Introduction to Markov Chains

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 500

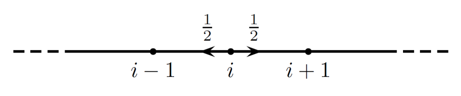
1. Further Theory
   - Martingales
   - Potential Theory
   - Electrical Networks
   - Brownian Motion

Subsection 1

Martingales

Example

- Consider the simple symmetric random walk $(X_n)_{n \geq 0}$ on $\mathbb{Z}$, which is a Markov chain with the following diagram



- The average value of the walk is constant.
- In precise terms we have $\mathbb{E}X_n = \mathbb{E}X_0$.
- Indeed, the average value of the walk at some future time is always simply the current value.
- This stronger property says that, for $n \geq m$,

$$\mathbb{E}(X_n - X_m | X_0 = i_0, \ldots, X_m = i_m) = 0.$$

- The stronger property expresses that $(X_n)_{n \geq 0}$ is a martingale.

Filtration

- Let us fix for definiteness a Markov chain $(X_n)_{n\geq 0}$.
- Write $\mathcal{F}_n$ for the collection of all sets depending only on $X_0, \ldots, X_n$.
- The sequence $(\mathcal{F}_n)_{n\geq 0}$ is called the **filtration** of $(X_n)_{n\geq 0}$.
- We think of $\mathcal{F}_n$ as representing the state of knowledge, or history, of the chain up to time n.

Martingales

- A process $(M_n)_{n\geq 0}$ is called **adapted** if M_n depends only on $X_0, \ldots, X_n$.
- A process $(M_n)_{n\geq 0}$ is called **integrable** if

$$\mathbb{E}|M_n| < \infty, \quad \text{for all } n.$$

- An adapted integrable process $(M_n)_{n\geq 0}$ is called a **martingale** if, for all n and all $A \in \mathcal{F}_n$,

$$\mathbb{E}[(M_{n+1} - M_n)1_A] = 0.$$

Martingales: Second Formulation

- Note that the collection $\mathcal{F}_n$ consists of countable unions of elementary events, such as

$$\{X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n\}.$$

- It follows that the martingale property is equivalent to saying that, for all n and all $i_0, \ldots, i_n$,

$$\mathbb{E}(M_{n+1} - M_n | X_0 = i_0, \ldots, X_n = i_n) = 0.$$

Martingales: Third Formulation

- Given an integrable random variable Y, we define
$$\mathbb{E}(Y|\mathcal{F}_n) = \sum_{i_0,\ldots,i_n} \mathbb{E}(Y|X_0 = i_0, \ldots, X_n = i_n) 1_{\{X_0=i_0,\ldots,X_n=i_n\}}.$$
- The random variable $\mathbb{E}(Y|\mathcal{F}_n)$ is called the **conditional expectation** of Y given $\mathcal{F}_n$.
- In passing from Y to $\mathbb{E}(Y|\mathcal{F}_n)$, we replace, on each elementary event $A \in \mathcal{F}_n$, the random variable Y by its average value $\mathbb{E}(Y|A)$.
- It is easy to check that an adapted integrable process $(M_n)_{n\geq 0}$ is a martingale if and only if, for all n,
$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = M_n.$$

Martingales: Third Formulation (Cont'd)

- Conditional expectation is a partial averaging.
- So, if we complete the process and average the conditional expectation, we should get the full expectation
$$\mathbb{E}(\mathbb{E}(Y|\mathcal{F}_n)) = \mathbb{E}(Y).$$
- In particular, for a martingale
$$\mathbb{E}(M_n) = \mathbb{E}(\mathbb{E}(M_{n+1}|\mathcal{F}_n)) = \mathbb{E}(M_{n+1}).$$
- So, by induction,
$$\mathbb{E}(M_n) = \mathbb{E}(M_0).$$
- This was already clear on taking $A = \Omega$ in our original definition of a martingale.

Optional Stopping Theorem

- Recall that a random variable $T : \Omega \to \{0, 1, 2, \ldots\} \cup \{\infty\}$ is a **stopping time** if

$$\{T = n\} \in \mathcal{F}_n, \quad \text{for all } n < \infty.$$

- An equivalent condition is that $\{T \leq n\} \in \mathcal{F}_n$, for all $n < \infty$.
- Recall that all sorts of hitting times are stopping times.

Theorem (Optional Stopping Theorem)

Let $(M_n)_{n \geq 0}$ be a martingale and let T be a stopping time. Suppose that at least one of the following conditions holds:

(i) $T \leq n$, for some n;

(ii) $T < \infty$ and $|M_n| \leq C$ whenever $n \leq T$.

Then $\mathbb{E}M_T = \mathbb{E}M_0$.

Optional Stopping Theorem (Cont'd)

- Assume that Condition (i) holds. Then

$$\begin{aligned} M_T - M_0 &= (M_T - M_{T-1}) + \cdots + (M_1 - M_0) \\ &= \sum_{k=0}^{n-1} (M_{k+1} - M_k) 1_{k<T}. \end{aligned}$$

  Since T is a stopping time, $\{k < T\} = \{T \leq k\}^c \in \mathcal{F}_k$.

  Since $(M_k)_{k \geq 0}$ is a martingale, $\mathbb{E}[(M_{k+1} - M_k)1_{k<T}] = 0$.

  Hence,

$$\mathbb{E}M_T - \mathbb{E}M_0 = \sum_{k=0}^{n-1} \mathbb{E}[(M_{k+1} - M_k)1_{k<T}] = 0.$$

Optional Stopping Theorem (Cont'd)

- Next, suppose Condition (ii) holds.

  The preceding argument applies to the stopping time $T \wedge n$.

  So
  $$\mathbb{E}M_{T\wedge n} = \mathbb{E}M_0.$$

  Then, for all n,
  $$\begin{aligned} |\mathbb{E}M_T - \mathbb{E}M_0| &= |\mathbb{E}M_T - \mathbb{E}M_{T\wedge n}| \\ &\leq \mathbb{E}|M_T - M_{T\wedge n}| \\ &\leq 2C\mathbb{P}(T > n). \end{aligned}$$

  But $\mathbb{P}(T > n) \to 0$ as $n \to \infty$.

  So
  $$\mathbb{E}M_T = \mathbb{E}M_0.$$

Application to Simple Symmetric Random Walk

- Consider the simple symmetric random walk $(X_n)_{n\geq 0}$.
- Suppose that $X_0 = 0$ and $a, b \in \mathbb{N}$ given.
- Take
$$T = \inf\{n \geq 0 : X_n = -a \text{ or } X_n = b\}.$$
- Then:
  - T is a stopping time;
  - $T < \infty$ by recurrence of finite closed classes.
- Thus, Condition (ii) of the Optional Stopping Theorem applies with $M_n = X_n$ and $C = a \vee b$.
- We deduce that
$$\mathbb{E}X_T = \mathbb{E}X_0 = 0.$$

Application to Simple Symmetric Random Walk (Cont'd)

- Now we can compute
$$p = \mathbb{P}(X_n \text{ hits } -a \text{ before } b).$$
- We have:
  - $X_T = -a$ with probability p;
  - $X_T = b$ with probability $1 - p$.
- So
$$0 = \mathbb{E}X_T = p(-a) + (1 - p)b.$$
- Thus, $p = \frac{b}{a+b}$.
- The intuition behind the result $\mathbb{E}X_T = 0$ is very clear:
  - A gambler, playing a fair game, leaves the casino once losses reach a or winnings reach b, whichever is sooner.
  - Since the game is fair, the average gain should be zero.

Comparison with Gambler's Ruin

- We discussed previously the counter-intuitive case of a gambler who keeps on playing a fair game against an infinitely rich casino, with the certain outcome of ruin.
- This game ends at the finite stopping time

$$T = \inf\{n \geq 0 : X_n = -a\},$$

  where a is the gambler's initial fortune.
- We have $X_T = -a$.
- So $\mathbb{E}X_T \neq 0 = \mathbb{E}X_0$.
- This does not contradict the Optional Stopping Theorem because neither Condition (i) nor Condition (ii) is satisfied.
- Thus, while intuition might suggest that $\mathbb{E}X_T = \mathbb{E}X_0$ is rather obvious, some care is needed as it is not always true.

Martingales and Markov Chains

- We recall that, given a function $f : I \to \mathbb{R}$ and a Markov chain $(X_n)_{n\geq 0}$ with transition matrix P, we have
$$(P^n f)(i) = \sum_{j\in I} p_{ij}^{(n)} f_j = \mathbb{E}_i(f(X_n)).$$

Theorem

Let $(X_n)_{n\geq 0}$ be a random process with values in I and let P be a stochastic matrix. Write $(\mathcal{F}_n)_{n\geq 0}$ for the filtration of $(X_n)_{n\geq 0}$. Then the following are equivalent:

(i) $(X_n)_{n\geq 0}$ is a Markov chain with transition matrix P;

(ii) For all bounded functions $f : I \to \mathbb{R}$, the following process is a martingale:
$$M_n^f = f(X_n) - f(X_0) - \sum_{m=0}^{n-1} (P - I) f(X_m).$$

Martingales and Markov Chains (Cont'd)

- Suppose Condition (i) holds. Let f be a bounded function.
  Clearly (M_n^f) is adapted.
  We show it is also integrable.
  We have
  $$|(Pf)(i)| = \left|\sum_{j \in I} p_{ij} f_j\right| \leq \sup_j |f_j|.$$
  So
  $$|M_n^f| \leq 2(n+1) \sup_j |f_j| < \infty.$$
  This shows that M_n^f is integrable for all n.
  Let $A = \{X_0 = i_0, \ldots, X_n = i_n\}$.
  By the Markov Property,
  $$\mathbb{E}(f(X_{n+1})|A) = \mathbb{E}_{i_n}(f(X_1)) = (Pf)(i_n).$$

Martingales and Markov Chains (Cont'd)

- So we get

$$\begin{aligned}\mathbb{E}(M_{n+1}^f - M_n^f | A) &= \mathbb{E}(f(X_{n+1}) - f(X_0) - \textstyle\sum_{m=0}^{n}(P-I)f(X_m) \\ &\quad - f(X_n) + f(X_0) + \textstyle\sum_{m=0}^{n-1}(P-I)f(X_m)|A) \\ &= \mathbb{E}(f(X_{n+1}) - (P-I)f(X_n) - f(X_n)|A) \\ &= \mathbb{E}[f(X_{n+1}) - (Pf)(X_n)|A] = 0.\end{aligned}$$

Thus, $(M_n^f)_{n\geq 0}$ is a martingale.

Conversely, suppose Condition (ii) holds.

Then, for all bounded functions f,

$$\mathbb{E}[f(X_{n+1}) - (Pf)(X_n)|X_0 = i_0, \ldots, X_n = i_n] = 0.$$

Take $f = 1_{\{i_{n+1}\}}$. Then we obtain

$$\mathbb{P}(X_{n+1} = i_{n+1}|X_0 = i_0, \ldots, X_n = i_n) = p_{i_n i_{n+1}}.$$

So $(X_n)_{n\geq 0}$ is Markov with transition matrix P.

More on Markov Chains and Martingales

Theorem

Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix P. Suppose that a function $f : \mathbb{Z}_+ \times I \to \mathbb{R}$ satisfies, for all $n \geq 0$:

- $E|f(n, X_n)| < \infty$;
- $(Pf)(n+1, i) = \sum_{j\in I} p_{ij} f(n+1, j) = f(n, i)$.

