leq \bigvee S \Rightarrow k \leq \bigvee T, \text{ for some finite subset } T \text{ of } S.$$
  The set of compact elements of L is denoted $K(L)$.

Finite and Compact Elements in a Complete Lattice

- Unlike compactness, finiteness makes sense in ordered sets in which joins exist for directed subsets, but not necessarily for all subsets.

Lemma

Let L be a complete lattice. Then $F(L) = K(L)$. Further, $k_1 \vee k_2 \in F(L)$ whenever $k_1, k_2 \in F(L)$.

- Assume first that $k \in K(L)$ and that $k \leq \bigsqcup D$, where D is directed. Then there exists a finite subset F of D, such that $k \leq \bigvee F$. Because D is directed, we can find $d \in D$ with $d \in F^u$. Then $k \leq d$, so $k \in F(L)$.

  Conversely, assume that $k \in F(L)$ and that $k \leq \bigvee S$. The set $D = \{\bigvee T : T \subseteq S \text{ and } T \text{ is finite}\}$ is directed. Moreover, $\bigsqcup D = \bigvee S$. Applying the finiteness condition, we find a finite subset T of S with $k \leq \bigvee T$.

Finite and Compact Elements (Cont'd)

- For the second part, assume $k_1, k_2 \in F(L)$.
  Suppose, there exists directed $D \subseteq L$, such that $k_1 \vee k_2 \leq \bigsqcup D$.
  Then $k_1 \leq \bigsqcup D$ and $k_2 \leq \bigsqcup D$.
  Since $k_1, k_2 \in F(K)$, there exist $d_1, d_2 \in D$, such that $k_1 \leq d_1$ and $k_2 \leq d_2$.
  Since D is directed, there exists $d \in D$, such that $d_1 \leq d$ and $d_2 \leq d$.
  Now we get $k_1 \vee k_2 \leq d_1 \vee d_2 \leq d$.
  Therefore, $k_1 \vee k_2 \in F(L)$.

Examples of Finite Elements in Complete Lattices

- In $\mathcal{P}(X)$ (X a set): the finite elements are all finite subsets of X.
- In $\mathcal{O}(P)$ (P an ordered set): the finite elements are all down-sets of the form $\downarrow F$, with F finite.
- In Sub$G$ (G a group): the finite elements are all finitely generated subgroups.
- In Sub$V$ (V a vector space): the finite elements are all finite-dimensional subspaces.
- In a complete lattice satisfying the (ACC): all elements are finite.
- In $[0,1]$: the only finite element is 0.
- Note that $\bot$ in a complete lattice is always finite.
- As a simple example of a non-finite element we have the top element of $\mathbb{N} \oplus \mathbf{1}$.

Natural Numbers under Divisibility

- Consider $\langle \mathbb{N}_0; \preccurlyeq \rangle$.

  Claim: No element other than 1 $(= \bot)$ is compact.

  0 $(= \top)$ is the join of the set of all primes. But 0 not the join of any finite set of primes. Thus, 0 is not compact.

  Now let $n \in \mathbb{N}_0$ with $n \notin \{0, 1\}$. Let S be the set of primes which do not divide n. Then S is infinite. So $\bigvee S = \top$ $(= 0)$, because any non-zero element of $\mathbb{N}_0$ has only finitely many prime divisors. Hence $n \preccurlyeq \bigvee S$ but $n \not\preccurlyeq \bigvee T$, for any finite subset T of S. Thus, n is not compact.

Algebraic Lattices

- A complete lattice L is said to be **algebraic** if, for each $a \in L$, $a = \bigvee\{k \in K(L) : k \leq a\}$.

Lemma

Let C be an algebraic closure operator on X and $\mathcal{L}_C$ the associated topped algebraic $\bigcap$-structure. Then $\mathcal{L}_C$ is an algebraic lattice in which an element A is finite (equivalently, compact) if and only if $A = C(Y)$, for some finite set $Y \subseteq X$.

