Mathematical Logic

(Based on lecture slides by Stan Burris)

George Voutsadakis[1]

[1] Mathematics and Computer Science
Lake Superior State University

LSSU Math 300

Subsection 1

Connectives, Formulas and Truth Tables

The Alphabet: Connectives and Variables

- The following are the basic **logical connectives** that we use to connect logical statements:

Symbol	Name	Symbol	Name
1	true	$\wedge$	and
0	false	$\vee$	or
$\neg$	not	$\rightarrow$	implies
		$\leftrightarrow$	iff

- In the same way that in algebra we use $x, y, z, \ldots$ to stand for unknown or varying numbers, in logic we use the **propositional variables** $P, Q, R, \ldots$ to stand for unknown or varying propositions or statements;
- Using the connectives and variables we can construct propositional formulas like

$$((P \rightarrow (Q \vee R)) \wedge ((\neg Q) \leftrightarrow (1 \vee P))).$$

Inductive (Recursive) Definition of Propositional Formulas

- **Propositional formulas** are formally built as follows:
  - Every propositional variable P is a propositional formula;
  - the constants 0 and 1 are propositional formulas;
  - if F is a propositional formula, then $(\neg F)$ is a propositional formula;
  - if F and G are propositional formulas, then
    - $(F \wedge G)$,
    - $(F \vee G)$,
    - $(F \rightarrow G)$ and
    - $(F \leftrightarrow G)$

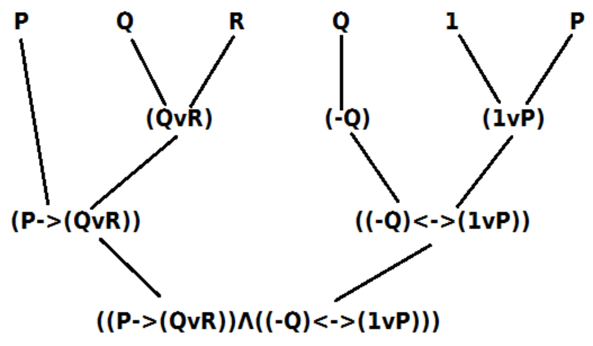
    are propositional formulas.

An Example of Recursively Building a Formula

- As an example, the formula

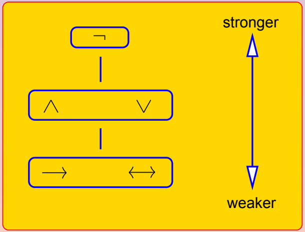
$$((P \to (Q \vee R)) \wedge ((\neg Q) \leftrightarrow (1 \vee P)))$$

of the previous page is recursively built as follows:



Priorities or Precedence of Logical Connectives

- You may remember from algebra that, when we write algebraic expressions, we impose certain precedence in the application of operation symbols so as to avoid writing too many parentheses. E.g., we agree that exponentiation applies before multiplication and division and those apply before addition and subtraction.
- Similarly, to simplify our writing of formulas in logic, we
  - drop the outer parentheses;
  - use the following precedence conventions:



Example of Precedence

- The formula

$$((P \to (Q \lor R)) \land ((\neg Q) \leftrightarrow (1 \lor P)))$$

can be rewritten without redundant parentheses as

$$(P \to Q \lor R) \land (\neg Q \leftrightarrow 1 \lor P)$$

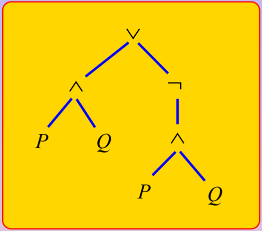
- On the other hand, we do not want to write a non-formula

$$P \land Q \lor R$$

since this writing is ambiguous!

Subformulas

- Consider the formula $(P \wedge Q) \vee \neg(P \wedge Q)$; its syntax tree is



- The **subformulas** of $(P \wedge Q) \vee \neg(P \wedge Q)$ are all formulas appearing in the tree, i.e.,

$$(P \wedge Q) \vee \neg(P \wedge Q)$$
$$P \wedge Q$$
$$\neg(P \wedge Q)$$
$$P$$
$$Q$$

Formal Inductive Definition of Subformulas

- The **subformulas** of a formula F are defined inductively by:
  - The only subformula of a propositional variable P is P itself;
  - The only subformula of a constant c is c itself (c is 0 or 1).
  - The subformulas of $\neg F$ are
    - $\neg F$ and
    - all subformulas of F;
  - The subformulas of $G \Box H$ are
    - $G \Box H$ and
    - all subformulas of G and
    - all subformulas of H;

  ($\Box$ denotes any of $\vee, \wedge, \rightarrow, \leftrightarrow$.)

Semantics Using Truth Values

- The **semantics** of a formula refers to the meaning of the formula;
- If we assign truth values to the variables in a propositional formula then we can calculate the truth value of the formula.
- This is based on the truth tables for the connectives:

P	$\neg P$
1	0
0	1

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

P	Q	$P \rightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

P	Q	$P \leftrightarrow Q$
1	1	1
1	0	0
0	1	0
0	0	1

Truth Tables for Arbitrary Formulas

- Given any propositional formula F we have a **truth table** for F.
- For instance, for $(P \vee Q) \to (P \leftrightarrow Q)$, we have the table

P	Q	$(P \vee Q) \to (P \leftrightarrow Q)$
1	1	1
1	0	0
0	1	0
0	0	1

- This is constructed starting from the truth assignments to the variables and inductively calculating values for subformulas:

P	Q	$P \vee Q$	$P \leftrightarrow Q$	$(P \vee Q) \to (P \leftrightarrow Q)$
1	1	1	1	1
1	0	1	0	0
0	1	1	0	0
0	0	0	1	1