Then $M_n = f(n, X_n)$ is a martingale.

- We have assumed that M_n is integrable for all n.

  Then, by the Markov Property

$$\begin{aligned}\mathbb{E}(M_{n+1} - M_n | X_0 = i_0, \ldots, X_n = i_n) \\ = \mathbb{E}_{i_n}[f(n+1, X_1) - f(n, X_0)] \\ = (Pf)(n+1, i_n) - f(n, i_n) = 0.\end{aligned}$$

  So $(M_n)_{n\geq 0}$ is a martingale.

Application to a Simple Random Walk

- Suppose $(X_n)_{n\geq 0}$ is a simple random walk on $\mathbb{Z}$, starting from 0.
- Define
$$\begin{aligned} f(i) &= i; \\ g(n,i) &= i^2 - n. \end{aligned}$$
- Now $|X_n| \leq n$ for all n.
- Thus:
  - $\mathbb{E}|f(X_n)| < \infty$;
  - $\mathbb{E}|g(n,X_n)| < \infty$.
- Also
$$\begin{aligned} (Pf)(i) &= \tfrac{i-1}{2} + \tfrac{i+1}{2} = i = f(i); \\ (Pg)(n+1,i) &= \tfrac{(i-1)^2}{2} + \tfrac{(i+1)^2}{2} - (n+1) = i^2 - n = g(n,i). \end{aligned}$$
- Hence both $X_n = f(X_n)$ and $Y_n = g(n,X_n)$ are martingales.

Application to a Simple Random Walk (Cont'd)

- Consider again, for $a, b \in \mathbb{N}$ the stopping time

$$T = \inf\{n \geq 0 : X_n = -a \text{ or } X_n = b\}.$$

- By the Optional Stopping Theorem

$$0 = \mathbb{E}(Y_0) = \mathbb{E}(Y_{T \wedge n}) = \mathbb{E}(X_{T \wedge n}^2) - \mathbb{E}(T \wedge n).$$

- Hence, $\mathbb{E}(T \wedge n) = \mathbb{E}(X_{T \wedge n}^2)$.
- Let $n \to \infty$.
  - The left side converges to $\mathbb{E}(T)$, by Monotone Convergence;
  - The right side converges to $\mathbb{E}(X_T^2)$ by Bounded Convergence.
- So we obtain

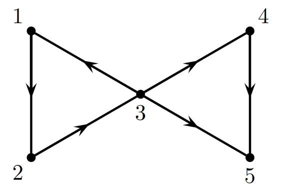
$$\mathbb{E}(T) = \mathbb{E}(X_T^2) = a^2 p + b^2(1 - p) \overset{p = \frac{b}{a+b}}{=} ab.$$

Subsection 2

Potential Theory

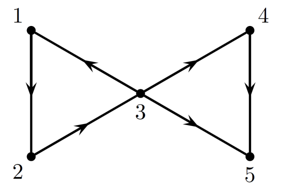
Example

- Consider the discrete-time random walk on the directed graph shown.
- At each step it chooses among the allowable transitions with equal probability.
- Suppose that on each visit to states $i = 1, 2, 3, 4$ a cost c_i is incurred, where $c_i = i$.



- What is the fair price to move from state 3 to state 4?
- We denote by ϕ_i the expected total cost starting from i.
- The fair price is always the difference in the expected total cost.

Example (Cont'd)



- Obviously, $\phi_5 = 0$.
- The effect of a single step gives:

$$\begin{aligned} \phi_1 &= 1 + \phi_2, \\ \phi_2 &= 2 + \phi_3, \\ \phi_3 &= 3 + \frac{1}{3}\phi_1 + \frac{1}{3}\phi_4, \\ \phi_4 &= 4. \end{aligned}$$

- Hence $\phi_3 = 8$.
- So the fair price to move from 3 to 4 is 4.

Example: A Variation

- Suppose our process is, instead, the continuous-time random walk $(X_t)_{t\geq 0}$ on the same directed graph.
- Assume it makes each allowable transition at rate 1.
- A cost is incurred at rate $c_i = i$ in state i for $i = 1, 2, 3, 4$.
- The total cost is now
$$\int_0^\infty c(X_s)ds.$$
- We wish to find the fair price to move from 3 to 4.

Example: A Variation (Cont'd)

- The expected cost incurred on each visit to i is given by

$$\frac{c_i}{q_i},$$

where

$$q_1 = 1, \quad q_2 = 1, \quad q_3 = 3, \quad q_4 = 1.$$

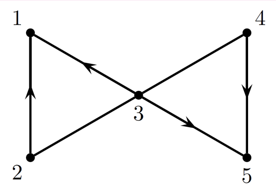
- So we see, as before:

$$\begin{array}{rcl} \phi_1 & = & 1 + \phi_2; \\ \phi_2 & = & 2 + \phi_3; \\ \phi_3 & = & \frac{3}{3} + \frac{1}{3}\phi_1 + \frac{1}{3}\phi_4; \\ \phi_4 & = & 4. \end{array}$$

- Hence $\phi_3 = 5$.
- So the fair price to move from 3 to 4 is 1.

Example: Another Variation

- We consider the discrete time random walk $(X_n)_{n\geq 0}$ on the modified graph shown.



- Where there is no arrow, transitions are allowed in both directions.
- Obviously, states 1 and 5 are absorbing.
- We impose a cost $c_i = i$ on each visit to i for $i = 2, 3, 4$.
- There is a final cost f_i on arrival at $i = 1$ or 5, where $f_i = i$.
- Thus, the total cost is now

$$\sum_{n=0}^{T-1} c(X_n) + f(X_T),$$

where T is the hitting time of $\{1, 5\}$.

Example: Another Variation (Cont'd)

- Write, as before, ϕ_i for the expected total cost starting from i.
- Then $\phi_1 = 1$ and $\phi_5 = 5$.
- Moreover:

$$\begin{aligned} \phi_2 &= 2 + \tfrac{1}{2}(\phi_1 + \phi_3); \\ \phi_3 &= 3 + \tfrac{1}{4}(\phi_1 + \phi_2 + \phi_4 + \phi_5); \\ \phi_4 &= 4 + \tfrac{1}{2}(\phi_3 + \phi_5). \end{aligned}$$

- On solving these equations we obtain

$$\phi_2 = 7, \quad \phi_3 = 9, \quad \phi_4 = 11.$$

- So in this case the fair price to move from 3 to 4 is -2.

Example

- Consider the simple discrete time random walk on $\mathbb{Z}$ with transition probabilities $p_{i,i-1} = q < p = p_{i,i+1}$.
- Let $c > 0$.
- Suppose that a cost c^i is incurred every time the walk visits state i.
- We would like to compute the expected total cost ϕ_0 incurred by the walk starting from 0.
- We must be prepared to find that $\phi_0 = \infty$ for some values of c, as the total cost is a sum over infinitely many times.
- Indeed, we know that the walk $X_n \to \infty$ with probability 1.
- So, for $c \geq 1$, we shall certainly have $\phi_0 = \infty$.

Example (Cont'd)

- Let ϕ_i denote the expected total cost starting from i.
- On moving one step to the right, all costs are multiplied by c.
- So we must have

$$\phi_{i+1} = c\phi_i.$$

- By considering what happens on the first step, we see

$$\phi_0 = 1 + p\phi_1 + q\phi_{-1} = 1 + \left(cp + \frac{q}{c}\right)\phi_0.$$

- Note that $\phi_0 = \infty$ always satisfies this equation.
- We shall see in the general theory that ϕ_0 is the minimal non-negative solution.

Example (Cont'd)

- Let us look for a finite solution.
- We obtained $\phi_0 = 1 + \left(cp + \frac{q}{c}\right)\phi_0$.
- Thus,
$$-(c^2p - c + q)\phi_0 = c.$$
- So
$$\phi_0 = \frac{c}{c - c^2p - q}.$$
- The quadratic $c^2p - c + q$ has roots at $\frac{q}{p}$ and 1, and takes negative values in between.
- Hence, the expected total cost is given by
$$\phi_0 = \begin{cases} \frac{c}{c-c^2p-q}, & \text{if } c \in (\frac{q}{p}, 1), \\ \infty, & \text{otherwise.} \end{cases}$$

The Potentials

- Let $(X_n)_{n\geq 0}$ be a discrete time chain with transition matrix P.
- Let $(X_t)_{t\geq 0}$ be a continuous time chain with generator matrix Q.
- As usual, we insist that $(X_t)_{t\geq 0}$ be minimal.
- We partition the state-space I into two disjoint sets D and ∂D.
- We call ∂D the **boundary**.

The Potentials (Cont'd)

- We suppose that we are given functions:
  - $(c_i : i \in D)$;
  - $(f_i : i \in \partial D)$.
- We denote by T the hitting time of ∂D.
- Then the associated **potential** is defined by:
  - In discrete time,

$$\phi_i = \mathbb{E}_i \left(\sum_{n<T} c(X_n) + f(X_T) 1_{T<\infty} \right);$$

  - In continuous time,

$$\phi_i = \mathbb{E}_i \left(\int_0^T c(X_t) dt + f(X_T) 1_{T<\infty} \right).$$

Positivity of Costs

- To be sure that the sums and integrals in the potential formulas are well defined, we shall assume for the most part that c and f are non-negative:
  - $c_i \geq 0$, for all $i \in D$;
  - $f_i \geq 0$, for all $i \in \partial D$.
- More generally, ϕ is the difference of the potentials associated with the positive and negative parts of c and f.
- So the positivity assumption is not too restrictive.
- In the explosive case we always set $c(\infty) = 0$.
- So no further costs are incurred after explosion.

Interpretation of Potential as Cost

- The most obvious interpretation of the potentials is in terms of cost.
- The chain wanders around in D until it hits the boundary.
  - Whilst in D, at state i say, it incurs a **cost** c_i per unit time;
  - When and if it hits the boundary, at j say, a **final cost** f_j is incurred.
- Note that we do not assume the chain will hit the boundary.
- We do not even assume that the boundary is nonempty.

Properties of Potential

Theorem

Suppose that $(c_i : i \in D)$ and $(f_i : i \in \partial D)$ are nonnegative. Set

$$\phi_i = \mathbb{E}_i \left(\sum_{n<T} c(X_n) + f(X_T) 1_{T<\infty} \right),$$

where T denotes the hitting time of ∂D. Then:

(i) The potential $\phi = (\phi_i : i \in I)$ satisfies

$$\begin{cases} \phi = P\phi + c & \text{in } D \\ \phi = f & \text{in } \partial D; \end{cases}$$

Properties of Potential (Cont'd)

Theorem (Cont'd)

(ii) If $\psi = (\psi_i : i \in I)$ satisfies

$$\begin{cases} \psi \geq P\psi + c & \text{in } D \\ \psi \geq f & \text{in } \partial D \end{cases}$$

and $\psi_i \geq 0$ for all i, then $\psi_i \geq \phi_i$ for all i;

(iii) If $\mathbb{P}_i(T < \infty) = 1$ for all i, then the system

$$\begin{cases} \phi = P\phi + c & \text{in } D \\ \phi = f & \text{in } \partial D; \end{cases}$$

has at most one bounded solution.

Properties of Potential (i)

(i) Obviously, $\phi = f$ on ∂D.