- We show that the finite elements are the closures of the finite sets: Let Y be a finite subset of X and let $A = C(Y)$. Take a directed set $\mathcal{D}$ in $\mathcal{L}_C$ with $A \subseteq \bigsqcup \mathcal{D}$. Then, since $\bigsqcup$ coincides with $\bigcup$ in $\mathcal{L}_C$, $Y \subseteq C(Y) = A \subseteq \bigsqcup \mathcal{D} = \bigcup \mathcal{D}$. As Y is finite and $\mathcal{D}$ directed, there exists $B \in \mathcal{D}$, such that $Y \subseteq B$. Then $A = C(Y) \subseteq C(B) = B$.
  If $A \in \mathcal{L}_C$ is a finite element, $A = \bigsqcup\{C(Y) : Y \subseteq A \text{ and } Y \text{ is finite}\}$. Since A is finite in $\mathcal{L}_C$, there exists a finite set $Y \subseteq A$ such that $A \subseteq C(Y)$. For the reverse, note $Y \subseteq A$ implies $C(Y) \subseteq C(A) = A$.

Algebraic Lattices and Topped $\bigcap$-Structures

Theorem

(i) Let $\mathcal{L}$ be a topped algebraic $\bigcap$-structure. Then $\mathcal{L}$ is an algebraic lattice.

(ii) Let L be an algebraic lattice and define $D_a := \{k \in K(L) : k \leq a\}$, for each $a \in L$. Then $\mathcal{L} := \{D_a : a \in L\}$ is a topped algebraic $\bigcap$-structure isomorphic to L.

(i) This has already been shown.

(ii) We omit the proof that $\mathcal{L}$ is a topped $\bigcap$-structure and prove that the map $\varphi : a \mapsto D_a$ is an isomorphism of L onto $\mathcal{L}$ and that $\mathcal{L}$ is algebraic.

Because L is algebraic, $D_a \subseteq D_b$ in $\mathcal{L}$ implies $a = \bigvee D_a \leq \bigvee D_b = b$ in L. The reverse implication holds always. Therefore φ is an order-isomorphism.

Algebraic Lattices and Topped $\bigcap$-Structures (Cont'd)

- Take a directed subset $\mathcal{D} = \{D_c : c \in C\}$ of $\mathcal{L}$. As φ is an order isomorphism, the indexing set C is a directed subset of L. Define

$$a = \bigsqcup C.$$

Claim: $\bigcup \mathcal{D} = D_a$ and so it belongs to L.

Indeed,

$$\begin{array}{rcl} k \in D_a & \Leftrightarrow & k \in K(L) = F(L) \text{ and } k \leq a = \bigsqcup C \\ & \Leftrightarrow & k \in F(L) \text{ and } k \leq c, \text{ for some } c \in C \\ & \Leftrightarrow & k \in D_c \text{ for some } c \in C \\ & \Leftrightarrow & k \in \bigcup \mathcal{D}. \end{array}$$

Hence L is closed under directed unions and so is algebraic.

Examples of Algebraic Topped $\bigcap$-Structures

- $\mathcal{P}(X)$, for any set X;
- Any complete lattice of sets and, in particular, the down-set lattice $\mathcal{O}(P)$, for any ordered set P;
- Sub$G$, for any group G;
- Sub$V$, for any vector space V;
- $\mathcal{I}(L)$, the ideal lattice of any lattice L with 0;
- Con$L$, for any lattice L.
- The chains $\boldsymbol{n}$, for $n \geq 1$, and $\mathbb{N} \oplus \mathbf{1}$ are algebraic lattices.
- Any lattice L with a bottom element and satisfying (ACC) is an algebraic lattice:
  - L is a complete lattice;
  - every element $x \in L$ is compact, and so is the join of $\downarrow x \cap K(L)$.

  An example of an infinite algebraic lattice of this type is $\langle \mathbb{N}_0; \preccurlyeq \rangle^\partial$.

  On the other hand, $\langle \mathbb{N}_0; \preccurlyeq \rangle$ is not algebraic, since its only compact element is $\perp$.