Truth Assignments

- A **truth assignment** or **truth evaluation** $\mathbf{e} = (e_1, \ldots, e_n)$ for the list $P_1, \ldots, P_n$ of propositional variables is a sequence of n truth values;
- Example: $\mathbf{e} = (1, 1, 0, 1)$ is a truth evaluation for the variables P, Q, R, S;
- Given a formula $F(P_1, \ldots, P_n)$ let $F(\mathbf{e})$ denote the propositional formula $F(e_1, \ldots, e_n)$;
- Example: If the formula has four variables, say $F(P, Q, R, S)$, then for the $\mathbf{e}$ above we have $F(\mathbf{e}) = F(1, 1, 0, 1)$;
- Let $\hat{F}(\mathbf{e})$ be the **truth value** of F at $\mathbf{e}$.
- Example: Consider the formula $F(P, Q, R, S) = \neg(P \vee R) \rightarrow (S \wedge Q)$ and the truth assignment $\mathbf{e} = (1, 1, 0, 1)$ for P, Q, R, S; Then, $F(\mathbf{e}) = \neg(1 \vee 0) \rightarrow (1 \wedge 1)$ and $\hat{F}(\mathbf{e}) = 1$.

Subsection 2

Equivalences, Tautologies and Contradictions

Equivalent Formulas

- Formulas F and G are called **(truth) equivalent**, written $F \sim G$, if they have the same truth tables;
- Examples:

$$\begin{array}{rcl} 1 & \sim & P \vee \neg P \\ 0 & \sim & \neg(P \vee \neg P) \\ P \wedge Q & \sim & \neg(\neg P \vee \neg Q) \\ P \rightarrow Q & \sim & \neg P \vee Q \\ P \leftrightarrow Q & \sim & \neg(\neg P \vee \neg Q) \vee \neg(P \vee Q) \end{array}$$

- These are the well-known expressions of the standard connectives in terms of just $\neg$ and $\vee$.

Proof of an Equivalence

- We show that $P \to Q \;\sim\; \neg Q \to \neg P \;\sim\; \neg P \vee Q$:

P	Q	$\neg Q$	$\neg P$	$P \to Q$	$\neg Q \to \neg P$	$\neg P \vee Q$
1	1	0	0	1	1	1
1	0	1	0	0	0	0
0	1	0	1	1	1	1
0	0	1	1	1	1	1

One More Equivalence

- We also show that $P \wedge (Q \vee R) \sim (P \wedge Q) \vee (P \wedge R)$.

P	Q	R	$Q \vee R$	$P \wedge Q$	$P \wedge R$	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Fundamental Truth Equivalences

1.	$P \vee P$	$\sim$	P	(Idempotent)
2.	$P \wedge P$	$\sim$	P	(Idempotent)
3.	$P \vee Q$	$\sim$	$Q \vee P$	(Commutative)
4.	$P \wedge Q$	$\sim$	$Q \wedge P$	(Commutative)
5.	$P \vee (Q \vee R)$	$\sim$	$(P \vee Q) \vee R$	(Associative)
6.	$P \wedge (Q \wedge R)$	$\sim$	$(P \wedge Q) \wedge R$	(Associative)
7.	$P \wedge (P \vee Q)$	$\sim$	P	(Absorption)
8.	$P \vee (P \wedge Q)$	$\sim$	P	(Absorption)
9.	$P \wedge (Q \vee R)$	$\sim$	$(P \wedge Q) \vee (P \wedge R)$	(Distributive)
10.	$P \vee (Q \wedge R)$	$\sim$	$(P \vee Q) \wedge (P \vee R)$	(Distributive)

To be continued after a break!

Augustus De Morgan

- Augustus De Morgan, born in Madurai, Madras Presidency, British Raj (1806-1871)

More Truth Equivalences

11.	$P \vee \neg P$	$\sim$	1	(Excluded Middle)
12.	$P \wedge \neg P$	$\sim$	0	
13.	$\neg\neg P$	$\sim$	P	
14.	$P \vee 1$	$\sim$	1	
15.	$P \wedge 1$	$\sim$	P	
16.	$P \vee 0$	$\sim$	P	
17.	$P \wedge 0$	$\sim$	0	
18.	$\neg(P \vee Q)$	$\sim$	$\neg P \wedge \neg Q$	(De Morgan's Law)
19.	$\neg(P \wedge Q)$	$\sim$	$\neg P \vee \neg Q$	(De Morgan's Law)
20.	$P \to Q$	$\sim$	$\neg P \vee Q$	
21.	$P \to Q$	$\sim$	$\neg Q \to \neg P$	(Contraposition)

More Truth Equivalences

$$
\begin{array}{lrcl}
22. & P \to (Q \to R) & \sim & (P \wedge Q) \to R \\
23. & P \to (Q \to R) & \sim & (P \to Q) \to (P \to R) \\
24. & P \leftrightarrow P & \sim & 1 \\
25. & P \leftrightarrow Q & \sim & Q \leftrightarrow P \\
26. & (P \leftrightarrow Q) \leftrightarrow R & \sim & P \leftrightarrow (Q \leftrightarrow R) \\
27. & P \leftrightarrow \neg Q & \sim & \neg(P \leftrightarrow Q) \\
28. & P \leftrightarrow (Q \leftrightarrow P) & \sim & Q \\
29. & P \leftrightarrow Q & \sim & (P \to Q) \wedge (Q \to P) \\
30. & P \leftrightarrow Q & \sim & (P \wedge Q) \vee (\neg P \wedge \neg Q) \\
31. & P \leftrightarrow Q & \sim & (P \vee \neg Q) \wedge (\neg P \vee Q)
\end{array}
$$

A Few More Useful Equivalences

32.	$1 \leftrightarrow P$	$\sim$	P
33.	$0 \leftrightarrow P$	$\sim$	$\neg P$
34.	$1 \rightarrow P$	$\sim$	P
35.	$P \rightarrow 1$	$\sim$	1
36.	$0 \rightarrow P$	$\sim$	1
37.	$P \rightarrow 0$	$\sim$	$\neg P$

Tautologies and Contradictions

- A formula F is called a **tautology** if $\hat{F}(\mathbf{e}) = 1$, for every truth assignment $\mathbf{e}$. This means the truth table for F looks like:

	F
	1
	$\vdots$
	1

Theorem (Truth Equivalence and Tautologies)

Two propositional formulas F and G are truth equivalent if and only if the formula $F \leftrightarrow G$ is a tautology.