For $i \in D$, by the Markov Property

$$\begin{aligned}&\mathbb{E}_i(\sum_{1 \leq n < T} c(X_n) + f(X_T)1_{T<\infty} | X_1 = j) \\ &= \mathbb{E}_j(\sum_{n<T} c(X_n) + f(X_T)1_{T<\infty}) \\ &= \phi_j.\end{aligned}$$

So we have

$$\begin{aligned}\phi_i &= c_i + \sum_{j \in I} p_{ij} \mathbb{E}(\sum_{1 \leq n < T} c(X_n) + f(X_T)1_{T<\infty} | X_1 = j) \\ &= c_i + \sum_{j \in I} p_{ij} \phi_j.\end{aligned}$$

Properties of Potential (ii)

(ii) Consider the expected cost up to time n:

$$\phi_i(n) = \mathbb{E}_i\left(\sum_{k=0}^{n} c(X_k)1_{k<T} + f(X_T)1_{T\le n}\right).$$

By Monotone Convergence, $\phi_i(n) \nearrow \phi_i$ as $n \to \infty$.

Also, by the argument used in Part (i), we find

$$\begin{cases} \phi(n+1) = c + P\phi(n) & \text{in } D \\ \phi(n+1) = f & \text{in } \partial D. \end{cases}$$

Suppose that ψ satisfies the system in (ii) and $\psi \ge 0 = \phi(0)$.

- In D, $\psi \ge P\psi + c \ge P\phi(0) + c = \phi(1)$;
- In ∂D, $\psi \ge f = \phi(1)$.

So $\psi \ge \phi(1)$.

Similarly and by induction, $\psi \ge \phi(n)$, for all n.

Hence $\psi \ge \phi$.

Properties of Potential (iii)

(iii) Suppose ψ satisfies the system in Part (ii).

We show that, then,

$$\psi_i \geq \phi_i(n-1) + \mathbb{E}_i(\psi(X_n)1_{T \geq n}),$$

with equality if equality holds in Part (ii).

This is another proof of Part (ii).

But also, in the case of equality, if $|\psi_i| \leq M$ and $\mathbb{P}_i(T < \infty) = 1$, for all i, then, as $n \to \infty$,

$$|\mathbb{E}_i(\psi(X_n)1_{T \geq n})| \leq M\mathbb{P}_i(T \geq n) \to 0.$$

So

$$\psi = \lim_{n \to \infty} \phi(n) = \phi.$$

This proves Part (iii).

Properties of Potential ((iii) Cont'd)

- For $i \in D$, we have
$$\psi_i \geq c_i + \sum_{j \in \partial D} p_{ij} f_j + \sum_{j \in D} p_{ij} \psi_j.$$
By repeated substitution for ψ on the right
$$\begin{aligned}
\psi_i \quad \geq \quad & c_i + \textstyle\sum_{j \in \partial D} p_{ij} f_j + \sum_{j \in D} p_{ij} c_j \\
& + \cdots + \textstyle\sum_{j_1 \in D} \cdots \sum_{j_{n-1} \in D} p_{ij_1} \cdots p_{j_{n-2} j_{n-1}} c_{j_{n-1}} \\
& + \textstyle\sum_{j_1 \in D} \cdots \sum_{j_{n-1} \in D} \sum_{j_n \in \partial D} p_{ij_1} \cdots p_{j_{n-1} j_n} f_{j_n} \\
& + \textstyle\sum_{j_1 \in D} \cdots \sum_{j_n \in D} p_{ij_1} \cdots p_{j_{n-1} j_n} \psi_{j_n} \\
= \quad & \mathbb{E}_i(c(X_0) 1_{T > 0} + f(X_1) 1_{T=1} + c(X_1) 1_{T > 1} \\
& + \cdots + c(X_{n-1}) 1_{T > n-1} + f(X_n) 1_{T=n} + \psi(X_n) 1_{T > n}) \\
= \quad & \phi_i(n-1) + \mathbb{E}_i(\psi(X_n) 1_{T \geq n}).
\end{aligned}$$
Equality holds when equality holds in Part (ii).

Recasting in Terms of Martingales

- We look at the calculation we have just done in terms of martingales.
- Consider
$$M_n = \sum_{k=0}^{n-1} c(X_k)1_{k<T} + f(X_T)1_{T<n} + \psi(X_n)1_{n\leq T}.$$
- Then
$$\begin{aligned}\mathbb{E}(M_{n+1}|\mathcal{F}_n) &= \textstyle\sum_{k=0}^{n-1} c(X_k)1_{k<T} + f(X_T)1_{T<n} \\ &\quad + (P\psi + c)(X_n)1_{T>n} + f(X_n)1_{T=n} \\ &\leq M_n,\end{aligned}$$
with equality if equality holds in Part (ii).
- We note that M_n is not necessarily integrable.
- Nevertheless, it still follows that
$$\psi_i = \mathbb{E}_i(M_0) \geq \mathbb{E}_i(M_n) = \phi_i(n-1) + \mathbb{E}_i(\psi(X_n)1_{T\geq n}),$$
with equality if equality holds in Part (ii).

Restricting to States Accessible from i

- For continuous time chains there is a result analogous to the preceding theorem.
- We have to state it slightly differently because, when ϕ takes infinite values, the preceding equations may involve subtraction of infinities, and therefore not make sense.
- Although the conclusion then appears to depend on the finiteness of ϕ, which is a priori unknown, we can still use the result to determine ϕ_i in all cases.
- To do this we restrict our attention to the set of states J accessible from i.
- If the linear equations have a finite non-negative solution on J, then $(\phi_j : j \in J)$ is the minimal such solution.
- If not, then $\phi_j = \infty$, for some $j \in J$, which forces $\phi_i = \infty$, since i leads to j.

Characterization of Potential in Continuous Time

Theorem

Assume that $(X_t)_{t\geq 0}$ is minimal, and that $(c_i : i \in D)$ and $(f_i : i \in \partial D)$ are non-negative. Set

$$\phi_i = \mathbb{E}_i \left(\int_0^T c(X_t)dt + f(X_T)1_{T<\infty} \right),$$

where T is the hitting time of ∂D. Then $\phi = (\phi_i : i \in I)$, if finite, is the minimal non-negative solution to

$$\begin{cases} -Q\phi = c & \text{in } D, \\ \phi = f & \text{in } \partial D. \end{cases}$$

If $\phi_i = \infty$ for some i, then this system has no finite non-negative solution. Moreover, if $\mathbb{P}_i(T < \infty) = 1$ for all i, then the system has at most one bounded solution.

Characterization of Potential in Continuous Time (Cont'd)

- We use the following notation related to the process $(X_t)_{t\geq 0}$:
  - $(Y_n)_{n\geq 0}$ is the jump chain;
  - $S_1, S_2, \ldots$ are the holding times;
  - Π is the jump matrix.

  We use the convention $0 \times \infty = 0$.

  We then have

$$\int_0^T c(X_t)dt + f(X_T)1_{T<\infty} = \sum_{n<N} c(Y_n)S_{n+1} + f(Y_N)1_{N<\infty},$$

  where N is the first time $(Y_n)_{n\geq 0}$ hits ∂D.

  Moreover,

$$\mathbb{E}(c(Y_n)S_{n+1}|Y_n = j) = \widetilde{c}_j = \begin{cases} \frac{c_j}{q_j} & \text{if } c_j > 0, \\ 0, & \text{if } c_j = 0. \end{cases}$$

Characterization of Potential in Continuous Time (Cont'd)

- So, by Fubini's Theorem

$$\phi_i = \mathbb{E}_i \left(\sum_{n<N} \widetilde{c}(Y_n) + f(Y_N) 1_{N<\infty} \right).$$

By the preceding theorem, ϕ is therefore the minimal non-negative solution to

$$\begin{cases} \phi = \Pi\phi + \widetilde{c} & \text{in } D, \\ \phi = f & \text{in } \partial D, \end{cases}$$

which has at most one bounded solution if $\mathbb{P}_i(N < \infty) = 1$, for all i.

But the finite solutions of the last system are exactly the finite solutions of the system in the statement.

Moreover, N is finite whenever T is finite.

So this proves the result.

Potentials With Discounted Costs

- **Potentials with discounted costs** are obtained by applying to future costs a discount factor $\alpha \in (0,1)$ or rate $\lambda \in (0,\infty)$, corresponding to an interest rate.

Theorem

Suppose that $(c_i : i \in I)$ is bounded. Set

$$\phi_i = \mathbb{E}_i \sum_{n=0}^{\infty} \alpha^n c(X_n).$$

Then $\phi = (\phi_i : i \in I)$ is the unique bounded solution to

$$\phi = \alpha P \phi + c.$$

Potentials With Discounted Costs (Cont'd)

- Suppose that $|c_i| \leq C$, for all i.
  Then
  $$|\phi_i| \leq C \sum_{n=0}^{\infty} \alpha^n = \frac{C}{1-\alpha}.$$
  So ϕ is bounded.
  By the Markov Property
  $$\mathbb{E}\left(\sum_{n=1}^{\infty} \alpha^{n-1} c(X_n) | X_1 = j\right) = \mathbb{E}_j \sum_{n=0}^{\infty} \alpha^n c(X_n) = \phi_j.$$
  Then
  $$\begin{array}{rcl} \phi_i & = & \mathbb{E}_i \sum_{n=0}^{\infty} \alpha^n c(X_n) \\ & = & c_i + \alpha \sum_{j \in I} p_{ij} \mathbb{E}(\sum_{n=1}^{\infty} \alpha^{n-1} c(X_n) | X_1 = j) \\ & = & c_i + \alpha \sum_{j \in I} p_{ij} \phi_j. \end{array}$$
  So $\phi = c + \alpha P \phi$.

Potentials With Discounted Costs (Cont'd)

- Suppose, next, that ψ is bounded and
$$\psi = c + \alpha P\psi.$$
Set
$$M = \sup_i |\psi_i - \phi_i|.$$
Then $M < \infty$.
But $\psi - \phi = \alpha P(\psi - \phi)$.
So
$$|\psi_i - \phi_i| \leq \alpha \sum_{j \in I} p_{ij} |\psi_j - \phi_j| \leq \alpha M.$$
Hence, $M \leq \alpha M$.
This forces $M = 0$ and $\psi = \phi$.

Characterizations of Potentials With Discounted Costs

Theorem

Assume that $(X_t)_{t\geq 0}$ is non-explosive. Suppose that $(c_i : i \in I)$ is bounded. Set

$$\phi_i = \mathbb{E}_i \int_0^\infty e^{-\lambda t} c(X_t) dt.$$

Then $\phi = (\phi_i : i \in I)$ is the unique bounded solution to

$$(\lambda I - Q)\phi = c.$$

Characterizations of Potentials With Discounts (Cont'd)

- Assume, for now, that c is non-negative.

  Introduce a new state ∂ with $c_\partial = 0$.

  Let T be an independent $E(\lambda)$ random variable.

  Define
  $$\widetilde{X}_t = \begin{cases} X_t & \text{for } t < T \\ \partial & \text{for } t \geq T. \end{cases}$$

  Then $(\widetilde{X}_t)_{t \geq 0}$ is a Markov chain on $I \cup \{\partial\}$, with modified transition rates
  $$\widetilde{q}_i = q_i + \lambda, \quad \widetilde{q}_{i\partial} = \lambda, \quad \widetilde{q}_\partial = 0.$$

  Also T is the hitting time of ∂, and is finite with probability 1.