Subsection 3

Galois Connections

Galois Connections

- Let P and Q be ordered sets. A pair $({}^{\triangleright}, {}^{\triangleleft})$ of maps ${}^{\triangleright} : P \to Q$ and ${}^{\triangleleft} : Q \to P$ (called **right** and **left**, respectively) is a **Galois connection** between P and Q if, for all $p \in P$ and $q \in Q$,

$$\text{(Gal)} \qquad p^{\triangleright} \le q \iff p \le q^{\triangleleft}.$$

- The map ${}^{\triangleright}$ is called the **lower adjoint of** ${}^{\triangleleft}$ and the map ${}^{\triangleleft}$ the **upper adjoint** of ${}^{\triangleright}$;

  The terms "lower" and "upper" refer to the side of $\le$ on which the map appears.

Order-Theoretic Examples

(1) Suppose that sets P and Q are ordered by the discrete order, $=$. Then ${}^{\triangleright}: P \to Q$ and ${}^{\triangleleft}: Q \to P$ set up a Galois connection between P and Q if and only if they are set-theoretic inverses of each other.

(2) Let P be an ordered set. For $A \subseteq P$, we have previously defined the sets of upper and lower bounds of A as

$$A^u = \{y \in P : (\forall x \in A)\ x \leq y\}, \quad A^\ell = \{y \in P : (\forall x \in A)\ y \leq x\}.$$

It is easy to see that $({}^u, {}^\ell)$ is a Galois connection between $\mathcal{P}(P)$ and $\mathcal{P}(P)^\partial$:

$$\begin{array}{rcl} A^u \supseteq B & \Leftrightarrow & (\forall y \in B)((\forall x \in A)\ x \leq y) \\ & \Leftrightarrow & (\forall x \in A)((\forall y \in B)\ y \geq x) \\ & \Leftrightarrow & A \subseteq B^\ell. \end{array}$$

(4) Let P be an ordered set. For $A \subseteq P$, define

$$A^{\triangleright} := P \backslash {\downarrow} A \quad \text{and} \quad A^{\triangleleft} := P \backslash {\uparrow} A.$$

Then $({}^{\triangleright}, {}^{\triangleleft})$ forms a Galois connection between $\mathcal{P}(P)^\partial$ and $\mathcal{P}(P)$.

Properties of Galois Connections

Lemma

Assume $({}^{\triangleright}, {}^{\triangleleft})$ is a Galois connection between ordered sets P and Q. Let $p, p_1, p_2 \in P$ and $q, q_1, q_2 \in Q$. Then:

(Gal1) $p \leq p^{\triangleright\triangleleft}$ and $q^{\triangleleft\triangleright} \leq q$; **Cancelation Rule**

(Gal2) $p_1 \leq p_2 \Rightarrow p_1^{\triangleright} \leq p_2^{\triangleright}$ and $q_1 \leq q_2 \Rightarrow q_1^{\triangleleft} \leq q_2^{\triangleleft}$;

(Gal3) $p^{\triangleright} = p^{\triangleright\triangleleft\triangleright}$ and $q^{\triangleleft} = q^{\triangleleft\triangleright\triangleleft}$. **Semi-inverse Rule**

Conversely, a pair of maps ${}^{\triangleright} : P \to Q$ and ${}^{\triangleleft} : Q \to P$ satisfying (Gal1) and (Gal2) for all $p, p_1, p_2 \in P$ and for all $q, q_1, q_2 \in Q$ sets up a Galois connection between P and Q.

Gal1: For $p \in P$, we have $p^{\triangleright} \leq p^{\triangleright}$ from which we obtain $p \leq p^{\triangleright\triangleleft}$ by putting $q = p^{\triangleright}$ in (Gal). Hence (Gal) implies (Gal1).

Gal2: For (Gal2), $p_1 \leq p_2$ implies, by (Gal1) and transitivity, $p_1 \leq p_2^{\triangleright\triangleleft}$, whence, by (Gal), $p_1^{\triangleright} \leq p_2^{\triangleright}$.

Properties of Galois Connections (Cont'd)

$q_1 \leq q_2$ implies, by (Gal1) and transitivity, $q_1^{\lhd\rhd} \leq q_2$, whence, by (Gal), $q_1^{\lhd} \leq q_2^{\lhd}$.

Gal3: We now prove (Gal3). Applying $^{\rhd}$ to the inequality $p \leq p^{\rhd\lhd}$ in (Gal1) we have, by (Gal2), $p^{\rhd} \leq p^{\rhd\lhd\rhd}$. Also, by (Gal) with $p^{\rhd\lhd}$ in place of p and $p^{\rhd}$ in place of q, $p^{\rhd\lhd} \leq p^{\rhd\lhd}$ implies $p^{\rhd\lhd\rhd} \leq p^{\rhd}$.