- A formula F is called a **contradiction** if $\hat{F}(\mathbf{e}) = 0$, for every truth assignment $\mathbf{e}$. How does the truth table of a contradiction looks like?

Subsection 3

Substitution

Substitutions

- **Substitution** means uniform substitution of formulas for variables.
- Given
  - a formula $F(P_1, \ldots, P_n)$ with variables $P_1, \ldots, P_n$, and
  - formulas $H_1, \ldots, H_n$,

  $F(H_1, \ldots, H_n)$ is the formula resulting from substituting H_i for each occurrence of P_i in $F(P_1, \ldots, P_n)$;
- If $F(P, Q)$ is the formula $P \to (Q \to P)$, then

$$F(\neg P \vee R, \neg P) = \neg P \vee R \to (\neg P \to \neg P \vee R).$$

Substitution Theorem

Substitution Theorem

Given

- formulas $F(P_1, \ldots, P_n)$ and $G(P_1, \ldots, P_n)$ with variables $P_1, \ldots, P_n$, and
- formulas $H_1, \ldots, H_n$,

if $F(P_1, \ldots, P_n) \sim G(P_1, \ldots, P_n)$, then $F(H_1, \ldots, H_n) \sim G(H_1, \ldots, H_n)$.

- Example: Consider one of De Morgan's Laws:

$$\neg(P \vee Q) \sim \neg P \wedge \neg Q.$$

By the Substitution Theorem, we may conclude:

$$\neg((P \rightarrow R) \vee (R \leftrightarrow Q)) \sim \neg(P \rightarrow R) \wedge \neg(R \leftrightarrow Q).$$

An Exercise on Substitution

- Which of the following propositional formulas are substitution instances of the formula $P \rightarrow (Q \rightarrow P)$? If a formula is indeed a substitution instance, give the formulas substituted for P, Q.
  1. $\neg R \rightarrow (R \rightarrow \neg R)$
     YES!
  2. $\neg R \rightarrow (\neg R \rightarrow \neg R)$
     YES!
  3. $\neg R \rightarrow (\neg R \rightarrow R)$
     No!
  4. $(P \wedge Q \rightarrow P) \rightarrow ((Q \rightarrow P) \rightarrow (P \wedge Q \rightarrow P))$
     YES!
  5. $((P \rightarrow P) \rightarrow P) \rightarrow ((P \rightarrow (P \rightarrow (P \rightarrow P))))$
     No!

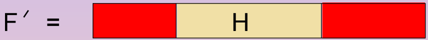
Subsection 4

Replacement

Replacement

- If a formula F has a subformula G, say

  then, when we **replace** the given occurrence of G by another formula H, the result looks like



- Some like to call this substitution as well. But then there are two kinds of substitution!
- So, for clarity it is better to call it **replacement**.
- Example: If we replace the second occurrence of $P \vee Q$ in the formula $F = (P \vee Q) \rightarrow (R \leftrightarrow (P \vee Q))$ by the formula $Q \vee P$, then we obtain the formula $F' = (P \vee Q) \rightarrow (R \leftrightarrow (Q \vee P))$.

Replacement Theorem

Replacement Theorem

Let F, G, H be formulas. If $G \sim H$, then $F(\cdots G \cdots) \sim F(\cdots H \cdots)$.

- Example: We know by De Morgan's Law that

$$\neg(Q \vee R) \sim \neg Q \wedge \neg R.$$

Thus, by the Replacement Theorem, we can conclude that

$$(P \rightarrow \neg(Q \vee R)) \wedge \neg Q \sim (P \rightarrow \neg Q \wedge \neg R) \wedge \neg Q$$

Simplification Through Replacement

- By replacing subformulas by equivalent formulas and using the Replacement Theorem, we can simplify formulas; This means obtaining equivalent formulas in simpler form;
- Example: Simplify the formula $(P \wedge Q) \vee \neg(\neg P \vee Q)$.

$$\begin{array}{rcl} (P \wedge Q) \vee \neg(\neg P \vee Q) & & (\text{apply De Morgan's Law}) \\ & \sim & (P \wedge Q) \vee (\neg\neg P \wedge \neg Q) \\ & & (\text{apply Double Negation Law}) \\ & \sim & (P \wedge Q) \vee (P \wedge \neg Q) \\ & & (\text{apply Distributive Law}) \\ & \sim & P \wedge (Q \vee \neg Q) \\ & & (\text{apply Disjunction Law}) \\ & \sim & P \wedge 1 \\ & & (\text{apply Conjunction Law}) \\ & \sim & P. \end{array}$$

By transitivity of $\sim$, we get $(P \wedge Q) \vee \neg(\neg P \vee Q) \sim P$.

Subsection 5

Adequate Sets of Connectives

Adequate Sets of Connectives

- A set of connectives is called **adequate** if every truth table is the truth table of some propositional formula using only the given set of connectives;
- The set of standard connectives $\{1, 0, \neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ is adequate;
- Given any truth table, we can construct a formula using only these connectives whose truth table agrees with the given table;

P	Q	R	F
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	1

Find a formula $F(P, Q, R)$ using only the standard connectives that has this truth table:

Answer:

$(P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

Minimal Adequate Sets

- From the previous example, we conclude that we only need the connectives $\vee, \wedge$ and $\neg$ to construct a formula for any given table;
- It follows that $\{\vee, \wedge, \neg\}$ is an adequate set of connectives;
- An adequate set of connectives is **minimal** if no proper subset of it is adequate;
- Is $\{\vee, \wedge, \neg\}$ minimal? The answer is "no" because, by De Morgan's Laws

$$P \vee Q \sim \neg(\neg P \wedge \neg Q) \quad \text{and} \quad P \wedge Q \sim \neg(\neg P \vee \neg Q).$$

Therefore, both $\{\wedge, \neg\}$ and $\{\vee, \neg\}$ are adequate sets of connectives.