Characterizations of Potentials With Discounts (Cont'd)

- By Fubini's Theorem

$$\phi_i = \mathbb{E}_i \int_0^T c(\widetilde{X}_t) dt.$$

Suppose $c_i \leq C$, for all i.

Then

$$\phi_i \leq C \int_0^\infty e^{-\lambda t} dt \leq \frac{C}{\lambda}.$$

So ϕ is bounded.

Hence, by a previous theorem, ϕ is the unique bounded solution to

$$-\widetilde{Q}\phi = c.$$

This yields the same solution as the equation in the statement (with a 0 appended).

Characterizations of Potentials With Discounts (Cont'd)

- Now suppose c takes negative values.

  We can apply the preceding argument to the potentials

  $$\phi_i^{\pm} = \mathbb{E}_i \int_0^\infty e^{-\lambda t} c^{\pm}(X_t) dt,$$

  where $c_i^{\pm} = (\pm c) \vee 0$.

  Then $\phi = \phi^+ - \phi^-$.

  So ϕ is bounded.

  We have $(\lambda I - Q)\phi^{\pm} = c^{\pm}$.

  So, subtracting, we get $(\lambda I - Q)\phi = c$.

  Finally, suppose ψ is bounded and $(\lambda I - Q)\psi = c$.

  Then $(\lambda I - Q)(\psi - \phi) = 0$.

  So $\psi - \phi$ is the unique bounded solution for the case when $c = 0$, which is 0.

Potentials Without Boundary

- We consider potentials with non-negative costs c, and without boundary.
- In discrete time, the potential is defined by

$$\phi_i = \mathbb{E}_i \sum_{n=0}^{\infty} c(X_n).$$

- In continuous time, it is defined by

$$\phi_i = \mathbb{E}_i \int_0^{\infty} c(X_t) dt.$$

The Green Matrix

- In discrete time, by Fubini's Theorem, we have

$$\phi_i = \sum_{n=0}^{\infty} \mathbb{E}_i c(X_n) = \sum_{n=0}^{\infty} (P^n c)_i = (Gc)_i,$$

where $G = (g_{ij} : i, j \in I)$ is the **Green matrix**

$$G = \sum_{n=0}^{\infty} P^n.$$

- Similarly, in continuous time

$$\phi = Gc,$$

with

$$G = \int_0^{\infty} P(t) dt.$$

The Fundamental Solution

- We found that:
  - $\phi_i = (Gc)_i$, where $G = \sum_{n=0}^{\infty} P^n$, in the discrete case;
  - $\phi = Gc$, where $G = \int_0^{\infty} P(t)dt$, in the continuous case.
- Thus, once we know the Green matrix, we have explicit expressions for all potentials of the Markov chain.
- The Green matrix is also called the **fundamental solution** of the systems of the previous theorems.

The Green Matrix, Transience and Recurrence

- The j-th column $(g_{ij} : i \in I)$ is itself a potential.
- We have:
  - $g_{ij} = \mathbb{E}_i \sum_{n=0}^{\infty} 1_{X_n = j}$ in discrete time;
  - $g_{ij} = \mathbb{E}_i \int_0^{\infty} 1_{X_t = j} dt$ in continuous time.
- Thus g_{ij} is the expected total time in j starting from i.
- These quantities are related to transience and recurrence.
- We know that $g_{ij} = \infty$ if and only if i leads to j and j is recurrent.
  - In discrete time
    $$g_{ij} = \frac{h_i^j}{1 - f_j},$$
    where h_i^j is the probability of hitting j from i, and f_j is the return probability for j.
  - In continuous time,
    $$g_{ij} = \frac{h_i^j}{q_j(1 - f_j)}.$$

The Case of Discounted Costs

- For potentials with discounted costs the situation is similar.
  - In discrete time,

$$\phi_i = \mathbb{E}_i \sum_{n=0}^{\infty} \alpha^n c(X_n) = \sum_{n=0}^{\infty} \alpha^n \mathbb{E}_i c(X_n) = (R_\alpha c)_i,$$

    where

$$R_\alpha = \sum_{n=0}^{\infty} \alpha^n P^n.$$

  - In continuous time,

$$\phi_i = \mathbb{E}_i \int_0^\infty e^{-\lambda t} c(X_t) dt = \int_0^\infty e^{-\lambda t} \mathbb{E}_i c(X_t) dt = (R_\lambda c)_i,$$

    where

$$R_\lambda = \int_0^\infty e^{-\lambda t} P(t) dt.$$

Resolvents

- We found that
  - $\phi_i = (R_\alpha c)_i$, where $R_\alpha = \sum_{n=0}^{\infty} \alpha^n P^n$, in discrete time;
  - $\phi_i = (R_\lambda c)_i$, where $R_\lambda = \int_0^\infty e^{-\lambda t} P(t) dt$, in continuous time.
- We call $(R_\alpha : \alpha \in (0,1))$ and $(R_\lambda : \lambda \in (0,\infty))$ the **resolvent** of the Markov chain.
- Unlike the Green matrix the resolvent is always finite.
- For finite state space we have:
  - $R_\alpha = (I - \alpha P)^{-1}$;
  - $R_\lambda = (\lambda I - Q)^{-1}$.

Harmonic Functions

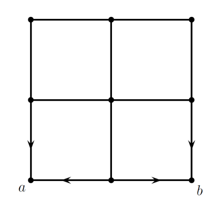
- We consider the general case, with boundary ∂D.
- Any bounded function $(\phi_i : i \in I)$ for which

$$\phi = P\phi, \quad \text{in } D,$$

  is called **harmonic** in D.

Example (Absorbing Boundary)

- Consider a random walk $(X_n)_{n\geq 0}$ on the graph shown.



- Each allowable transition is made with equal probability.
- States a and b are absorbing.
- We set $\partial D = \{a, b\}$.
- Let h_i^a denote the absorption probability for a, starting from i.
- By a method used previously we find

$$h^a = \begin{pmatrix} \frac{3}{5} & \frac{1}{2} & \frac{2}{5} \\ \frac{7}{10} & \frac{1}{2} & \frac{3}{10} \\ 1 & \frac{1}{2} & 0 \end{pmatrix},$$

where we have written the vector h^a as a matrix, corresponding in an obvious way to the state space.

Example (Absorbing Boundary Cont'd)

- The linear equations for the vector h^a read
$$\begin{cases} h^a = Ph^a, & \text{in } D \\ h^a_a = 1, h^a_b = 0. & \end{cases}$$
- Thus we can find two non-negative functions h^a and h^b, harmonic in D, but with different boundary values.
- The most general non-negative harmonic function ϕ in D satisfies $\begin{cases} \phi = P\phi & \text{in } D \\ \phi = f & \text{in } \partial D, \end{cases}$ where $f_a, f_b \geq 0$.
- This implies
$$\phi = f_a h^a + f_b h^b.$$
- Thus the boundary points a and b give us extremal generators h^a and h^b of the set of all nonnegative harmonic functions.

Example (No Boundary)

- Consider the random walk $(X_n)_{n\geq 0}$ on $\mathbb{Z}$ which:
  - Jumps towards 0 with probability q;
  - Jumps away from 0 with probability $p = 1 - q$;
  - At 0 it jumps to -1 or 1 with probability $\frac{1}{2}$.
- We choose $p > q$ so that the walk is transient.
- In fact, starting from 0, we can show that $(X_n)_{n\geq 0}$ is equally likely to end up drifting to the left or to the right, at speed $p - q$.
- Consider the problem of determining for $(X_n)_{n\geq 0}$ the set C of all non-negative harmonic functions ϕ.
- We must have:

$$\begin{array}{rcll} \phi_i & = & p\phi_{i+1} + q\phi_{i-1}, & \text{for } i = 1, 2, \ldots \\ \phi_0 & = & \frac{1}{2}\phi_1 + \frac{1}{2}\phi_{-1}, & \\ \phi_i & = & q\phi_{i+1} + p\phi_{i-1}, & \text{for } i = -1, -2, \ldots. \end{array}$$

Example (No Boundary Cont'd)

- The first equation has general solution

$$\phi_i = A + B\left(1 - \left(\frac{q}{p}\right)^i\right), \quad i = 0, 1, 2, \ldots.$$

- It is non-negative provided $A + B \geq 0$.
- Similarly, the third equation has general solution

$$\phi_i = A' + B'\left(1 - \left(\frac{q}{p}\right)^{-i}\right), \quad i = 0, -1, -2, \ldots.$$

- It is non-negative provided $A' + B' \geq 0$.
- To obtain a general harmonic function we must match the values ϕ_0 and satisfy

$$\phi_0 = \frac{\phi_1 + \phi_{-1}}{2}.$$

Example (No Boundary Cont'd)

- We found:
  - $\phi_i = A + B(1 - (\frac{q}{p})^i)$, for $i = 0, 1, 2, \ldots$;
  - $\phi_i = A' + B'(1 - (\frac{q}{p})^{-i})$, for $i = 0, -1, -2, \ldots$;
  - $\phi_0 = \frac{\phi_1 + \phi_{-1}}{2}$.
- This forces $A = A'$ and $B + B' = 0$.
- It follows that all non-negative harmonic functions have the form

$$\phi = f^- h^- + f^+ h^+,$$

where $f^-, f^+ \geq 0$, $h_i^- = h_{-i}^+$ and

$$h_i^+ = \begin{cases} \frac{1}{2} + \frac{1}{2}(1 - (\frac{q}{p})^i) & \text{for } i = 0, 1, 2, \ldots, \\ \frac{1}{2} - \frac{1}{2}(1 - (\frac{q}{p})^{-i}) & \text{for } i = -1, -2, \ldots. \end{cases}$$

Generalized Boundary and Limiting Behavior

- In the first example the generators of C were in one-to-one correspondence with the points of the boundary - the possible places for the chain to end up.
- In the last example there is no boundary, but *the generators of C still correspond to the two possibilities for the long-time behavior of the chain*.
- We have
$$h_i^+ = \mathbb{P}_i(X_n \to \infty \text{ as } n \to \infty).$$
- This suggests that the set of non-negative harmonic functions may be used to identify a generalized notion of boundary for Markov chains.
  - Sometimes it just consists of points in the state space.
  - More generally, it corresponds to the varieties of possible limiting behavior for X_n as $n \to \infty$.

The Case of Absorbing Boundary

- Consider a Markov chain $(X_n)_{n\geq 0}$ with absorbing boundary ∂D.
- Set $h_i^\partial = \mathbb{P}_i(T < \infty)$, where T is the hitting time of ∂D.
- Then by the methods used in the discrete case, we have

$$\begin{cases} h^\partial = Ph^\partial, & \text{in } D, \\ h^\partial = 1, & \text{in } \partial D. \end{cases}$$

- Note that $h_i^\partial = 1$, for all i, always gives a possible solution.
- Hence, if the system has a unique bounded solution, then

$$h_i^\partial = \mathbb{P}_i(T < \infty) = 1, \quad \text{for all } i.$$

The Case of Absorbing Boundary (Cont'd)

- Conversely, suppose

$$\mathbb{P}_i(T < \infty) = 1, \quad \text{for all } i.$$

- Then, as we showed in a previous theorem, the system has a unique bounded solution.
- Indeed, we showed more generally that this condition implies that

$$\begin{cases} \phi = P\phi + c, & \text{in } D \\ \phi = f, & \text{in } \partial D \end{cases}$$

has at most one bounded solution.