Lastly, assume that (Gal1) and (Gal2) hold universally.

- Let $p^{\rhd} \leq q$. By (Gal2), $p^{\rhd\lhd} \leq q^{\lhd}$. Also, (Gal1) gives $p \leq p^{\rhd\lhd}$. Hence $p \leq q^{\lhd}$ by transitivity.
- Let $p \leq q^{\lhd}$. By (Gal2), $p^{\rhd} \leq q^{\lhd\rhd}$. Also, (Gal1) gives $q^{\lhd\rhd} \leq q$. Hence $p^{\rhd} \leq q$ by transitivity.

From a Galois Connection to a Closure Operator

- Let $(^{\triangleright}, ^{\triangleleft})$ be a Galois connection between ordered sets P and Q^{∂} (note that we have Q^{∂} instead of Q here). Then:
  (i) $c = ^{\triangleright\triangleleft}: P \to P$ and $k = ^{\triangleleft\triangleright}: Q \to Q$ are closure operators. (In this notation, the left-hand map in each composition is performed first.)
  (ii) Let $P_c := \{p \in P : p^{\triangleright\triangleleft} = p\}$ and $Q_k := \{q \in Q : q^{\triangleleft\triangleright} = q\}$. Then $^{\triangleright} : P_c \to Q_k^{\partial}$ and $^{\triangleleft} : Q_k^{\partial} \to P_c$ are mutually inverse order isomorphisms.
- Note that, indeed, for all $p, p' \in P$:
  - $p \leq p^{\triangleright\triangleleft}$;
  - $p \leq p'$ implies $p^{\triangleright\triangleleft} \leq p'^{\triangleright\triangleleft}$;
  - $p^{\triangleright\triangleleft\triangleright\triangleleft} = p^{\triangleright\triangleleft}$.
- To check (ii), use (Gal3) to get that:
  - $^{\triangleright}$ maps P_c onto Q_k^{∂};
  - $^{\triangleleft}$ maps Q_k^{∂} onto P_c;
  - these maps are inverse to each other.

  Since they are also order-preserving (by (Gal2)), they are order-isomorphisms.

From a Closure Operator to a Galois Connection

- Every closure operator arises as the composite of the left and right maps of a Galois connection:
  - Let $c : P \to P$ be a closure operator;
  - Define $Q := P_c$;
  - Let ${}^{\triangleright} : P \to P_c$ be given by $p^{\triangleright} := c(p)$;
  - Let ${}^{\triangleleft} : P_c \to P$ be the inclusion map.

  Then $c = {}^{\triangleright\triangleleft}$.
- Correspondences which provide alternative ways in which complete lattices arise:
  - Every topped $\bigcap$-structure is a complete lattice. Up to isomorphism, every complete lattice arises this way.
  - There is a bijective correspondence between closure operators on a set X and topped $\bigcap$-structures on X.
  - Every Galois connection $({}^{\triangleright}, {}^{\triangleleft})$ gives rise to a pair of closure operators, ${}^{\triangleright\triangleleft}$ and ${}^{\triangleleft\triangleright}$, and thence to an isomorphic pair of complete lattices.

Galois Connections and Preservation of Joins and Meets

Proposition

Let $({}^{\triangleright}, {}^{\triangleleft})$ be a Galois connection between ordered sets P and Q. Then ${}^{\triangleright}$ preserves existing joins and ${}^{\triangleleft}$ preserves existing meets.

- We first define $z := \bigvee_P S$ and show that $z^{\triangleright}$ is an upper bound for $S^{\triangleright}$. By (Gal2),
$$(\forall s \in S)\ s \leq z \ \Rightarrow\ (\forall s \in S)\ s^{\triangleright} \leq z^{\triangleright}.$$
Now let q be any upper bound for $S^{\triangleright}$. Then
$$\begin{array}{rcll}(\forall s \in S)\ s^{\triangleright} \leq q & \Leftrightarrow & (\forall s \in S)\ s \leq q^{\triangleleft} & \text{(by (Gal))} \\ & \Rightarrow & \bigvee_P S \leq q^{\triangleleft} & \text{(by definition of } \bigvee_P S) \\ & \Leftrightarrow & (\bigvee_P S)^{\triangleright} \leq q. & \text{(by (Gal))}\end{array}$$
We conclude that $z^{\triangleright}$ is the least upper bound of $S^{\triangleright}$.