More on Minimality

- Is the set of connectives $\{\neg, \rightarrow\}$ adequate?
  Yes! How can we show this?
  We must show that every connective in an adequate set can be expressed using only those two!

Sets with a Single Standard Connective

No single standard connective is adequate.

- This is an interesting statement.
  How can we prove something like this?
- The strategy is to show that for each standard connective, there is some other standard connective that cannot be expressed using the first standard connective.

Inadequacy of a Single Standard Connective

- If we have a single constant 0 or 1 then we cannot express $\neg$;
- If we have just $\neg$ we cannot express $\wedge$;
- If we have just $\Box$ ($\Box$ can be any of $\wedge, \vee, \rightarrow, \leftrightarrow$), then we cannot express $\neg$;
  This means that it is not possible to find a formula $F(P)$ using just the connective $\Box$ that is equivalent to $\neg P$.
  To see this, we first have two find out what can be expressed with $F(P)$ using only a single connective $\Box$.
  The following table summarizes what can be expressed:

$\Box =$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
$F(P) \sim$	P	P	1 or P	1 or P

More on Table

- The table:

$\square =$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
$F(P) \sim$	P	P	1 or P	1 or P

- For example, if we start with $\rightarrow$, then any formula $F(P)$ in one variable P, using just the connective $\rightarrow$, is equivalent to either 1 or P;
- This can be proved by using a form of induction, called **structural induction** because it inducts on the increasingly complex structure of formulas taking into account the rules for constructing them;
- In the next slide we use structural induction to show that

> any formula constructed just using P and $\rightarrow$ is truth equivalent to the formula 1 or the formula P.

$\to$ cannot express $\neg$

- If only P and $\to$ are used to construct $F(P)$, then $F(P)$ can only be
  - P or $P \to P$;
  - $G(P) \to H(P)$, for some formulas $G(P)$ and $H(P)$, using only P and $\to$ and of simpler structure than $F(P)$ itself;
- So, to see that $F(P)$ is truth equivalent to 1 or P, we start with P and $P \to P$ (Basis of Structural Induction):
  - $P \sim P$;
  - $P \to P \sim 1$: This is because of the following truth-table:

P	$P \to P$
1	1
0	1

$\to$ cannot express $\neg$ (Cont'd)

- We finish by showing that
  if $G(P), H(P)$ are truth-equivalent to 1 or P (Structural Induction Hypothesis),
  then $F(P) = G(P) \to H(P)$ must also be equivalent to 1 or P (Step of Structural Induction): This is because of the following truth-table:

$G(P)$	$H(P)$	$G(P) \to H(P)$
1	1	1
1	P	P
P	1	1
P	P	1

- Since $F(P)$ is equivalent to either 1 or P, it cannot be equivalent to $\neg P$.
  Thus, no formula constructed just by using P and $\to$, no matter how complex, can express the formula $\neg P$.

Ernst Schröder

- Ernst Schröder, born in Mannheim, Baden, Germany (1841-1902)

Schröder's $\curlywedge$ Connective

- Schröder found in 1880 the $\curlywedge$ connective with truth table

P	Q	$P \curlywedge Q$
1	1	0
1	0	0
0	1	0
0	0	1

- This connective is adequate because it can express
  - $\neg P \sim P \curlywedge P$

P	$P \curlywedge P$
1	0
0	1

  - $P \wedge Q \sim (P \curlywedge P) \curlywedge (Q \curlywedge Q)$

P	Q	$P \curlywedge P$	$Q \curlywedge Q$	$(P \curlywedge P) \curlywedge (Q \curlywedge Q)$
1	1	0	0	1
1	0	0	1	0
0	1	1	0	0
0	0	1	1	0

Henry Maurice Sheffer

- Henry Maurice Sheffer, born in western Ukraine (1882-1964)

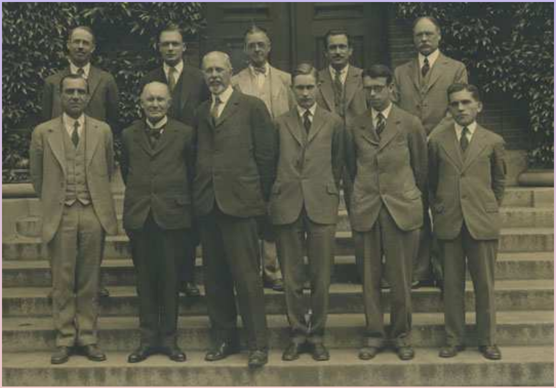


Figure : The Harvard Philosophy Faculty 1929: Sheffer Last in Front Row

The Sheffer Stroke $|$

- Sheffer found in 1913 the **Sheffer stroke** $|$ connective:

P	Q	$P \mid Q$
1	1	0
1	0	1
0	1	1
0	0	1

- This connective is adequate because it can express
  - $\neg P \sim P \mid P$

P	$P \mid P$
1	0
0	1

  - $P \vee Q \sim (P \mid P) \mid (Q \mid Q)$

P	Q	$P \mid P$	$Q \mid Q$	$(P \mid P) \mid (Q \mid Q)$
1	1	0	0	1
1	0	0	1	1
0	1	1	0	1
0	0	1	1	0