The Case of Absorbing Boundary (Cont'd)

- Recall that
$$\phi_i = \mathbb{E}_i\left(\sum_{n<T} c(X_n) + f(X_T)1_{T<\infty}\right)$$
is the minimal solution.
- Thus, any bounded solution is given by this formula.
- Suppose from now on that $\mathbb{P}_i(T < \infty) = 1$, for all i.
- Let ϕ be a bounded non-negative function, harmonic in D, with boundary values $\phi_i = f_i$, for $i \in \partial D$.
- Then, by Monotone Convergence,
$$\phi_i = \mathbb{E}_i(f(X_T)) = \sum_{j\in\partial D} f_j \mathbb{P}_i(X_T = j).$$
- Hence, every bounded harmonic function is determined by its boundary values.

The Case of Absorbing Boundary (Cont'd)

- We have

$$\phi = \sum_{j \in \partial D} f_j h^j,$$

where

$$h_i^j = \mathbb{P}_i(X_T = j).$$

- The hitting probabilities for boundary states form a set of extremal generators for the set of all bounded non-negative harmonic functions.

Subsection 3

Electrical Networks

Electrical Networks

- An electrical network has a countable set I of **nodes**.
- Each node i has a **capacity** $\pi_i > 0$.
- Some nodes are joined by **wires**.
- The wire between i and j has **conductivity** $a_{ij} = a_{ji} \geq 0$.
- When there is no wire joining i to j we take $a_{ij} = 0$.
- In practice, each "wire" contains a resistor, which determines the conductivity as the reciprocal of its resistance.

Ohm's Law

- Each node i holds a certain **charge** χ_i.
- This determines its **potential** ϕ_i by
$$\chi_i = \phi_i \pi_i.$$
- A **current** or **flow of charge** is any matrix $(\gamma_{ij} : i, j \in I)$ with
$$\gamma_{ij} = -\gamma_{ji}.$$
- Physically, the current γ_{ij} from i to j obeys **Ohm's Law**:
$$\gamma_{ij} = a_{ij}(\phi_i - \phi_j).$$
- Thus, charge flows from nodes of high to nodes of low potential.

External Connections and Equilibrium

- The first problem in electrical networks is to determine equilibrium flows and potentials, subject to given external conditions.
- The nodes are partitioned into two sets D and ∂D.
- External connections are made at the nodes in ∂D and possibly at some of the nodes in D.
- These have the effect that:
  - Each node $i \in \partial D$ is held at a given potential f_i;
  - A given current g_i enters the network at each node $i \in D$.
- If $g_i = 0$, then a node has no external connection.
- In equilibrium, current may also enter or leave through ∂D.
- Here, however, it is not the current but the potential which is determined externally.

Equilibrium Flow

- Given a flow $(\gamma_{ij} : i, j \in I)$ we shall write γ_i for the **total flow from** i **to the network**:
$$\gamma_i = \sum_{j \in I} \gamma_{ij}.$$
- In equilibrium the charge at each node is constant,
$$\gamma_i = g_i, \quad \text{for } i \in D.$$
- Therefore, by Ohm's Law, any equilibrium potential $\phi = (\phi_i : i \in I)$ must satisfy
$$\begin{cases} \sum_{j \in I} a_{ij}(\phi_i - \phi_j) = g_i, & i \in D, \\ \phi_i = f_i, & i \in \partial D. \end{cases}$$
- There is a simple correspondence between electrical networks and reversible Markov chains in continuous time, given by
$$a_{ij} = \pi_i q_{ij}, \quad i \neq j.$$

Equilibrium Potentials and Non-Uniqueness

- We assume that *the total conductivity at each node is finite*:

$$a_i = \sum_{j \neq i} a_{ij} < \infty.$$

- Then $a_i = \pi_i q_i = -\pi_i q_{ii}$.
- The capacities π_i are the components of an invariant measure.
- The symmetry of a_{ij} corresponds to the detailed balance equations.
- The equations for an equilibrium potential may now be written in a form familiar from the preceding section:

$$\begin{cases} -Q\phi = c & \text{in } D, \\ \phi = f & \text{in } \partial D, \end{cases} \quad \text{where } c_i = \frac{g_i}{\pi_i}.$$

- Note that ct and f have the same physical dimensions.
- We know that these equations may fail to have a unique solution.
- So there may be more than one equilibrium potential.

Equilibrium Potentials: Conditions for Uniqueness

- For simplification purposes, we shall assume that:
  - I is finite;
  - The network is connected;
  - ∂D is non-empty.
- This is enough to ensure uniqueness of potentials.
- Then, by a previous theorem, the equilibrium potential is given by

$$\phi_i = \mathbb{E}_i \left(\int_0^T c(X_t)dt + f(X_T) \right),$$

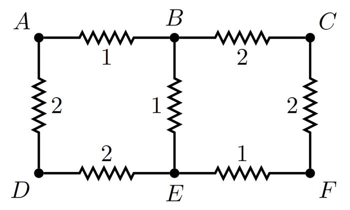
where T is the hitting time of ∂D.

Equilibrium Potentials: Empty Boundary

- The case where ∂D is empty may be reduced to the nonempty boundary case.
- A necessary condition for the existence of an equilibrium is
$$\sum_{i \in I} g_i = 0.$$
- Pick one node k.
- Set
$$\partial D = \{k\}.$$
- Replace the condition $\gamma_k = g_k$ by
$$\phi_k = 0.$$
- The new problem is equivalent to the old, but now ∂D is non-empty.

Example

- We determine the equilibrium current in the network shown.
- A unit current enters at A and leaves at F.
- The conductivities are as shown.



- We obtain the system of equations:

$$\begin{array}{rcl} \phi_A - \phi_B + 2\phi_A - 2\phi_D & = & 1 \\ \phi_B - \phi_A + 2\phi_B - 2\phi_C + \phi_B - \phi_E & = & 0 \\ 2\phi_C - 2\phi_B + 2\phi_C - 2\phi_F & = & 0 \\ 2\phi_D - 2\phi_A + 2\phi_D - 2\phi_E & = & 0 \\ \phi_E - \phi_B + 2\phi_E - 2\phi_D + \phi_E - \phi_F & = & 0 \\ 2\phi_F - 2\phi_C + \phi_F - \phi_E & = & -1 \end{array}$$

Example (Cont'd)

- They can be rewritten as:

$$\begin{aligned} 3\phi_A - \phi_B - 2\phi_D &= 1 \\ -\phi_A + 4\phi_B - 2\phi_C - \phi_E &= 0 \\ -2\phi_B + 4\phi_C - 2\phi_F &= 0 \\ -2\phi_A + 4\phi_D - 2\phi_E &= 0 \\ -\phi_B - 2\phi_D + 4\phi_E - \phi_F &= 0 \\ -2\phi_C - \phi_E + 3\phi_F &= -1 \end{aligned}$$

Setting $\phi_F = 0$, we get:

$$\begin{aligned} 3\phi_A - \phi_B - 2\phi_D &= 1 \\ -\phi_A + 3\phi_B - \phi_E &= 0 \\ -2\phi_B + 4\phi_C &= 0 \\ -2\phi_A + 4\phi_D - 2\phi_E &= 0 \\ -\phi_B - 2\phi_D + 4\phi_E &= 0 \\ \phi_F &= 0 \end{aligned}$$

Example (Cont'd)

- The last four give:

$$\begin{aligned} \phi_B &= 2\phi_C \\ \phi_A &= 2\phi_D - \phi_E \\ \phi_B &= -2\phi_D + 4\phi_E \\ \phi_F &= 0 \end{aligned}$$

- Plugging into the first two we get:

$$\begin{aligned} 6\phi_D - 7\phi_E &= 1 \\ -8\phi_D + 12\phi_E &= 0 \end{aligned}$$

- Solving the latter, we get $\phi_E = \frac{1}{2}$, $\phi_D = \frac{3}{4}$.
- Finally, $\phi_A = 1$, $\phi_B = \frac{1}{2}$ and $\phi_C = \frac{1}{4}$.

Remarks

- Note that the node capacities did not affect the problem.
- Let us arbitrarily assign to each node a capacity 1.
- Then there is an associated Markov chain.
- Let T be the hitting time of $\{A, F\}$.
- According to

$$\phi_i = \mathbb{E}_i\left(\int_0^T c(X_t)dt + f(X_T)\right),$$

  the equilibrium potential is given by

$$\phi_i = \mathbb{E}_i(1_{X_T=A}) = \mathbb{P}_i(X_T = A).$$

- Different node capacities result in different Markov chains.
- However, the jump chain and hitting probabilities remain the same.

Potentials and Flows and in terms of Markov Chains

Theorem

Consider a finite network with external connections at two nodes A and B, and the associated Markov chain $(X_t)_{t\geq 0}$.

(a) The unique equilibrium potential ϕ with $\phi_A = 1$ and $\phi_B = 0$ is given by

$$\phi_i = \mathbb{P}_i(T_A < T_B),$$

where T_A and T_B are the hitting times of A and B.

Charges in terms of Markov Chains (Cont'd)

Theorem (Cont'd)

(b) The unique equilibrium flow γ with $\gamma_A = 1$ and $\gamma_B = -1$ is given by

$$\gamma_{ij} = \mathbb{E}_A(\Gamma_{ij} - \Gamma_{ji}),$$

where Γ_{ij} is the number of times that $(X_t)_{t\geq 0}$ jumps from i to j before hitting B.

(c) The charge χ associated with γ, subject to $\chi_B = 0$, is given by

$$\chi_i = \mathbb{E}_A \int_0^{T_B} 1_{\{X_t = i\}} dt.$$

Proof of the Theorem

- The formula for ϕ is a special case of

$$\phi_i = \mathbb{E}_i \left(\int_0^T c(X_t)dt + f(X_T) \right),$$

where $c = 0$ and $f = 1_{\{A\}}$.

We prove Parts (b) and (c) together.

Suppose $X_0 = A$.

Then we have

$$\sum_{j \neq i} (\Gamma_{ij} - \Gamma_{ji}) = \begin{cases} 1, & \text{if } i = A \\ 0, & \text{if } i \notin \{A, B\}, \\ -1, & \text{if } i = B. \end{cases}$$

So, if $\gamma_{ij} = \mathbb{E}_A(\Gamma_{ij} - \Gamma_{ji})$, then γ is a unit flow from A to B.

Proof of the Theorem (Cont'd)

- We found that, if $X_0 = A$ and

$$\gamma_{ij} = \mathbb{E}_A(\Gamma_{ij} - \Gamma_{ji}),$$

then γ is a unit flow from A to B.

We have

$$\Gamma_{ij} = \sum_{n=0}^{\infty} 1_{\{Y_n = i, Y_{n+1} = j, n < N_B\}},$$

where N_B is the hitting time of B for the jump chain $(Y_n)_{n \geq 0}$.

So, by the Markov Property of the jump chain,

$$\begin{array}{rcl} \mathbb{E}_A(\Gamma_{ij}) & = & \sum_{n=0}^{\infty} \mathbb{P}_A(Y_n = i, Y_{n+1} = j, n < N_B) \\ & = & \sum_{n=0}^{\infty} \mathbb{P}_A(Y_n = i, n < N_B)\pi_{ij}. \end{array}$$

Proof of the Theorem (Cont'd)

- Set
$$\chi_i = \mathbb{E}_A \int_0^{T_B} 1_{\{X_t=i\}} dt.$$
Consider the associated potential $\psi_i = \frac{\chi_i}{\pi_i}$.
Then
$$\chi_i q_{ij} = \chi_i q_i \pi_{ij} = \sum_{n=0}^{\infty} \mathbb{P}_A(Y_n = i, n < N_B)\pi_{ij} = \mathbb{E}_A(\Gamma_{ij}).$$
So
$$(\psi_i - \psi_j)a_{ij} = \chi_i q_{ij} - \chi_j q_{ij} = \gamma_{ij}.$$
Hence $\psi = \phi$, γ is the equilibrium unit flow and χ the associated charge, as required.