Order Preserving Maps and Galois Connections

Lemma

Let P and Q be ordered sets and $\varphi : P \to Q$ an order preserving map. Then the following are equivalent:

(i) There exists an order-preserving map $\varphi^\# : Q \to P$, such that both $\varphi^\# \circ \varphi \geq \mathrm{id}_P$ and $\varphi \circ \varphi^\# \leq \mathrm{id}_Q$;

(ii) For each $q \in Q$, there exists a (necessarily unique) $s \in P$, such that $\varphi^{-1}(\downarrow q) = \downarrow s$.

[(i)⇒(ii)] Claim: $\varphi^{-1}(\downarrow q) = \downarrow \varphi^\#(q)$.

We have

$$\begin{array}{lcll} p \in \varphi^{-1}(\downarrow q) & \Leftrightarrow & \varphi(p) \leq q & \\ & \Rightarrow & (\varphi^\# \circ \varphi)(p) \leq \varphi^\#(q) & (\text{since } \varphi^\# \text{ is order-preserving}) \\ & \Rightarrow & p \leq \varphi^\#(q) & (\text{from } \varphi^\# \circ \varphi \geq \mathrm{id}_P \text{ \& transitivity}) \\ & \Rightarrow & p \in \downarrow \varphi^\#(q). & \end{array}$$

Order Preserving Maps and Galois Connections (Converse)

For the other direction, let $p \in {\downarrow}\varphi^{\#}(q)$. This yields $\varphi(p) \leq (\varphi \circ \varphi^{\#})(q)$ from which we can deduce that $\varphi(p) \leq q$, so that $p \in \varphi^{-1}({\downarrow}q)$.

[(ii)⇒(i)] For each $q \in Q$, we have a unique element $s \in P$, depending on q, such that $\varphi^{-1}({\downarrow}q) = {\downarrow}s$. Define $\varphi^{\#}(q) := s$. Restated, this means that

$$(\forall q \in Q)(\forall p \in P)\varphi(p) \leq q \iff p \leq \varphi^{\#}(q).$$

We now see that the pair $(\varphi, \varphi^{\#})$ is a Galois connection between P and Q, so that the properties in (i) follow.

- The proof says that, in a Galois connection $(^{\triangleright}, ^{\triangleleft})$, each of $^{\triangleright}$ and $^{\triangleleft}$ uniquely determines the other:

$$\begin{aligned} p^{\triangleright} &= \min\{q \in Q : p \leq q^{\triangleleft}\}; \\ q^{\triangleleft} &= \max\{p \in P : p^{\triangleright} \leq q\}. \end{aligned}$$

Preservation of Joins and Meets

Proposition

Let P and Q be ordered sets and $\varphi : P \to Q$ be a map.

(i) Assume P is a complete lattice. Then φ preserves arbitrary joins if and only if φ possesses an upper adjoint $\varphi^\#$ (that is, $(\varphi, \varphi^\#)$ is a Galois connection).

(ii) Assume Q is a complete lattice. Then φ preserves arbitrary meets if and only if φ possesses a lower adjoint $\varphi^\flat$ (that is, $(\varphi^\flat, \varphi)$ is a Galois connection).

(i) The backward implication has been shown.

For the forward implication, assume that φ preserves arbitrary joins. Note first that φ is order-preserving. It therefore suffices to show that condition (ii) in the preceding lemma is satisfied.

Preservation of Joins and Meets (Cont'd)

- Let $q \in Q$.

  Claim: $s := \bigvee_P \{p \in P : \varphi(p) \leq q\} (= \bigvee_P \varphi^{-1}(\downarrow q))$ is such that $\varphi^{-1}(\downarrow q) = \downarrow s$.

  It follows immediately that $\varphi^{-1}(\downarrow q) \subseteq \downarrow s$.