Subsection 6

Disjunctive and Conjunctive Forms

Associativity

- Since the associative law holds for $\vee$ and $\wedge$ it is common practice to drop parentheses in situations such as

$$P \wedge ((Q \wedge R) \wedge S),$$

yielding

$$P \wedge Q \wedge R \wedge S;$$

- Likewise we often write

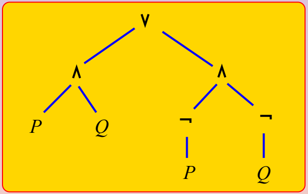
$$P \vee Q \vee R \vee S$$

instead of

$$(P \vee Q) \vee (R \vee S).$$

Disjunctive Normal Form (DNF)

- Any formula F can be transformed into a **disjunctive form**, e.g., $P \leftrightarrow Q \sim (P \wedge Q) \vee (\neg P \wedge \neg Q)$;
- If every variable or its negation appears in each conjunction then we call it a **disjunctive normal form**.
- Such conjunctions are called **DNF-constituents**.
- The above disjunctive form is actually a disjunctive normal form, with the DNF-constituents $P \wedge Q$ and $\neg P \wedge \neg Q$.
- The formula tree for this DNF form is



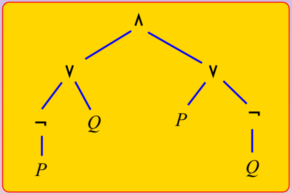
- Notice that
  - the negations are all next to the leaves of the tree;
  - And there is no $\wedge$ above a $\vee$.

Disjunctive Form

- Being in **disjunctive form** really means:
  1. negations only appear next to variables;
  2. no $\wedge$ is above a $\vee$.
- So we can have degenerate cases of the disjunctive form:
  - P;
  - $P \vee \neg Q$;
  - $P \wedge \neg Q$;

Conjunctive Form

- And we have conjunctive forms such as $P \leftrightarrow Q \sim (\neg P \vee Q) \wedge (P \vee \neg Q)$;
- The formula tree is given by



- Being in **conjunctive form** means:
  1. negations only appear next to variables;
  2. no $\vee$ is above a $\wedge$.

Important Remarks on CNF and DNF

- The formula
$$F(P,Q) = P \vee \neg Q$$
is in both disjunctive and conjunctive form;
- It is in conjunctive normal form, but not in disjunctive normal form;
- The set of variables used affects the normal forms;
- The formula $F(P) = \neg P$ is in both CNF and DNF;
- However, the formula $F(P,Q) = \neg P$ is in neither:
  - Its CNF is $(\neg P \vee Q) \wedge (\neg P \vee \neg Q)$;
  - Its DNF is $(\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$.

Rules for Transforming a Formula into DNF and CNF

- To transform a given formula F into a disjunctive form we apply the following equivalences:
  - $F \to G \rightsquigarrow \neg F \vee G$
  - $F \leftrightarrow G \rightsquigarrow (F \to G) \wedge (G \to F)$
  - $\neg(F \vee G) \rightsquigarrow \neg F \wedge \neg G$
  - $\neg(F \wedge G) \rightsquigarrow \neg F \vee \neg G$
  - $\neg\neg F \rightsquigarrow F$
  - $F \wedge (G \vee H) \rightsquigarrow (F \wedge G) \vee (F \wedge H)$
  - $(F \vee G) \wedge H \rightsquigarrow (F \wedge H) \vee (G \wedge H)$
- These rules are applied until no further applications are possible.

Example and Additional Rules

- Consider $P \wedge (P \rightarrow Q)$;
- Rewrite $P \wedge (\neg P \vee Q) \rightsquigarrow (P \wedge \neg P) \vee (P \wedge Q)$;
- Now this formula clearly gives a disjunctive form, but not a normal form. (Why?)
- We can simplify it considerably, but to do this we need to invoke additional rewrite rules.
  - $0 \wedge F \rightsquigarrow 0$
  - $\neg 1 \rightsquigarrow 0$
  - $\cdots \wedge F \wedge \cdots \wedge \cdots \wedge \neg F \wedge \cdots \rightsquigarrow 0$
  - $\cdots \wedge F \wedge \cdots \wedge \cdots \wedge F \wedge \cdots \rightsquigarrow \cdots \wedge F \wedge \cdots$
- Applying them, we get $(P \wedge \neg P) \vee (P \wedge Q) \rightsquigarrow 0 \vee (P \wedge Q) \rightsquigarrow P \wedge Q$;
- One more rule is needed, to handle the exceptional case that the above rules reduce the formula to the constant 1. In this case we rewrite 1 as a join of all possible DNF constituents.

Exceptional Cases

- Sometimes, after applying all these rules, one still does not have a disjunctive normal form;
- Example: If we start with $(P \wedge Q) \vee \neg P$ then none of the rules apply. To get a DNF we need to replace $\neg P$ with $(\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$. Then

$$(P \wedge Q) \vee \neg P \sim (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q).$$

- Now we have a disjunctive normal form.

Using Truth Tables to find Normal Forms

- The second method to find normal forms is to use truth tables;
- Rows of truth table of F yield the constituents according to:
  - The DNF-constituents chosen from rows for which F is true;
  - The CNF-constituents chosen from rows for which F is false.
- **Example:** Consider $F(P,Q) = (\neg P \vee Q) \wedge \neg P$. Truth table:

P	Q	$(\neg P \vee Q) \wedge \neg P$
1	1	0
1	0	0
0	1	1
0	0	1

P	Q	$(P \leftrightarrow Q) \vee (P \rightarrow Q)$
1	1	1
1	0	0
0	1	1
0	0	1

  Therefore, its DNF is $(\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$.
- **Example:** Consider $F(P,Q) = (P \leftrightarrow Q) \vee (P \rightarrow Q)$. Truth table:
  Therefore, its CNF is $\neg P \vee Q$.