Energy

- Suppose:
  - $\phi = (\phi_i : i \in I)$ is a potential;
  - $\gamma = (\gamma_{ij} : i, j \in I)$ is a flow.
- Define the following quantities:

$$E(\phi) = \frac{1}{2}\sum_{i,j\in I}(\phi_i - \phi_j)^2 a_{ij}; \quad I(\gamma) = \frac{1}{2}\sum_{i,j\in I}\gamma_{ij}^2 a_{ij}^{-1}.$$

- The $\frac{1}{2}$ signifies that each wire is counted once.

Energy and Ohm's Law

- When ϕ and γ are related by Ohm's law we have

$$\begin{aligned} E(\phi) &= \frac{1}{2}\sum_{i,j}(\phi_i - \phi_j)^2 a_{ij} \\ &= \frac{1}{2}\sum_{i,j}(\phi_i - \phi_j)\gamma_{ij} \\ &= \frac{1}{2}\sum_{i,j}\frac{\gamma_{ij}^2}{a_{ij}} \\ &= I(\gamma). \end{aligned}$$

- $E(\phi)$ is found physically to give the rate of dissipation of energy, as heat, by the network.
- We will see that certain equilibrium potentials and flows determined by Ohm's law minimize these energy functions.
- This characteristic of energy minimization can indeed replace Ohm's law as the fundamental physical principle.

Potential, Flow and Energy

Theorem

The equilibrium potential and flow may be determined as follows.

(a) The equilibrium potential $\phi = (\phi_i : i \in I)$, with boundary values $\phi_i = f_i$, for $i \in \partial D$, and no current sources in D, is the unique solution to

$$\text{minimize } E(\phi) \text{ subject to } \phi_i = f_i, \text{ for } i \in \partial D.$$

(b) The equilibrium flow $\gamma = (\gamma_{ij} : i, j \in I)$, with current sources $\gamma_i = g_i$, for $i \in D$, and boundary potential zero, is the unique solution to

$$\text{minimize } I(\gamma) \text{ subject to } \gamma_i = g_i \text{ for } i \in D.$$

Potential, Flow and Energy (Part (a))

- For any potential $\phi = (\phi_i : i \in I)$ and any flow $\gamma = (\gamma_{ij} : i, j \in I)$ we have
$$\sum_{i,j \in I} (\phi_i - \phi_j)\gamma_{ij} = 2 \sum_{i \in I} \phi_i \gamma_i.$$

(a) Denote by $\phi = (\phi_i : i \in I)$ and by $\gamma = (\gamma_{ij} : i, j \in I)$ the equilibrium potential and flow.

By hypothesis, $\gamma_i = 0$, for $i \in D$.

We can write any potential in the minimization problem in the form $\phi + \varepsilon$, where $\varepsilon = (\varepsilon_i : i \in I)$, with $\varepsilon_i = 0$, for $i \in \partial D$.

Then
$$\sum_{i,j \in I} (\varepsilon_i - \varepsilon_j)(\phi_i - \phi_j) a_{ij} = \sum_{i,j \in I} (\varepsilon_i - \varepsilon_j)\gamma_{ij} = 2 \sum_{i \in I} \varepsilon_i \gamma_i = 0.$$

So $E(\phi + \varepsilon) = E(\phi) + E(\varepsilon) \geq E(\phi)$.

Equality holds only if $\varepsilon = 0$.

Potential, Flow and Energy (Part (b))

(b) Denote by $\phi = (\phi_i : i \in I)$ and by $\gamma = (\gamma_{ij} : i, j \in I)$ the equilibrium potential and flow.

By hypothesis, $\phi_i = 0$, for $i \in \partial D$.

We can write any flow in the minimization problem in the form $\gamma + \delta$, where $\delta = (\delta_{ij} : i, j \in I)$ is a flow, with $\delta_i = 0$, for $i \in D$.

Then

$$\sum_{i,j \in I} \gamma_{ij} \delta_{ij} a_{ij}^{-1} = \sum_{i,j \in I} (\phi_i - \phi_j) \delta_{ij} = 2 \sum_{i \in I} \phi_i \delta_i = 0.$$

So

$$I(\gamma + \delta) = I(\gamma) + I(\delta) \geq I(\delta).$$

Equality holds only if $\delta = 0$.

Reformulation of Part (a)

- The following reformulation of Part (a) of the preceding result states that harmonic functions minimize energy.

Corollary

Suppose that $\phi = (\phi_i : i \in I)$ satisfies

$$\begin{cases} Q\phi = 0 & \text{in } D, \\ \phi = f & \text{in } \partial D. \end{cases}$$

Then ϕ is the unique solution to

"minimize $E(\phi)$ subject to $\phi = f$ in ∂D".

Effective Conductivities

- Let $J \subseteq I$.
- We say that $\overline{a} = (\overline{a}_{ij} : i, j \in J)$ is an **effective conductivity** on J if, for all potentials $f = (f_i : i \in J)$, the external currents into J when J is held at potential f are the same for $(J, \overline{a})$ as for (I, a).
- We know that f determines an equilibrium potential $\phi = (\phi_i : i \in I)$ by
$$\begin{cases} \sum_{j \in I} (\phi_i - \phi_j) a_{ij} = 0 & \text{for } i \notin J \\ \phi_i = f_i & \text{for } i \in J. \end{cases}$$
- Then $\overline{a}$ is an effective conductivity if, for all f, for $i \in J$ we have
$$\sum_{j \in I} (\phi_i - \phi_j) a_{ij} = \sum_{j \in J} (f_i - f_j) \overline{a}_{ij}.$$

Effective Conductivities and Energy

- For a conductivity matrix $\overline{a}$ on J, for a potential $f = (f_i : i \in J)$ and a flow $\delta = (\delta_{ij} : i, j \in J)$, we set

$$\overline{E}(f) = \frac{1}{2} \sum_{i,j \in J} (f_i - f_j)^2 \overline{a}_{ij}$$

and

$$\overline{I}(\delta) = \frac{1}{2} \sum_{i,j \in J} \delta_{ij}^2 \overline{a}_{ij}^{-1}.$$

Existence and Uniqueness of Effective Conductivity

Theorem

There is a unique effective conductivity $\overline{a}$ given by $\overline{a}_{ij} = a_{ij} + \sum_{k \notin J} a_{ik}\phi_k^j$, where for each $j \in J$, $\phi^j = (\phi_i^j : i \in I)$ is the potential defined by

$$\begin{cases} \sum_{k \in I}(\phi_i^j - \phi_k^j)a_{ik} = 0 & \text{for } i \notin J, \\ \phi_i^j = \delta_{ij} & \text{for } i \in J. \end{cases}$$

Moreover, $\overline{a}$ is characterized by the **Dirichlet variational principle**

$$\overline{E}(f) = \inf_{\phi_i = f_i \text{ on } J} E(\phi),$$

and also by the **Thompson variational principle**

$$\inf_{\delta_i = g_i \text{ on } J} \overline{I}(\delta) = \inf_{\gamma_i = \begin{cases} g_i \text{ on } J \\ 0 \text{ off } J \end{cases}} I(\gamma).$$

Proof of Existence and Uniqueness

- Let $f = (f_i : i \in J)$ be given.
  Define $\phi = (\phi_i : i \in I)$ by
  $$\phi_i = \sum_{j \in J} f_j \phi_i^j.$$
  Then we have, for $i \notin J$,
  $$\begin{aligned} \textstyle\sum_{j \in I} a_{ij}(\phi_i - \phi_j) &= \textstyle\sum_{j \in I} a_{ij}[\sum_{k \in J} f_k \phi_i^k - \sum_{\ell \in J} f_\ell \phi_j^\ell] \\ &= \textstyle\sum_{j \in I} a_{ij} \sum_{k \in J} f_k (\phi_i^k - \phi_j^k) \\ &= \textstyle\sum_{k \in J} f_k \sum_{J \in I} a_{ij}(\phi_i^k - \phi_j^k) = 0. \end{aligned}$$
  Moreover, for $i \in J$, $\phi_i = \sum_{j \in I} f_j \phi_i^j = \sum_{j \in J} f_j \delta_{ij} = f_i$.
  So ϕ is the equilibrium potential given by
  $$\begin{cases} \sum_{j \in I} a_{ij}(\phi_i - \phi_j) = 0 & \text{for } i \notin J, \\ \phi_i = f_i & \text{for } i \in J. \end{cases}$$

Proof of Existence and Uniqueness (Cont'd)

- By a previous corollary, ϕ solves

$$\text{minimize } E(\phi) \text{ subject to } \phi_i = f_i \text{ for } i \in J.$$

We have, for $i \in J$,

$$\sum_{j \in I} a_{ij}\phi_j = \sum_{j \in J} a_{ij} f_j + \sum_{k \notin J} \sum_{j \in J} a_{ik}\phi_k^j f_j = \sum_{j \in J} \overline{a}_{ij} f_j.$$

In particular, taking $f \equiv 1$ we obtain $\sum_{j \in I} a_{ij} = \sum_{j \in J} \overline{a}_{ij}$.
Hence we have equality of external currents:

$$\sum_{j \in I} (\phi_i - \phi_j) a_{ij} = \sum_{j \in J} (f_i - f_j) \overline{a}_{ij}.$$

Proof of Existence and Uniqueness (Cont'd)

- Moreover, we also have equality of energies.

$$\begin{array}{rcl}\sum_{i,j\in I}(\phi_i-\phi_j)^2a_{ij} & = & 2\sum_{i\in I}\phi_i\sum_{j\in I}(\phi_i-\phi_j)a_{ij}\\ & = & 2\sum_{i\in J}f_i\sum_{j\in J}(f_i-f_j)\overline{a}_{ij}\\ & = & \sum_{i,j\in J}(f_i-f_j)^2\overline{a}_{ij}.\end{array}$$

Finally, let $g_{ij}=(f_i-f_j)\overline{a}_{ij}$ and $\gamma_{ij}=(\phi_i-\phi_j)a_{ij}$.

$$\begin{array}{rcl}\sum_{i,j\in I}\gamma_{ij}^2a_{ij}^{-1} & = & \sum_{i,j\in I}(\phi_i-\phi_j)^2a_{ij}\\ & = & \sum_{i,j\in J}(f_i-f_j)^2\overline{a}_{ij}\\ & = & \sum_{i,j\in J}g_{ij}^2\overline{a}_{ij}^{-1}.\end{array}$$

So, by the preceding theorem, for any flow $\delta=(\delta_{ij}:i,j\in I)$ with $\delta_i=g_i$ for $i\in J$ and $\delta_i=0$ for $i\notin J$,

$$\sum_{i,j\in I}\delta_{ij}^2a_{ij}^{-1}\geq\sum_{i,j\in J}g_{ij}^2\overline{a}_{ij}^{-1}.$$

Effective Conductivity and Associated Markov Chain

- Consider again the associated Markov chain $(X_t)_{t\geq 0}$.
- Define the **time spent in** J

$$A_t = \int_0^t 1_{\{X_s \in J\}} ds.$$

- Define a time-changed process $(\overline{X}_t)_{t\geq 0}$ by

$$\overline{X}_t = X_{\tau(t)},$$

  where $\tau(t) = \inf\{s \geq 0 : A_s > t\}$.
- We obtain $(\overline{X}_t)_{t\geq 0}$ by observing $(X_t)_{t\geq 0}$ whilst in J, and stopping the clock whilst $(X_t)_{t\geq 0}$ makes excursions outside J.
- This is really a transformation of the jump chain.

Effective Conductivity and Markov Chain (Cont'd)

- By applying the strong Markov property to the jump chain we find that $(\overline{X}_t)_{t\geq 0}$ is itself a Markov chain, with jump matrix $\overline{\Pi}$ given by

$$\overline{\pi}_{ij} = \pi_{ij} + \sum_{k\notin J} \pi_{ik}\phi_k^j, \quad i,j \in J,$$

  where $\phi_k^j = \mathbb{P}_k(X_T = j)$ and T denotes the hitting time of J.
- Hence $(\overline{X}_t)_{t\geq 0}$ has Q-matrix given by

$$\overline{q}_{ij} = q_{ij} + \sum_{k\notin J} q_{ik}\phi_k^j.$$

- Since $\phi^j = (\phi_k^j : k \in I)$ is the unique solution to the system in the preceding theorem, this shows that $\pi_i \overline{q}_{ij} = \overline{a}_{ij}$.
- So $(\overline{X}_t)_{t\geq 0}$ is the Markov chain on J associated with the effective conductivity $\overline{a}$.

Subsection 4

Brownian Motion

The Idea of Brownian Motion

- Imagine a symmetric random walk in Euclidean space which takes infinitesimal jumps with infinite frequency and you will have some idea of Brownian motion.
- A discrete approximation to Euclidean space $\mathbb{R}^d$ is provided by

$$c^{-1/2}\mathbb{Z}^d = \{c^{-1/2}z : z \in \mathbb{Z}^d\},$$

  where c is a large positive number.
- The simple symmetric random walk $(X_n)_{n\geq 0}$ on $\mathbb{Z}^d$ is a Markov chain.
- We shall show that the scaled-down and speeded-up process

$$X_t^{(c)} = c^{-1/2}X_{ct}$$

  is a good approximation to Brownian motion.

The Rescaling

- We explain why space is rescaled by the square root of the time scaling.
- A desideratum is that $X_t^{(c)}$ converges, in some sense, as $c \to \infty$ to a non-degenerate limit.
- A least requirement is that $\mathbb{E}[|X_t^{(c)}|^2]$ converges to a non-degenerate limit.
- For $ct \in \mathbb{Z}^+$, we have

$$\mathbb{E}[|X_{ct}|^2] = ct\mathbb{E}[|X_1|^2].$$

- So the square root scaling gives

$$\mathbb{E}[|X_t^{(c)}|^2] = \mathbb{E}[|c^{-1/2}X_{ct}|^2] = c^{-1}\mathbb{E}[|X_{ct}|^2] = t\mathbb{E}[|X_1|^2].$$

- This is independent of c.

Gaussian Distributions

- A real-valued random variable is said to have **Gaussian distribution with mean** 0 **and variance** t if it has density function
$$\phi_t(x) = (2\pi t)^{-1/2} \exp\{-x^2/2t\} = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}.$$
- The fundamental role of Gaussian distributions in probability derives from the Central Limit Theorem.

The Central Limit Theorem

Theorem (Central Limit Theorem)

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed real-valued random variables with mean 0 and variance $t \in (0, \infty)$. Then, for all bounded continuous functions f, as $n \to \infty$ we have

$$\mathbb{E}\left[f\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}}\right)\right] \to \int_{\mathbb{R}} f(x)\phi_t(x)dx.$$

- We shall take this result and a few other standard properties of the Gaussian distribution for granted in this section.

Brownian Motion

- A real-valued process $(X_t)_{t \geq 0}$ is said to be **continuous** if

$$\mathbb{P}(\{\omega : t \mapsto X_t(\omega) \text{ is continuous}\}) = 1.$$

- A continuous real-valued process $(B_t)_{t \geq 0}$ is called a **Brownian motion** if:
  - $B_0 = 0$
  - For all $0 = t_0 < t_1 < \cdots < t_n$, the increments

$$B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}$$

    are independent Gaussian random variables of mean 0 and variance $t_1 - t_0, \ldots, t_n - t_{n-1}$.
- The conditions made on $(B_t)_{t \geq 0}$ are enough to determine all the probabilities associated with the process.
- To put it properly, the law of Brownian motion, which is a measure on the set of continuous paths, is uniquely determined.

Wiener's Theorem: Existence of Brownian Motion

Theorem (Wiener's Theorem)

Brownian motion exists.

- For $N = 0, 1, 2, \ldots$, denote by D_N the set of integer multiples of 2^{-N} in $[0, \infty)$, and denote by D the union of these sets.
  We say $(B_t : t \in D_N)$ is a **Brownian motion indexed by** D_N if:
  - $B_0 = 0$;
  - For all $0 = t_0 < t_1 < \cdots < t_n$ in D_N, the increments $B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent Gaussian random variables of mean 0 and variance $t_1 - t_0, \ldots, t_n - t_{n-1}$.

  We suppose given, for each $t \in D$, an independent Gaussian random variable Y_t of mean 0 and variance 1.
  For $t \in D_0 = \mathbb{Z}^+$, set
  $$B_t = Y_1 + Y_2 + \cdots + Y_t.$$

Wiener's Theorem (Strategy)

- Note that $(B_t : t \in D_0)$, with

$$B_t = Y_1 + Y_2 + \cdots + Y_t, \quad t \in D_0 = \mathbb{Z}^+,$$

  is a Brownian motion indexed by D_0.
- We shall show how to extend this process successively to Brownian motions $(B_t : t \in D_N)$ indexed by D_N.
- Then $(B_t : t \in D)$ is a Brownian motion indexed by D.
- $(B_t : t \in D)$ extends continuously to $t \in [0, \infty)$.
- Finally, we check that this extension is a Brownian motion.

Wiener's Theorem: Extension to D_N

- Suppose we have constructed
$$(B_t : t \in D_{N-1}),$$
a Brownian motion indexed by D_{N-1}.
For $t \in D_N \backslash D_{N-1}$, set
$$r = t - 2^{-N} \quad \text{and} \quad s = t + 2^{-N}.$$
Note that $r, s \in D_{N-1}$.
Define
$$Z_t = 2^{-(N+1)/2} Y_t, \quad B_t = \frac{1}{2}(B_r + B_s) + Z_t.$$
We obtain two new increments:
$$\begin{aligned} B_t - B_r &= \tfrac{1}{2}(B_s - B_r) + Z_t; \\ B_s - B_t &= \tfrac{1}{2}(B_s - B_r) - Z_t. \end{aligned}$$

Wiener's Theorem: Extension to D_N (Cont'd)

- We compute

$$\begin{array}{rcl} \mathbb{E}[(B_t - B_r)^2] & = & \mathbb{E}[(B_s - B_t)^2] \\ & = & \frac{1}{4}2^{-(N-1)} + 2^{-(N+1)} \\ & = & 2^{-N}; \\ \mathbb{E}[(B_t - B_r)(B_s - B_t)] & = & \frac{1}{4}2^{-(N-1)} - 2^{-(N+1)} \\ & = & 0. \end{array}$$

The two new increments, being Gaussian, are therefore independent and of the required variance.

Moreover, being constructed from $B_s - B_r$ and Y_t, they are certainly independent of increments over intervals disjoint from (r, s).

Hence, $(B_t : t \in D_N)$ is a Brownian motion indexed by D_N.

By induction, we obtain a Brownian motion $(B_t : t \in D)$.

Wiener's Theorem: Extension to $t \geq 0$ (Cont'd)

- For each N denote by
$$(B_t^{(N)})_{t\geq 0}$$
the continuous process obtained by linear interpolation from $(B_t : t \in D_N)$.
Set
$$Z_t^{(N)} = B_t^{(N)} - B_t^{(N-1)}.$$
For $t \in D_{N-1}$ we have $Z_t^{(N)} = 0$.
For $t \in D_N \backslash D_{N-1}$, by construction, we have
$$\begin{aligned} Z_t^{(N)} &= B_t - \frac{1}{2}(B_{t-2^{-N}} + B_{t+2^{-N}}) \\ &= Z_t \\ &= 2^{-(N+1)/2} Y_t, \end{aligned}$$
with Y_t Gaussian of mean 0 and variance 1.

Wiener's Theorem: Extension to $t \geq 0$ (Cont'd)

- Set
$$M_N = \sup_{t \in [0,1]} |Z_t^{(N)}|.$$
Now $(Z_t^{(N)})_{t \geq 0}$ interpolates linearly between its values on D_N.
So we obtain
$$M_N = \sup_{t \in (D_N \setminus D_{N-1}) \cap [0,1]} 2^{-(N+1)/2} |Y_t|.$$

Wiener's Theorem: Extension to $t \geq 0$ (Cont'd)

- There are 2^{N-1} points in $(D_N \backslash D_{N-1}) \cap [0,1]$.

  So, for $\lambda > 0$, we have

$$\mathbb{P}(M_N > \lambda 2^{-(N+1)/2}) \leq 2^{N-1}\mathbb{P}(|Y_1| > \lambda).$$

  For a random variable $X \geq 0$ and $p > 0$, we have the formula

$$\mathbb{E}(X^p) = \mathbb{E}\int_0^\infty 1_{\{X>\lambda\}} p\lambda^{p-1} d\lambda = \int_0^\infty p\lambda^{p-1}\mathbb{P}(X > \lambda) d\lambda.$$

  Hence,

$$\begin{array}{rcl} 2^{p(N+1)/2}\mathbb{E}(M_N^p) & = & \int_0^\infty p\lambda^{p-1}\mathbb{P}(2^{(N+1)/2}M_N > \lambda) d\lambda \\ & \leq & 2^{N-1}\int_0^\infty p\lambda^{p-1}\mathbb{P}(|Y_1| > \lambda) d\lambda \\ & = & 2^{N-1}\mathbb{E}(|Y_1|^p). \end{array}$$

Wiener's Theorem: Extension to $t \geq 0$ (Cont'd)

- Hence, for any $p > 2$,

$$\begin{aligned} \mathbb{E}\sum_{N=0}^{\infty} M_n &= \sum_{N=0}^{\infty} \mathbb{E}(M_N) \\ &\leq \sum_{N=0}^{\infty} \mathbb{E}(M_N^p)^{1/p} \\ &\leq \mathbb{E}(|Y_1|^p)^{1/p} \sum_{N=0}^{\infty} (2^{(p-2)/2p})^{-N} \\ &< \infty. \end{aligned}$$

It follows that, with probability 1, as $N \to \infty$,

$$B_t^{(N)} = B_t^{(0)} + Z_t^{(1)} + \cdots + Z_t^{(N)}$$

converges uniformly in $t \in [0,1]$.

Wiener's Theorem: Extension to $t \geq 0$ (Cont'd)

- By a similar argument with probability 1, as $N \to \infty$,

$$B_t^{(N)} = B_t^{(0)} + Z_t^{(1)} + \cdots + Z_t^{(N)}$$

converges uniformly for t in any bounded interval.

Now $B_t^{(N)}$ eventually equals B_t for any $t \in D$.

But the uniform limit of continuous functions is continuous.

So $(B_t : t \in D)$ has a continuous extension $(B_t)_{t \geq 0}$, as claimed.

Wiener's Theorem: Extension to $t \geq 0$ (Cont'd)

- It remains to show that the increments of $(B_t)_{t \geq 0}$ have the required joint distribution.

  Consider given $0 < t_1 < \cdots < t_n$.

  We can find sequences $(t_k^m)_{m \in \mathbb{N}}$ in D such that:
  - $0 < t_1^m < \cdots < t_n^m$, for all m;
  - $t_k^m \to t_k$, for all k.

  Set $t_0 = t_0^m = 0$.

  We know that the increments

$$B_{t_1^m} - B_{t_0^m}, \ldots, B_{t_n^m} - B_{t_{n-1}^m}$$

  are Gaussian of mean 0 and variance $t_1^m - t_0^m, \ldots, t_n^m - t_{n-1}^m$.

  Hence, using continuity of $(B_t)_{t \geq 0}$, we can let $m \to \infty$ to obtain the desired distribution for the increments $B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}$.

Brownian Motion as a Scaling Limit of Random Walks

Theorem

Let $(X_n)_{n\geq 0}$ be a discrete time, real valued random walk with steps of mean 0 and variance $\sigma^2 \in (0, \infty)$. For $c > 0$ consider the rescaled process

$$X_t^{(c)} = c^{-1/2} X_{ct},$$

where the value of X_{ct}, when ct is not an integer, is found by linear interpolation. Then, for all m, for all bounded continuous functions $f : \mathbb{R}^m \to \mathbb{R}$ and all $0 \leq t_1 < \cdots < t_m$, we have

$$\mathbb{E}[f(X_{t_1}^{(c)}, \ldots, X_{t_m}^{(c)})] \to \mathbb{E}[f(\sigma B_{t_1}, \ldots, \sigma B_{t_m})],$$

as $c \to \infty$, where $(B_t)_{t\geq 0}$ is a Brownian motion.

Brownian Motion and Random Walks (Cont'd)

- The claim is that, as $c \to \infty$, $(X_{t_1}^{(c)}, \ldots, X_{t_m}^{(c)})$ converges weakly to $(\sigma B_{t_1}, \ldots, \sigma B_{t_m})$.

  We take for granted some basic properties of weak convergence.

  First define $\widetilde{X}_t^{(c)} = c^{-1/2} X_{[ct]}$, with $[ct]$ the integer part of ct.

  Then

  $$|(X_{t_1}^{(c)}, \ldots, X_{t_m}^{(c)}) - (\widetilde{X}_{t_1}^{(c)}, \ldots, \widetilde{X}_{t_m}^{(c)})| \leq c^{-1/2} |(Y_{[ct_1]+1}, \ldots, Y_{[ct_n]+1})|,$$

  where Y_n denotes the n-th step of $(X_n)_{n \geq 0}$.

  The right side converges weakly to 0.

  So it suffices to prove the claim with $\widetilde{X}_t^{(c)}$ replacing $X_t^{(c)}$.

  Consider the increments

  $$U_k^{(c)} = \widetilde{X}_{tk}^{(c)} - \widetilde{X}_{t_{k-1}}^{(c)}, \ Z_k = \sigma(B_{t_k} - B_{t_{k-1}}), \ k = 1, \ldots, m.$$

  We have $\widetilde{X}_0^{(c)} = B_0 = 0$. So it suffices to show that $(U_1^{(c)}, \ldots, U_m^{(c)})$ converges weakly to $(Z_1, \ldots, Z_m)$.

Brownian Motion and Random Walks (Cont'd)

- But both sets of increments are independent.

  So it suffices to show that $U_k^{(c)}$ converges weakly to Z_k, for each k.

  Now, with $N_k(c) = [ct_k] - [ct_{k-1}]$, we have

$$\begin{array}{rcl} U_k^{(c)} & = & c^{-1/2} \sum_{n=[ct_{k-1}]+1}^{[ct_k]} Y_n \\ & \sim & (c^{-1/2} N_k(c)^{1/2}) N_k(c)^{-1/2} (Y_1 + \cdots + Y_{N(c)}). \end{array}$$

  By the Central Limit Theorem, we have:
  - $N_k(c)^{-1/2}(Y_1 + \cdots + Y_{N(c)})$ converges weakly to $(t_k - t_{k-1})^{-1/2} Z_k$,;
  - $(c^{-1/2} N_k(c)^{1/2}) \to (t_k - t_{k-1})^{1/2}$.

  Hence, we obtain

$$\begin{array}{rcl} U_k^{(c)} & \sim & (c^{-1/2} N_k(c)^{1/2}) N_k(c)^{-1/2} (Y_1 + \cdots + Y_{N(c)}) \\ & \stackrel{w}{\to} & ((t_k - t_{k-1})^{-1/2} Z_k)((t_k - t_{k-1})^{1/2}) \\ & = & Z_k. \end{array}$$

Brownian Motion in $\mathbb{R}^d$

- Let $(B_t^1)_{t\geq 0},\ldots,(B_t^d)_{t\geq 0}$ be d independent Brownian motions
- Consider the $\mathbb{R}^d$-valued process

$$B_t=(B_t^1,\ldots,B_t^d).$$

- We call $(B_t)_{t\geq 0}$ a **Brownian motion in $\mathbb{R}^d$**.
- There is a multidimensional version of the Central Limit Theorem which leads to a multidimensional version of the preceding theorem.
- Thus, if $(X_n)_{n\geq 0}$ is a random walk in $\mathbb{R}^d$, with steps of mean 0 and covariance matrix $V=\mathbb{E}(X_1X_1^T)$, and if V is finite, then for all bounded continuous functions $f:(\mathbb{R}^d)^m\to\mathbb{R}$, as $c\to\infty$, we have

$$\mathbb{E}[f(X_{t_1}^{(c)},\ldots,X_{t_m}^{(c)})]\to\mathbb{E}[f(\sqrt{V}B_{t_1},\ldots,\sqrt{V}B_{t_m})].$$

Scaling Invariance

- Brownian motion $(B_t)_{t\geq 0}$ satisfies the following scaling invariance property, which can checked from the definition.
- For any $c > 0$, the process $(B_t^{(c)})_{t\geq 0}$ defined by

$$B_t^{(c)} = c^{-1/2} B_{ct}$$

  is a Brownian motion.
- Thus Brownian motion appears as a fixed point of the scaling transformation.
- The scaling transformation attracts all other finite variance symmetric random walks as $c \to \infty$.

Transition Density in Brownian Motion

- **Brownian motion starting from** x is any process $(B_t)_{t\geq 0}$ such that:
  - $B_0 = x$;
  - $(B_t - B_0)_{t\geq 0}$ is a Brownian motion (starting from 0).
- In looking in Brownian motion for the structure of a Markov process we look for:
  - A **transition semigroup** $(P_t)_{t\geq 0}$;
  - A **generator** G.
- For any bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}$ we have

$$\begin{array}{rcl} \mathbb{E}_x[f(B_t)] & = & \mathbb{E}_0[f(x+B_t)] \\ & = & \int_{\mathbb{R}^d} f(x+y)\phi_t(y_1)\cdots\phi_t(y_d)dy_1\cdots dy_d \\ & = & \int_{\mathbb{R}^d} p(t,x,y)f(y)dy, \end{array}$$

  where $p(t,x,y) = (2\pi t)^{-d/2}\exp\{-|y-x|^2/2t\}$.
- This is the **transition density** for Brownian motion.

Transition Semigroup in Brownian Motion

- The **transition semigroup** is given by

$$(P_t f)(x) = \int_{\mathbb{R}^d} p(t,x,y) f(y) dy = \mathbb{E}_x[f(B_t)].$$

To check the semigroup property $P_s P_t = P_{s+t}$, note that

$$\begin{array}{rcl} \mathbb{E}_x[f(B_{s+t})] & = & \mathbb{E}_x[f(B_s + (B_{s+t} - B_s))] \\ & = & \mathbb{E}_x[P_t f(B_s)] \\ & = & (P_s P_t f)(x). \end{array}$$

Here, we first took the expectation over the independent increment $B_{s+t} - B_s$.

Generator in Brownian Motion

- For $t > 0$ it is easy to check that

$$\frac{\partial}{\partial t} p(t, x, y) = \frac{1}{2} \Delta_x p(t, x, y),$$

where $\Delta_x = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$.

Hence, if f has two bounded derivatives, we have

$$\begin{array}{rcl} \frac{\partial}{\partial t}(P_t f)(x) & = & \int_{\mathbb{R}^d} \frac{1}{2} \Delta_x p(t, x, y) f(y) dy \\ & = & \int_{\mathbb{R}^d} \frac{1}{2} \Delta_y p(t, x, y) f(y) dy \\ & = & \int_{\mathbb{R}^d} p(t, x, y)(\frac{1}{2} \Delta f)(y) dy \\ & = & \mathbb{E}_x[(\frac{1}{2} \Delta f)(B_t)] \overset{t \searrow 0}{\longrightarrow} \frac{1}{2} \Delta f(x). \end{array}$$

By analogy with continuous-time chains, the generator, a term we have not defined precisely, should be given by $G = \frac{1}{2} \Delta$.

Comparison with Markov Chains and the Laplacian

- Where formerly we considered vectors $(f_i : i \in I)$, now there are functions $f : \mathbb{R}^d \to \mathbb{R}$, required to have various degrees of local regularity, such as measurability and differentiability.
- Where formerly we considered matrices P_t and Q, now we have linear operators on functions:
  - P_t is an integral operator;
  - G is a differential operator.
- We explain the appearance of the Laplacian Δ by reference to the random walk approximation.
- Denote by $(X_n)_{n \geq 0}$ the simple symmetric random walk in $\mathbb{Z}^d$.
- Consider, for $N = 1, 2, \ldots$, the rescaled process

$$X_t^{(N)} = N^{-1/2} X_{N_t}, \quad t = 0, \frac{1}{N}, \frac{2}{N}, \ldots.$$

The Laplacian (Cont'd)

- For a bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$, set
$$(P_t^{(N)} f)(x) = \mathbb{E}_x[f(X_t^{(N)})], \quad x \in N^{-1/2}\mathbb{Z}^d.$$
- The closest thing to a derivative in t at 0, for $(P_t^{(N)})_{t=0,\frac{1}{N},\frac{2}{N},\ldots}$, is
$$\begin{aligned} N(P_{1/N}^{(N)} f - f)(x) &= N\mathbb{E}_x[f(X_{1/N}^{(N)}) - f(X_0^{(N)})] \\ &= N\mathbb{E}_{N^{1/2}x}[f(N^{-1/2}X_1) - f(N^{-1/2}X_0)] \\ &= \tfrac{N}{2}\{f(x - N^{-1/2}) - 2f(x) + f(x + N^{-1/2})\}. \end{aligned}$$
- Assume that f has two bounded derivatives.
- By Taylor's Theorem, as $N \to \infty$,
$$f(x - N^{-1/2}) - 2f(x) + f(x + N^{-1/2}) = N^{-1}(\Delta f(x) + o(N)).$$
- So $N(P_{1/N}^{(N)} f - f)(x) \to \frac{1}{2}\Delta f(x)$.