  Since φ preserves arbitrary joins,

$$\varphi(s) = \bigvee_Q \{\varphi(p) : p \in P \text{ with } \varphi(p) \leq q\}.$$

  Hence, $\varphi(s) \leq q$. For any $p \in \downarrow s$, we have $\varphi(p) \leq q$, because φ is order-preserving and $\leq$ is transitive. Therefore $\downarrow s \subseteq \varphi^{-1}(\downarrow q)$.

Subsection 4

Completions

Completion

- Let P be an ordered set. If $\varphi : P \hookrightarrow L$ and L is a complete lattice, then we say that L is a **completion** of P (via the order embedding φ).
- We saw that the map $\varphi : x \mapsto \downarrow x$ is an order embedding of P into $\mathcal{O}(P)$.

  We also saw that $\mathcal{O}(P)$ is a complete lattice.

  Thus, $\mathcal{O}(P)$ is a completion of P.

  This completion is unnecessarily large. For example, if P is a complete lattice, then P is a completion of itself (via the identity map) while $\mathcal{O}(P)$ is much larger.
- Another completion of an ordered set is the ideal completion.

Review of ^{u}, $^\ell$ and their Properties

- Let $A \subseteq P$. Then A "upper" and A "lower" are defined by

$$A^u := \{x \in P : (\forall a \in A)\ a \le x\} \quad \text{and} \quad A^\ell := \{x \in P : (\forall a \in A)\ a \ge x\}.$$

For subsets A and B of P, we have:

(i) $A \subseteq A^{u\ell}$ and $A \subseteq A^{\ell u}$;
(ii) if $A \subseteq B$, then $A^u \supseteq B^u$ and $A^\ell \supseteq B^\ell$;
(iii) $A^u = A^{u\ell u}$ and $A^\ell = A^{\ell u \ell}$.

Further, A^u is an up-set and A^ℓ is a down-set.

The Dedekind-MacNeille Completion

- Let P be an ordered set. We define

$$\mathrm{DM}(P) := \{A \subseteq P : A^{u\ell} = A\}.$$

- This is the topped $\bigcap$-structure on P corresponding to the closure operator

$$C(A) := A^{u\ell}$$

  on P.

  Therefore the ordered set $\langle \mathrm{DM}(P); \subseteq \rangle$ is a complete lattice.

- It is known as the **Dedekind-MacNeille completion** of P.

  It is also referred to as the **completion by cuts** or the **normal completion** of P.

Dedekind-McNeill Completion and Down-Sets

Lemma

Let P be an ordered set.

(i) For all $x \in P$, we have $(\downarrow x)^{u\ell} = \downarrow x$ and hence $\downarrow x \in \mathrm{DM}(P)$.

(ii) If $A \subseteq P$ and $\bigvee A$ exists in P, then $A^{u\ell} = \downarrow(\bigvee A)$.

(i) Let $y \in (\downarrow x)^u$. Then $z \leq y$, for all $z \in \downarrow x$. In particular, $x \leq y$ (as $x \in \downarrow x$) and, hence, $y \in \uparrow x$. Thus, $(\downarrow x)^u \subseteq \uparrow x$.
If $y \in \uparrow x$, then $y \geq x$. So, by transitivity, $y \geq z$, for all $z \in \downarrow x$, that is, $y \in (\downarrow x)^u$. Thus $\uparrow x \subseteq (\downarrow x)^u$.
Therefore, $(\downarrow x)^u = \uparrow x$ and, by duality, $(\uparrow x)^\ell = \downarrow x$.
Thus, $(\downarrow x)^{u\ell} = (\uparrow x)^\ell = \downarrow x$.

(ii) Let $A \subseteq P$. Assume that A exists in P. Of course $\bigvee A \in A^u$. Thus $x \in A^{u\ell}$ implies that $x \leq \bigvee A$ and hence $x \in \downarrow(\bigvee A)$. Consequently, $A^{u\ell} \subseteq \downarrow(\bigvee A)$. Since $\bigvee A$ is the least upper bound of A we have $\bigvee A \leq y$, for all $y \in A^u$ and hence $\bigvee A \in A^{u\ell}$. Since $A^{u\ell}$ is a down-set this gives $\downarrow(\bigvee A) \subseteq A^{u\ell}$. Hence, $A^{u\ell} = \downarrow(\bigvee A)$, as required.