Uniqueness of Normal Forms and Test for Equivalence

- A formula has many disjunctive forms, and many conjunctive forms;
- But it has only one disjunctive normal form and only one conjunctive normal form;
- This happens because normal forms are determined by the truth table of a formula.
- A consequence of this uniqueness property is the following:

Equivalence Test based on Normal Forms

Two formulas are equivalent iff they have the same disjunctive (or conjunctive) normal forms.

Subsection 7

Valid Arguments, Tautologies and Satisfiability

Logical Arguments

- A (logical) argument draws conclusions from premisses.
- **Example:**

$$\begin{array}{l} P \vee Q \vee R \\ \neg P \\ \neg Q \\ \therefore\ R \end{array}$$

- The general form of a **logical argument** is

$$F_1, \ldots, F_n \therefore F.$$

  $F_1, \ldots, F_n$ are the **premises** and F is the **conclusion**.
- Some arguments are valid and some are not; we define validity carefully in the following slide.

Valid Arguments

- An argument

$$F_1, \ldots, F_n \therefore F$$

  is **valid** (or **correct**) if the conclusion is true whenever the premisses are true.
- Schematically, $F_1, \ldots, F_n \therefore F$ is valid if

F_1	$\cdots$	F_n	F
1	$\cdots$	1	

implies

F_1	$\cdots$	F_n	F
1	$\cdots$	1	1

Validity of Arguments and Tautology of Implication

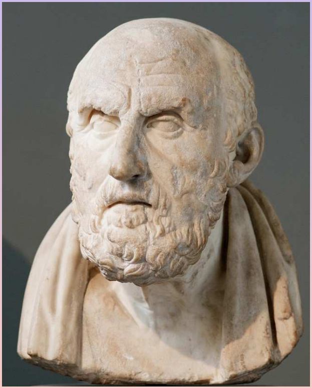
Proposition (Relating $\therefore$ and $\rightarrow$)

The argument $F_1, \ldots, F_n \therefore F$ is valid iff the implication $F_1 \wedge \cdots \wedge F_n \rightarrow F$ is a tautology.

- Both statements mean that F is true whenever $F_1, \ldots, F_n$ are true.
- A nice example to follow after a break!

Chrysippos from Soli

- Chrysippos, born in Soli, Cilicia (279-206 B.C.)



Chrysippos' Smart Hunting Dog

- When running after a rabbit, the dog found that the path suddenly split in three directions.
  - The dog sniffed the first path and found no scent;
  - Then it sniffed the second path and found no scent;
  - Then, without bothering to sniff the third path, it ran down that path.
- Reasoning of the smart canine:
  - The rabbit went this way or that way or the other way.
  - Not this way;
  - Not that way;
  - Therefore the other way.
- The dog's argument:

$$\begin{array}{l} P \vee Q \vee R \\ \neg P \\ \neg Q \\ \therefore\ R \end{array}$$

Validity of Chrysippos' Dog Reasoning

- Is the dog's argument
$$\begin{array}{l} P \vee Q \vee R \\ \neg P \\ \neg Q \\ \therefore\ R \end{array}$$
a valid argument?
- This is verified by the following truth table:

P	Q	R	$P \vee Q \vee R$	$\neg P$	$\neg Q$	R
1	1	1	1	0	0	1
1	1	0	1	0	0	0
1	0	1	1	0	1	1
1	0	0	1	0	1	0
0	1	1	1	1	0	1
0	1	0	1	1	0	0
0	0	1	11	11	11	11
0	0	0	0	1	1	0

Satisfiability of a Set of Formulas

- A set $\mathcal{S}$ of propositional formulas is **satisfiable** if there is a truth evaluation **e** for the variables in $\mathcal{S}$ that makes every formula in $\mathcal{S}$ true;
- In that case, we say that **e satisfies** $\mathcal{S}$;
- The expression $\text{Sat}(\mathcal{S})$ means "$\mathcal{S}$ is satisfiable";
- The expression $\neg\text{Sat}(\mathcal{S})$ means "$\mathcal{S}$ is not satisfiable";
- Thus a finite set $\{F_1, \ldots, F_n\}$ of formulas is satisfiable if, when we look at the combined truth table for the F_i's, we can find a line that looks as follows:

P_1	$\cdots$	P_m	F_1	$\cdots$	F_n
e_1	$\cdots$	e_m	1	$\cdots$	1

Example of Satisfiability I

- Consider the set $\mathcal{S} = \{P \to Q, Q \to R, R \to P\}$;
- Because

P	Q	R	$P \to Q$	$Q \to R$	$R \to P$
1	1	1	11	11	11
1	1	0	1	0	1
1	0	1	0	1	1
1	0	0	0	1	1
0	1	1	1	1	0
0	1	0	1	0	1
0	0	1	1	1	0
0	0	0	11	11	11

$\mathcal{S}$ is satisfiable and both $\mathbf{e} = (1,1,1)$ and $\mathbf{e}' = (0,0,0)$ satisfy $\mathcal{S}$.

Example of Satisfiability II

- Consider the set $\mathcal{S} = \{P \leftrightarrow \neg Q, Q \leftrightarrow R, R \leftrightarrow P\}$;
- Because

P	Q	R	$P \leftrightarrow \neg Q$	$Q \leftrightarrow R$	$R \leftrightarrow P$
1	1	1	0	1	1
1	1	0	0	0	0
1	0	1	1	0	1
1	0	0	1	1	0
0	1	1	1	1	0
0	1	0	1	0	1
0	0	1	0	0	0
0	0	0	0	1	1

$\mathcal{S}$ is not satisfiable.

Valid Arguments and Non-Satisfiable Formulas

Theorem

Let $F_1, \ldots, F_n, F$ be formulas. Then the following assertions are equivalent:

- The argument $F_1, \ldots, F_n \therefore F$ is valid;
- The set $\{F_1, \ldots, F_n, \neg F\}$ is not satisfiable;
- The formula $F_1 \wedge \cdots \wedge F_n \rightarrow F$ is a tautology;
- The formula $F_1 \wedge \cdots \wedge F_n \wedge \neg F$ is not satisfiable.

- All of those statements say in essence that F is true whenever $F_1, \ldots, F_n$ are true!

Summary of Info Provided by Combined Truth Tables

- From a **combined truth table** , such as

P	Q	R	F_1	F_2	F_3	F_4
1	1	1	1	1	0	1
1	1	0	0	0	1	0
1	0	1	1	1	0	0
1	0	0	0	0	0	0
0	1	1	1	0	0	0
0	1	0	0	0	1	1
0	0	1	1	1	0	1
0	0	0	0	0	0	1

  one may draw conclusions about:

- Normal Forms;
- Equivalence of Formulas;
- Tautologies;
- Contradictions;
- Satisfiability of Formulas;
- Valid Arguments.

An Applied Example: The Two Tribes on the Island of Tufa

- The island of Tufa has two tribes:
  - the Tu's who always tell the truth;
  - the Fa's who always lie.
- A traveler encountered three residents A, B, and C of Tufa, and each made a statement to the traveler:
  - A: "A or B tells the truth if C lies."
  - B: "If A or C tell the truth, then it is not the case that exactly one of us is telling the truth."
  - C: "A or B is lying iff A or C is telling the truth."
- How can we determine, as best possible, which tribes A, B, and C belong to?

Tribes on the Island of Tufa: Solution I

- Statements of the Tufa residents:
  - A: "A or B tells the truth if C lies."
  - B: "If A or C tell the truth, then it is not the case that exactly one of us is telling the truth."
  - C: "A or B is lying iff A or C is telling the truth."
- Let
  - A be the statement "A is telling the truth" (equivalently "A is a Tu");
  - B be the statement "B is telling the truth" (equivalently "B is a Tu");
  - C be the statement "C is telling the truth" (equivalently "C is a Tu");
- Then in symbolic form the three people have made the following statements:
  - A says: $\neg C \rightarrow (A \vee B)$;
  - B says: $A \vee C \rightarrow \neg((\neg A \wedge \neg B \wedge C) \vee (\neg A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge \neg C))$;
  - C says: $(\neg A \vee \neg B) \leftrightarrow (A \vee C)$.

Tribes on the Island of Tufa: Solution I

- Since a person tells the truth iff what he says is true, we obtain the following statements:
  - $A \leftrightarrow (\neg C \rightarrow (A \vee B))$;
  - $B \leftrightarrow (A \vee C \rightarrow \neg((\neg A \wedge \neg B \wedge C) \vee (\neg A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge \neg C)))$;
  - $C \leftrightarrow ((\neg A \vee \neg B) \leftrightarrow (A \vee C))$.
- Letting these three propositional formulas be F, G, and H, we have the truth table:

A	B	C	F	G	H
1	1	1	1	1	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	0	1	0
0	1	0	0	1	0
0	0	1	0	1	1
0	0	0	1	0	1

From lines 2 and 4 we see that A must be a Tu and C must be a Fa.
However, we do not know for sure which tribe B belongs to.

Subsection 8

Compactness

The Compactness Theorem

Compactness Theorem for Propositional Logic

Suppose $\mathcal{S}$ is a set of propositional formulas. $\mathcal{S}$ is satisfiable iff every finite subset $\mathcal{S}_0 \subseteq \mathcal{S}$ is satisfiable.

- Note that the theorem is trivial if $\mathcal{S}$ is finite;
- Note, also, that the left to right implication of the theorem is trivial, even when $\mathcal{S}$ is infinite; After all, if a set of formulas is satisfiable, every subset of the set is also satisfiable;
- The proof of the right to left implication takes some time; We will not present it here, but you may find it on page 75 of our textbook;
- We will instead showcase the usefulness of the theorem by presenting two applications.

Philip Hall

- Philip Hall, born in Hampstead, London, England (1904-1982)

William “Bill” Thomas Tutte

- William “Bill” Thomas Tutte, born in Newmarket, Suffolk, England (1917-2002)

Application of Compactness I: The Matching Problem

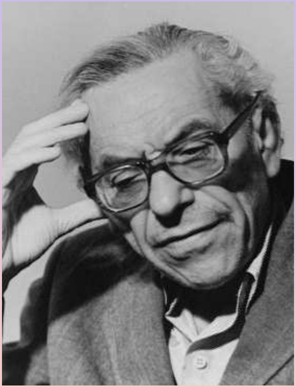
- Suppose that A, B are sets and $R \subseteq A \times B$ a relation from A to B, such that
  - every element of A is related to at least one element of B;
  - every element of A is related to only finitely many elements of B;
- Suppose that, for every finite subset $A_0 \subseteq A$, it is possible to find a **matching** $f : A_0 \to B$, i.e.,
  a one-to-one function f, satisfying $(a, f(a)) \in R$, for all $a \in A_0$;
- We can use the Compactness Theorem to show that there exists a matching for all of A;
  We do this carefully in the following slide.

The Matching Problem: The Solution

- For all $(a, b) \in A \times B$, we introduce a propositional variable P_{ab}; (The intuition is that P_{ab} will have value 1 if $f(a) = b$ and 0, otherwise.)
- Let $\mathcal{S}$ be the set of propositional formulas consisting of
  1. $P_{ab_1} \vee \cdots \vee P_{ab_n}$, where b_i ranges over all $b \in B$, such that aRb holds, and $a \in A$;
  2. $\neg P_{ab_1} \vee \neg P_{ab_2}$, for all $b_1, b_2 \in B$, $b_1 \neq b_2$, and $a \in A$;
  3. $\neg P_{a_1 b} \vee \neg P_{a_2 b}$, for all $a_1, a_2 \in A$, $a_1 \neq a_2$, and $b \in B$;
- The formulas of type
  1. say "each a is matched to at least one of the b's, such that aRb holds";
  2. say "each a is matched to at most one b";
  3. say "different a's are not matched to the same b";
- The postulated hypothesis asserts the existence of a matching for every finite subset of A; i.e., that every finite subset of $\mathcal{S}$ is satisfiable;
- By the Compactness Theorem $\mathcal{S}$ is also satisfiable; A satisfying assignment of truth values to P_{ab}'s translates directly to a matching for all A.

Paul Erdös

- Paul Erdös, born in Budapest, Austria-Hungary (1913-1996)



Nicolaas Govert "Dick" de Bruijn

- Nicolaas Govert "Dick" de Bruijn, born in The Hague (Den Haag), South Holland, Netherlands (1918-2012)

Application of Compactness II: Graph Coloring

- Suppose that $\mathbf{G} = (G, r)$ is a **graph**, i.e., G is a set of **vertices** and r is an **edge relation**, assumed to be
  - **irreflexive**, i.e., $(a, a) \notin r$, for all $a \in G$;
  - **symmetric**, i.e., $(a, b) \in r$ implies $(b, a) \in r$, for all $a, b \in G$;
- If $(a, b) \in r$, we say (a, b) is an **edge**, a, b **belong** to edge (a, b) or a, b are **adjacent**;
- A graph $\mathbf{G}' = (G', r')$ is a **subgraph** of $\mathbf{G} = (G, r)$ if
  - $G' \subseteq G$ is nonempty;
  - $(a, b) \in r'$ iff $(a, b) \in r$, for all $a, b \in G'$;
- Given a positive integer k, a k-**coloring** of $\mathbf{G} = (G, r)$ is an assignment of colors to the vertices of G from a collection of k colors $\{c_1, \ldots, c_k\}$ with the property that
  - adjacent vertices are not assigned the same color.

Erdös-De Bruijn Theorem

Let k be fixed. If every finite subgraph of a graph $\mathbf{G} = (G, r)$ has a k-coloring, then $\mathbf{G}$ itself has a k-coloring.

The Erdös-De Bruijn Theorem

- We use the Compactness Theorem to prove Erdös-De Bruijn;
- For all $a \in G$ and all $1 \leq i \leq k$, we introduce a variable P_{ai};
  (Intuition: P_{ai} will have value 1 if vertex a gets color i and 0, otherwise.)
- Let $\mathcal{S}$ be the set of propositional formulas consisting of
  1. $P_{a1} \vee \cdots \vee P_{ak}$, $a \in G$;
  2. $\neg P_{ai} \vee \neg P_{aj}$, $a \in G, 1 \leq i < j \leq k$;
  3. $\neg P_{ai} \vee \neg P_{bi}$, for all $a, b \in G$, with $(a, b) \in r$ and $1 \leq i \leq k$;
- The formulas of type
  1. say "each a is assigned at least one color";
  2. say "no a is assigned two different colors";
  3. say "adjacent a's are not assigned the same color";
- The hypothesis asserts the existence of a k-coloring for every finite subgraph of **G**; i.e., that every finite subset of $\mathcal{S}$ is satisfiable;
- By the Compactness Theorem $\mathcal{S}$ is also satisfiable; A satisfying assignment of truth values to P_{ai}'s translates directly to a k-coloring of **G** itself.

Subsection 9

Epilogue: Other Propositional Logics

Other Propositional Logics

- The term propositional logic alludes to the fact that the variables $P, Q, \ldots$ stand for propositions and that the connectives $\neg, \wedge, \ldots$ combine propositions;
- We studied Classical Propositional Logic, which has two distinctive features:
  - Its connectives are the classical connectives;
  - Its propositions are evaluated to either 1 (true) or 0 (false);
- This is by no means the only propositional logic!
- By allowing different sets of connectives (syntax) or different evaluations (semantics) we may construct and study a huge variety of other very important and interesting propositional logics!

Luitzen Egbertus Jan Brouwer

- Luitzen Egbertus Jan Brouwer, born in Overschie, Netherlands (1881-1966)

Arend Heyting

- Arend Heyting, born in Amsterdam, Netherlands (1898-1980)

Constructive Mathematics

- In Classical Propositional Logic (that we have studied in detail), the formula (Law of the Excluded Middle) $P \vee \neg P$ is a tautology.
- So, if one shows that $\neg P$ assumes the value 0, i.e., that the negation of P cannot hold, then one may conclude that P must assume the value 1, i.e., that P must hold!
- Many mathematicians objected to this type of reasoning on philosophical grounds.
  - They claimed that, e.g., to prove the existence of an object, it should not be enough to show that its nonexistence leads to a contradiction!
  - They insisted that to show that an object exists one must construct such an object!
- The propositional logic on which this type of mathematics, called **constructive mathematics**, is based is not classical propositional logic, but rather **intuitionistic logic**.
- One of the founders of intuitionism was Brouwer; Heyting was one of his students, also a strong intuitionist.

Jan Łukasiewicz

- Jan Łukasiewicz, born in Lwów (Lemberg in German), Galicia, Austria-Hungary (1878-1956)

Łukasiewicz's Three-Valued Logic

- Think of a propositional logic that is also based on the connectives $\neg, \vee, \wedge, \ldots$;
- But, instead of its variables being only allowed the values 0 and 1, the variables may be assigned the values 0, 1 and u, the latter standing for Unknown;
- The evaluations of the formulas are based on the following truth tables for these connectives:

P	$\neg P$
1	0
u	u
0	1

$\vee$	0	u	1
0	0	u	1
u	u	u	1
1	1	1	1

$\wedge$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

- Then $P \vee \neg P$ is not a tautology anymore!
- Since the Law of the Excluded Middle is not a law of this logic, the kind of reasoning by contradiction, that intuitionists strongly object to, is avoided!