The Dedekind-MacNeille Completion Theorem

Theorem

Let P be an ordered set and define $\varphi : P \to \mathrm{DM}(P)$ by $\varphi(x) = {\downarrow}x$, for all $x \in P$.

(i) $\mathrm{DM}(P)$ is a completion of P via the map φ.

(ii) φ preserves all joins and meets which exist in P.

(i) As we saw above, $\langle \mathrm{DM}(P); \subseteq \rangle$ is a complete lattice and the order-embedding φ maps P into $\mathrm{DM}(P)$.

(ii) Let $A \subseteq P$ and assume that $\bigvee A$ exists in P. We must show that $\varphi(\bigvee A) = \bigvee \varphi(A)$, that is, ${\downarrow}(\bigvee A) = \bigvee \{{\downarrow}a : a \in A\}$ in $\mathrm{DM}(P)$.

- Clearly, ${\downarrow}(\bigvee A)$ is an upper bound for $\{{\downarrow}a : a \in A\}$.
- Now choose any $B \in \mathrm{DM}(P)$ which is an upper bound for $\{{\downarrow}a : a \in A\}$. Since $a \in {\downarrow}a \subseteq B$, for all $a \in A$, we have $A \subseteq B$. Hence, ${\downarrow}(\bigvee A) = A^{u\ell} \subseteq B^{u\ell} = B$.

The Dedekind-MacNeille Completion Theorem (Cont'd)

- Now assume that $\bigwedge A$ exists in P. We must show that

$$\downarrow(\bigwedge A) = \bigwedge\{\downarrow a : a \in A\}.$$

Since $\mathrm{DM}(P)$ is a topped $\bigcap$-structure, we have in $\mathrm{DM}(P)$

$$\bigwedge\{\downarrow a : a \in A\} = \bigcap\{\downarrow a : a \in A\}.$$

This yields the result.

Characterization of the Dedekind-MacNeille Completion

Theorem

Let P be an ordered set and let $\varphi : P \to \mathrm{DM}(P)$ be the order-embedding of P into its Dedekind-MacNeille completion given by $\varphi(x) = \downarrow x$.

(i) $\varphi(P)$ is both join-dense and meet-dense in $\mathrm{DM}(P)$.

(ii) Let L be a complete lattice and assume that P is a subset of L which is both join-dense and meet-dense in L. Then $L \cong \mathrm{DM}(P)$ via an order-isomorphism which agrees with φ on P.

Theorem

Let L be a lattice with no infinite chains. Then $L \cong \mathrm{DM}(\mathcal{J}(L) \cup \mathcal{M}(L))$. Moreover, $\mathcal{J}(L) \cup \mathcal{M}(L)$ is the smallest subset of L which is both join-dense and meet-dense in L.

Examples I

(1) Every real number $x \in \mathbb{R}$ satisfies $\bigvee_{\mathbb{R}}(\downarrow x \cap \mathbb{Q}) = x = \bigwedge_{\mathbb{R}}(\uparrow x \cap \mathbb{Q})$. Hence $\mathbb{Q}$ is both join-dense and meet-dense in $\mathbb{R} \cup \{-\infty, \infty\}$. Consequently $\mathbb{R} \cup \{-\infty, \infty\}$ is (order-isomorphic to) the Dedekind-MacNeille completion of $\mathbb{Q}$.

(2) $\mathrm{DM}(\mathbb{N}) \cong \mathbb{N} \oplus \mathbf{1}$.

(3) For any set X, the complete lattice $\mathcal{P}(X) \cong \mathrm{DM}(P)$, where $P = \{\{x\} : x \in X\} \cup \{X \backslash \{x\} : x \in X\}$.

(4) The Dedekind-MacNeille completion of an n-element antichain (for $n \geq 1$) is order-isomorphic to the lattice M_n.

Examples II

(5) Each pair of diagrams consists of an ordered set P_i along with its Dedekind-MacNeille completion $L_i \cong \text{DM}(P_i)$ or, alternatively, as a lattice L_i with a distinguished subset P_i such that $L_i \cong \text{DM}(P_i)$: