Introduction to Measure Theory

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Subsection 1

Set Inclusion

Sets, Membership and Subsets

- The word **set** will mean a subset of a given set X, called a **space** or the **whole** or **entire space** or the **universe** under consideration.
- The elements of X are called **points**.
- If x is a point of X and E is a subset of X, the notation $x \in E$ means that x belongs to E, i.e., that one of the points of E is x.
- The negated statement that "x does not belong to E" will be denoted by $x \notin E$.

  Example:
  - For every point x of X, we have $x \in X$;
  - For no point x of X do we have $x \notin X$.
- If E and F are subsets of X, the notation $E \subseteq F$ or $F \supseteq E$ means that E is a **subset** of F, i.e., that every point of E belongs to F.

  In particular, we have $E \subseteq E$ for every set E.

Equality and the Empty Set

- Two sets E and F are called **equal** if and only if they contain exactly the same points. Equivalently, if and only if $E \subseteq F$ and $F \subseteq E$.
- As a consequence:

  The only way to prove that two sets are equal is to show, in two steps, that every point of either set belongs also to the other.

- We admit into the class of sets a set containing no points, called the **empty set** and denoted by $\emptyset$.
- Note that:
  - For every set E, we have $\emptyset \subseteq E \subseteq X$.
  - For every point x, we have $x \notin \emptyset$.

Classes

- Besides sets of points, we also consider sets of sets.

  Example:

  Let X be the real line.

  Then an interval is a set, i.e., a subset of X.

  The set of all intervals is a set of sets.
- To enhance clarity, we always use the word **class** for a set of sets.
- The same notations and terminology are used for classes as for sets.

  Example:

  If E is a set and $\boldsymbol{E}$ is a class of sets, then $E \in \boldsymbol{E}$ means that the set E belongs to (is a member of, is an element of) the class $\boldsymbol{E}$.

  If $\boldsymbol{E}$ and $\boldsymbol{F}$ are classes, then $\boldsymbol{E} \subseteq \boldsymbol{F}$ means that every set of $\boldsymbol{E}$ belongs also to $\boldsymbol{F}$, i.e., that $\boldsymbol{E}$ is a subclass of $\boldsymbol{F}$.

Collections

- On rare occasions we shall also consider sets of classes.
- We use the word **collection** for sets of classes.

  Example:

  Let X be the Euclidean plane.

  Let $\boldsymbol{E}_y$ be the class of all intervals on the horizontal line at distance y from the origin.

  Each $\boldsymbol{E}_y$ is a class.

  The set of all these classes is a collection.

Some Properties

(1) The relation $\subseteq$ between sets (subsets of X) is always reflexive and transitive. It is symmetric if and only if X is empty.

(2) Let $\boldsymbol{X}$ be the class of all subsets of X (including, of course, $\emptyset$ and X).
Let x be a point of X.
Let E be a subset of X (a member of $\boldsymbol{X}$).
Let $\boldsymbol{E}$ be a class of subsets of X (a subclass of $\boldsymbol{X}$).
If u and v vary independently over the five symbols $x, E, X, \boldsymbol{E}, \boldsymbol{X}$, then some of the fifty relations of the forms

$$u \in v \quad \text{or} \quad u \subseteq v$$

are necessarily true, some are possibly true, some are necessarily false, and some are meaningless.

- $u \in v$ is meaningless unless the right term is a subset of a space of which the left term is a point;
- $u \subseteq v$ is meaningless unless u and v are both subsets of the same space.

Subsection 2

Unions and Intersections

Unions

- Let $\boldsymbol{E}$ be any class of subsets of X.
- The **union** of the sets of $\boldsymbol{E}$ is the set of all those points of X which belong to at least one set of the class $\boldsymbol{E}$.
- The union of $\boldsymbol{E}$ is denoted by

$$\bigcup \boldsymbol{E} \quad \text{or} \quad \bigcup\{E : E \in \boldsymbol{E}\}.$$

Set-Building Using Properties

- Suppose we are given any set of objects, generically denoted by x.
- Assume, for each x, $\pi(x)$ is a proposition concerning x.
  Then the symbol
  $$\{x : \pi(x)\}$$
  denotes the set of those points x for which $\pi(x)$ is true.
- Suppose $\{\pi_n(x)\}$ is a sequence of propositions concerning x.
  The set of those points x for which $\pi_n(x)$ is true, for every n, is
  $$\{x : \pi_1(x), \pi_2(x), \ldots\}.$$
- Suppose, to every element γ of a certain index set Γ there corresponds a proposition $\pi_\gamma(x)$ concerning x.
  Then the set of all those points x for which the proposition $\pi_\gamma(x)$ is true, for every γ in Γ, is denoted by
  $$\{x : \pi_\gamma(x), \gamma \in \Gamma\}.$$

Examples of Set-Builder Notation

- We have $\{x : x \in E\} = E$ and $\{E : E \in \boldsymbol{E}\} = \boldsymbol{E}$.
- Consider also the following sets:
  - $\{t : 0 \leq t \leq 1\}$ (the closed unit interval);
  - $\{(x, y) : x^2 + y^2 = 1\}$ (the circumference of the unit circle in the plane);
  - $\{n^2 : n = 1, 2, \ldots\}$ (the set of those positive integers which are squares).
- In accordance with the preceding notation, the upper and lower bounds (supremum and infimum) of a set E of real numbers are denoted by

$$\sup\{x : x \in E\} \quad \text{and} \quad \inf\{x : x \in E\}.$$

respectively.

Pairs and Singletons

- The brace $\{\dots\}$ notation will be reserved for the formation of sets.

  Example: If x and y are points, then $\{x, y\}$ denotes the set whose only elements are x and y.
- It is important logically to distinguish between:
  - The point x and the set $\{x\}$ whose only element is x.
  - The set E and the class $\{E\}$ whose only element is E.

  Example:
  - The empty set $\emptyset$ contains no points;
  - The class $\{\emptyset\}$ contains exactly one set, the empty set.

Unions of Special Classes of Sets

- For the union of special classes of sets special notations are used:
  - If $\boldsymbol{E} = \{E_1, E_2\}$, then $\bigcup \boldsymbol{E} = \bigcup\{E_i : i = 1, 2\}$ is denoted by $E_1 \cup E_2$.
  - If $\boldsymbol{E} = \{E_1, \ldots, E_n\}$ is a finite class of sets, then $\bigcup \boldsymbol{E} = E_1 \cup \cdots \cup E_n$ or $\bigcup_{i=1}^n E_i$.
  - If $\{E_n\}$ is an infinite sequence of sets, then the union of the terms of this sequence is denoted by

    $$E_1 \cup E_2 \cup \cdots \quad \text{or} \quad \bigcup_{i=1}^{\infty} E_i.$$

  - If, to every element γ of an index set Γ there corresponds a set E_γ, then the union of the class of sets $\{E_\gamma : \gamma \in \Gamma\}$ is denoted by $\bigcup_{\gamma \in \Gamma} E_\gamma$ or $\bigcup_\gamma E_\gamma$.

    If the index set Γ is empty, we adopt the convention $\bigcup_\gamma E_\gamma = \emptyset$.
- The relations of the empty set $\emptyset$ and the whole space X to the formation of unions are given by: $E \cup \emptyset = E$ and $E \cup X = X$.
- More generally, we have $E \subseteq F$ if and only if $E \cup F = F$.

Intersection

- Let $\boldsymbol{E}$ be any class of subsets of X.
- The **intersection** of the sets of $\boldsymbol{E}$ is the set of all those points of X which belong to every set of $\boldsymbol{E}$.
- It is denoted by
$$\bigcap \boldsymbol{E} \quad \text{or} \quad \bigcap\{E : E \in \boldsymbol{E}\}.$$
- For the intersection of special classes of sets special notations are used:
  - If $\boldsymbol{E} = \{E_1, E_2\}$, then $\bigcap \boldsymbol{E} = \bigcap\{E_i : i = 1, 2\}$ is denoted by $E_1 \cap E_2$.
  - If $\boldsymbol{E} = \{E_1, \ldots, E_n\}$ is a finite class of sets, then $\bigcap \boldsymbol{E} = E_1 \cap \cdots \cap E_n$ or $\bigcap_{i=1}^n E_i$.
  - If $\{E_n\}$ is an infinite sequence of sets, then the intersection of the terms of this sequence is denoted by $E_1 \cap E_2 \cap \cdots$ or $\bigcap_{i=1}^\infty E_i$.
  - If, to every element γ of an index set Γ there corresponds a set E_γ, then the intersection of the class $\{E_\gamma : \gamma \in \Gamma\}$ is denoted by $\bigcap_{\gamma \in \Gamma} E_\gamma$ or $\bigcap_\gamma E_\gamma$.

Intersection of the Empty Class

- If the index set Γ is empty, we adopt the convention $\bigcap_{\gamma\in\Gamma} E_\gamma = X$.
- There are several heuristic motivations for this convention:
  - Suppose Γ_1 and Γ_2 are two (non empty) index sets for which $\Gamma_1 \subseteq \Gamma_2$. Then clearly $\bigcap_{\gamma\in\Gamma_1} E_\gamma \supseteq \bigcap_{\gamma\in\Gamma_2} E_\gamma$.
    Therefore to the smallest possible Γ, i.e., the empty, we should make correspond the largest possible intersection.
  - Consider the identity

    $$\bigcap_{\gamma\in\Gamma_1\cup\Gamma_2} E_\gamma = \bigcap_{\gamma\in\Gamma_1} E_\gamma \cap \bigcap_{\gamma\in\Gamma_2} E_\gamma,$$

    valid for all non empty index sets Γ_1 and Γ_2.
    If we want it valid for arbitrary Γ_1 and Γ_2, then we must have, for every Γ,

    $$\bigcap_{\gamma\in\Gamma} E_\gamma = \bigcap_{\gamma\in\Gamma\cup\emptyset} E_\gamma = \bigcap_{\gamma\in\Gamma} E_\gamma \cap \bigcap_{\gamma\in\emptyset} E_\gamma.$$

    Setting $E_\gamma = X$, γ in Γ, we get $\bigcap_{\gamma\in\emptyset} E_\gamma = X$.

Properties of Intersection and Disjoint Sets

- Union and intersection are sometimes called **join** and **meet**, respectively.
- The relations of $\emptyset$ and X to the formation of intersections are given by the identities

$$E \cap \emptyset = \emptyset \quad \text{and} \quad E \cap X = E.$$

- More generally, we have $E \subseteq F$ it and only if $E \cap F = E$.
- Two sets E and F are called **disjoint** if they have no points in common, i.e. if $E \cap F = \emptyset$.
- A **disjoint class** is a class $\boldsymbol{E}$ of sets, such that every two distinct sets of $\boldsymbol{E}$ are disjoint.
- If $\boldsymbol{E}$ is a disjoint class, the union of the sets of $\boldsymbol{E}$ is referred to as a **disjoint union**.

Characteristic Functions

- If E is any subset of X, the function χ_E, defined, for all x in X, by the relations
$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$$
is called the **characteristic function** of the set E.
- The correspondence between sets and their characteristic functions is one to one.
- Moreover, all properties of sets and set operations may be expressed by means of characteristic functions.
- Example: Note that $E = \{x : \chi_E(x) = 1\}$.

Additional Properties of Set Theoretic Operations

(1) The formation of unions and intersections is commutative and associative, i.e.,

$$E \cup F = F \cup E, \quad E \cup (F \cup G) = (E \cup F) \cup G;$$
$$E \cap F = F \cap E, \quad E \cap (F \cap G) = (E \cap F) \cap G.$$

(2) Each of the two operations, the formation of unions and the formation of intersections, is distributive with respect to the other, i.e.,

$$E \cap (F \cup G) = (E \cap F) \cup (E \cap G);$$
$$E \cup (F \cap G) = (E \cup F) \cap (E \cup G).$$

- More generally, we have:

$$F \cap \bigcup \{E : E \in \boldsymbol{E}\} = \bigcup \{F \cap E : E \in \boldsymbol{E}\};$$
$$F \cup \bigcap \{E : E \in \boldsymbol{E}\} = \bigcap \{F \cup E : E \in \boldsymbol{E}\}.$$

Additional Properties of Set Theoretic Operations (Cont'd)

(3) Does the class of all subsets of X form a group with respect to either of the operations $\cup$ and $\cap$?

(4) Note that:

- $\chi_\emptyset(x) \equiv 0$;
- $\chi_X(x) \equiv 1$;
- The relation $\chi_E(x) \leq \chi_F(x)$ is valid, for all x in X, if and only if $E \subseteq F$;
- If $E \cap F = A$ and $E \cup F = B$, then

$$\begin{aligned} \chi_A &= \chi_E \chi_F; \\ \chi_B &= \chi_E + \chi_F - \chi_A \\ &= \chi_E + \chi_F - \chi_E \chi_F. \end{aligned}$$

Subsection 3

Limits, Complements and Differences

Limits

- Let $\{E_n\}$ be a sequence of subsets of X.
- The set E^* of all those points of X which belong to E_n for infinitely many values of n is called the **superior limit** of the sequence.

  It is denoted by
  $$E^* = \limsup_n E_n.$$
- The set E_* of all those points of X which belong to E_n for all but a finite number of values of n is called the **inferior limit** of the sequence.

  It is denoted by
  $$E_* = \liminf_n E_n.$$
- If it so happens that $E^* = E_*$, we write $\lim_n E_n$ to denote this set.

Monotone Sequences

- Let $\{E_n\}$ is a sequence of subsets of X.
- If the sequence is such that:
  - $E_n \subseteq E_{n+1}$, for $n = 1, 2, \ldots$, it is called **increasing**;
  - $E_n \supseteq E_{n+1}$, for $n = 1, 2, \ldots$, it is called **decreasing**.

  Both increasing and decreasing sequences will be referred to as **monotone**.
- It is easy to verify that if $\{E_n\}$ is a monotone sequence, then $\lim_n E_n$ exists and is equal to:
  - $\bigcup_n E_n$, if the sequence is increasing;
  - $\bigcap_n E_n$ if the sequence is decreasing.

Complementation

- The **complement** of a subset E of X is the set of all those points of X which do not belong to E.
  It is denoted by E'.
- The operation of forming complements satisfies the following algebraic identities:

$$E \cap E' = \emptyset, \quad E \cup E' = X;$$
$$\emptyset' = X; \quad (E')' = E; \quad X' = \emptyset;$$
$$\text{if } E \subseteq F, \text{ then } E' \supseteq F'.$$

- We also have the **De Morgan Laws**:
  - The complement of the union is the intersection of the complements:

$$(\bigcup\{E : E \in \boldsymbol{E}\})' = \bigcap\{E' : E \in \boldsymbol{E}\};$$

  - The complement of the intersection is the union of the complements:

$$(\bigcap\{E : E \in \boldsymbol{E}\})' = \bigcup\{E' : E \in \boldsymbol{E}\}.$$

The Duality Principle

Duality Principle

Any valid identity among sets, obtained by forming unions, intersections, and complements, remains valid if the symbols

$$\bigcap, \quad \subseteq, \quad \emptyset$$

are interchanged with

$$\bigcup, \quad \supseteq, \quad X,$$

respectively (equality and complementation unchanged).

Difference and Symmetric Difference

- If E and F are subsets of X, the **difference** between E and F, in symbols $E - F$, is the set of all those points of E which do not belong to F.
- Note that:
  - $X - F = F'$;
  - $E - F = E \cap F'$.

  So $E - F$ is also called the **relative complement** of F in E.
- The operation of forming differences, similarly to the operation of forming complements, interchanges $\bigcup$ with $\bigcap$ and $\subseteq$ with $\supseteq$.
  Example: $E - (F \cup G) = (E - F) \cap (E - G)$.
- The difference $E - F$ is called **proper** if $E \supseteq F$.
- The **symmetric difference** of two sets E and F, denoted by $E \triangle F$, is defined by

$$E \triangle F = (E - F) \cup (F - E) = (E \cap F') \cup (E' \cap F).$$

Subsection 4

Rings and Algebras

Boolean Rings of Sets

- A **ring**, or **Boolean ring**, of sets is a non empty class $\boldsymbol{R}$ of sets, such that

$$\text{if } E \in \boldsymbol{R} \text{ and } F \in \boldsymbol{R}, \text{ then } E \cup F \in \boldsymbol{R} \text{ and } E - F \in \boldsymbol{R}.$$

- In other words a ring is a non empty class of sets which is closed under the formation of unions and differences.
- The empty set belongs to every ring $\boldsymbol{R}$.

  Suppose $\boldsymbol{R}$ is a ring of sets. By definition, it is nonempty. Let $E \in \boldsymbol{R}$. But $\boldsymbol{R}$ is closed under difference. Hence, $\emptyset = E - E \in \boldsymbol{R}$.
- A non empty class of sets closed under the formation of unions and proper differences is a ring.

  Let $\boldsymbol{R}$ be a nonempty class closed under unions and proper differences.

  It suffices to show that it is closed under arbitrary differences.

  Let $E, F \in \boldsymbol{R}$. Then $E - F = (E \cup F) - F \in \boldsymbol{R}$.

Boolean Rings of Sets (Cont'd)

- A ring is closed under the formation of symmetric differences and intersections:

  Let $\boldsymbol{R}$ be a ring and $E, F \in \boldsymbol{R}$.

  Then, $E \triangle F = (E - F) \cup (F - E) \in \boldsymbol{R}$.

  Moreover, $E \cap F = (E \cup F) - (E \triangle F) \in \boldsymbol{R}$.

- If $\boldsymbol{R}$ is a ring and $E_i \in \boldsymbol{R}$, $i = 1, \ldots, n$, then

$$\bigcup_{i=1}^{n} E_i \in \boldsymbol{R} \quad \text{and} \quad \bigcap_{i=1}^{n} E_i \in \boldsymbol{R}.$$

  We use mathematical induction together with the associative laws of union and intersection. E.g., in the case of union, we have:

  - For $n = 1$, $\bigcup_{i=1}^{1} E_i = E_1 \in \boldsymbol{R}$.
  - Assume $\bigcup_{i=1}^{n-1} E_i \in \boldsymbol{R}$.
  - Then $\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n-1} E_i \cup E_n \in \boldsymbol{R}$.

Examples of Boolean Rings

- Two extreme but useful examples of rings are:
  - The class $\{\emptyset\}$ containing the empty set only;
  - The class of all subsets of X.
- For an arbitrary set X, the class of all finite sets in X form a ring.
- Let $X = \{x : -\infty < x < +\infty\}$ be the real line.

  Let $\boldsymbol{R}$ be the class of all finite unions of bounded, left closed, and right open intervals, i.e. the class of all sets of the form

$$\bigcup_{i=1}^{n}\{x : -\infty < a_i \leq x < b_i < +\infty\}.$$

  Then $\boldsymbol{R}$ is a ring.

On the Definition of Rings: Union and Intersection

- Union and intersection are treated asymmetrically in the definition of rings.
  - It is true that a ring is closed under the formation of intersections.
  - It is not true that a class of sets closed under the formation of intersections and differences is necessarily closed also under the formation of unions.
    Consider, e.g., $X = \{a, b\}$ and $\boldsymbol{X} = \{\emptyset, \{a\}, \{b\}\}$.
- If a non empty class of sets is closed under the formation of intersections, proper differences and disjoint unions, then it is a ring.
  Suppose $\boldsymbol{R}$ is a nonempty class, closed under the formation of intersections, proper differences and disjoint unions.
  It suffices to show that $\boldsymbol{R}$ is closed under unions.
  Let $E, F \in \boldsymbol{R}$. Then

$$E \cup F = [E - (E \cap F)] \cup [F - (E \cap F)] \cup (E \cap F) \in \boldsymbol{R}.$$

Definition of Rings: Intersection and Symmetric Difference

- It is easily possible to give a definition of rings which is more nearly symmetric in its treatment of union and intersection.
- A ring may be defined as a non empty class of sets closed under the formation of intersections and symmetric differences.
  Suppose $\boldsymbol{R}$ is a nonempty class closed under intersections and symmetric differences.
  It suffices to show it is closed under unions and differences.
  Let $E, F \in \boldsymbol{R}$. Then we have:
  $$\begin{aligned} E \cup F &= (E \triangle F) \triangle (E \cap F) \in \boldsymbol{R}; \\ E - F &= E \triangle (E \cap F) \in \boldsymbol{R}. \end{aligned}$$
- In the latter form of the definition, if we replace intersection by union we obtain a true statement:

  A non empty class of sets closed under the formation of unions and symmetric differences is a ring.

Boolean Algebras of Sets

- An **algebra**, or **Boolean algebra**, of sets is a non empty class $\boldsymbol{R}$ of sets such that:
  - (a) if $E \in \boldsymbol{R}$ and $F \in \boldsymbol{R}$, then $E \cup F \in \boldsymbol{R}$;
  - (b) if $E \in \boldsymbol{R}$, then $E' \in \boldsymbol{R}$.
- Every algebra is a ring.

  Let $\boldsymbol{R}$ be an algebra.

  It suffices to show that it is closed under differences.

  Let $E, F \in \boldsymbol{R}$. Then

$$E - F = E \cap F' = (E' \cup F)' \in \boldsymbol{R}.$$

Boolean Algebras of Sets and Rings of Sets

- The relation between the general concept of ring and the more special concept of algebra is simple:

Proposition

A ring is an algebra if and only if it contains X.

- Assume $\boldsymbol{R}$ is a ring that contains X.
  We must show it is closed under complement.
  Let $E \in \boldsymbol{R}$. Then $E' = X - E \in \boldsymbol{R}$.
  Hence, $\boldsymbol{R}$ is an algebra.
- Suppose, conversely, $\boldsymbol{R}$ is an algebra.
  By the previous slide, it is a ring.
  Since $\boldsymbol{R} \neq \emptyset$, there exists $E \in \boldsymbol{R}$.
  By hypothesis, $X = E \cup E' \in \boldsymbol{R}$.
  Thus $\boldsymbol{R}$ is a ring containing X.

Subsection 5

Generated Rings and σ-Rings

Ring Generated by a Class of Sets

Theorem

If $\boldsymbol{E}$ is any class of sets, then there exists a unique ring $\boldsymbol{R}_0$, such that:

- $\boldsymbol{E} \subseteq \boldsymbol{R}_0$;
- If $\boldsymbol{R}$ is any other ring containing $\boldsymbol{E}$, then $\boldsymbol{R}_0 \subseteq \boldsymbol{R}$.

- The ring $\boldsymbol{R}_0$, the smallest ring containing $\boldsymbol{E}$, is called the **ring generated by** $\boldsymbol{E}$ and will be denoted by $\boldsymbol{R}(\boldsymbol{E})$.
- The class of all subsets of X is a ring.

  Hence, at least one ring containing $\boldsymbol{E}$ always exists.

  The intersection of any collection of rings is also a ring.

  Thus, the intersection of all rings containing $\boldsymbol{E}$ is clearly the smallest ring containing $\boldsymbol{E}$.

  I.e., $\boldsymbol{R}_0 = \bigcap\{\boldsymbol{R} : \boldsymbol{R} \text{ a ring and } \boldsymbol{E} \subseteq \boldsymbol{R}\}$.

Finite Coverings of Sets in $R(E)$

Theorem

If $\boldsymbol{E}$ is any class of sets, then, every set in $\boldsymbol{R}(\boldsymbol{E})$ may be covered by a finite union of sets in $\boldsymbol{E}$.

- The class $\boldsymbol{R}$ of all sets which may be covered by a finite union of sets in $\boldsymbol{E}$ is a ring.

  Let $E, F \in \boldsymbol{R}$. By hypothesis, there exist $\{E_1, \ldots, E_m\} \subseteq \boldsymbol{E}$ and $\{F_1, \ldots, F_n\} \subseteq \boldsymbol{E}$, such that

  $$E \subseteq E_1 \cup \cdots \cup E_m, \quad F \subseteq F_1 \cup \cdots \cup F_n.$$

  - Thus, $E \cup F \subseteq E_1 \cup \cdots \cup E_m \cup F_1 \cup \cdots \cup F_n$. So $E \cup F \in \boldsymbol{R}$.
  - Also, $E - F \subseteq E \subseteq E_1 \cup \cdots \cup E_m$. So $E - F \in \boldsymbol{R}$.

  Hence $\boldsymbol{R}$ is a ring.

  $\boldsymbol{R}$ clearly contains $\boldsymbol{E}$, since every set in $\boldsymbol{E}$ is covered by itself.

  Since $\boldsymbol{R}$ is a ring containing $\boldsymbol{E}$, by definition of $\boldsymbol{R}(\boldsymbol{E})$, $\boldsymbol{R} \subseteq \boldsymbol{R}(\boldsymbol{E})$.

Countability of a Ring Generated by a Countable Class

Theorem

If $\boldsymbol{E}$ is a countable class of sets, then $\boldsymbol{R}(\boldsymbol{E})$ is countable.

- For any class $\boldsymbol{C}$ of sets, we write $\boldsymbol{C}^*$ for the class of all finite unions of differences of sets of $\boldsymbol{C}$. It is clear that, if $\boldsymbol{C}$ is countable, then so is $\boldsymbol{C}^*$. Moreover, if $\emptyset \in \boldsymbol{C}$, then $\boldsymbol{C} \subseteq \boldsymbol{C}^*$.
  Assume, without any loss of generality, that $\emptyset \in \boldsymbol{E}$, and set:
  - $\boldsymbol{E}_0 = \boldsymbol{E}$;
  - $\boldsymbol{E}_n = \boldsymbol{E}^*_{n-1}$, $n = 1, 2, \ldots$.

  Clearly, $\boldsymbol{E} \subseteq \bigcup_{n=0}^{\infty} \boldsymbol{E}_n \subseteq \boldsymbol{R}(\boldsymbol{E})$. Also, $\bigcup_{n=0}^{\infty} \boldsymbol{E}_n$ is countable.
  We must show that $\bigcup_{n=1}^{\infty} \boldsymbol{E}_n$ is a ring.
  We have $\boldsymbol{E} = \boldsymbol{E}_0 \subseteq \boldsymbol{E}_1 \subseteq \boldsymbol{E}_2 \subseteq \cdots$. So, if A, B are any two sets in $\bigcup_{n=1}^{\infty} \boldsymbol{E}_n$, there exists $n > 0$, such that both $A, B \in \boldsymbol{E}_n$.
  - We have $A - B \in \boldsymbol{E}_{n+1}$.
  - $\emptyset \in \boldsymbol{E} \subseteq \boldsymbol{E}_n$. Hence, $A \cup B = (A - \emptyset) \cup (B - \emptyset) \in \boldsymbol{E}_{n+1}$.

  We have proved that both $A - B$ and $A \cup B$ belong to $\bigcup_{n=1}^{\infty} \boldsymbol{E}_n$.

σ-Rings

- A σ-**ring** is a non empty class $\boldsymbol{S}$ of sets such that:
  (a) if $E \in \boldsymbol{S}$ and $F \in \boldsymbol{S}$, then $E - F \in \boldsymbol{S}$;
  (b) if $E_i \in \boldsymbol{S}$, $i = 1, 2, \ldots$, then $\bigcup_{i=1}^{\infty} E_i \in \boldsymbol{S}$.
- Equivalently a σ-ring is a ring closed under the formation of countable unions.
- If $\boldsymbol{S}$ is a σ-ring and if $E_i \in \boldsymbol{S}$, $i = 1, 2, \ldots$, then $\bigcap_{i=1}^{\infty} E_i \in \boldsymbol{S}$, i.e. a σ-ring is closed under the formation of countable intersections.

  Set $E = \bigcup_{i=1}^{\infty} E_i$.

  Then $\bigcap_{i=1}^{\infty} E_i = E - \bigcup_{i=1}^{\infty} (E - E_i) \in \boldsymbol{S}$.
- Thus, if $\boldsymbol{S}$ is a σ-ring and $E_i \in \boldsymbol{S}$, $i = 1, 2, \ldots$, then:
  - $\liminf_i E_i = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_m \in \boldsymbol{S}$;
  - $\limsup_i E_i = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m \in \boldsymbol{S}$.

σ-Ring Generated by a Class of Sets

- The truth and proof of the theorem asserting the existence of a ring generated by a class of sets remain unaltered if we replace "ring" by "σ-ring".
- Thus, we define the **σ-ring $\boldsymbol{S}(\boldsymbol{E})$ generated by** any class $\boldsymbol{E}$ of sets as the smallest σ-ring containing $\boldsymbol{E}$.

Theorem

If $\boldsymbol{E}$ is any class of sets and E is any set in $\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{E})$, then there exists a countable subclass $\boldsymbol{D}$ of $\boldsymbol{E}$, such that $E \in \boldsymbol{S}(\boldsymbol{D})$.

- Consider the collection of those σ-subrings of $\boldsymbol{S}$ which are generated by some countable subclass of $\boldsymbol{E}$.

  The union of this collection is a σ-ring containing E and contained in $\boldsymbol{S}$.

  It is therefore identical with $\boldsymbol{S}$.

Restriction of a σ-Ring to a Subspace

- For every class $\boldsymbol{E}$ of subsets of X and every subset A of X, we shall denote by $\boldsymbol{E} \cap A$ the class of all sets of the form $E \cap A$ with $E \in \boldsymbol{E}$.

Theorem

If $\boldsymbol{E}$ is any class of sets and if A is any subset of X, then $\boldsymbol{S}(\boldsymbol{E}) \cap A = \boldsymbol{S}(\boldsymbol{E} \cap A)$.

- Consider the class

$$\boldsymbol{C} = \{B \cup (C - A) : B \in \boldsymbol{S}(\boldsymbol{E} \cap A) \text{ and } C \in \boldsymbol{S}(\boldsymbol{E})\}.$$

  $\boldsymbol{C}$ is a σ-ring.
  - $(B_1 \cup (C_1 - A)) - (B_2 \cup (C_2 - A)) = (B_1 - B_2) \cup ((C_1 - C_2) - A)$;
  - $\bigcup_{i=1}^{\infty}(B_i \cup (C_i - A)) = \bigcup_{i=1}^{\infty} B_i \cup (\bigcup_{i=1}^{\infty} C_i - A)$.

Restriction of a σ-Ring to a Subspace (Cont'd)

- Moreover, $\boldsymbol{E} \subseteq \boldsymbol{C}$.

  Suppose $E \in \boldsymbol{E}$. Then $E = (E \cap A) \cup (E - A)$ and $E \cap A \in \boldsymbol{E} \cap A \subseteq \boldsymbol{S}(\boldsymbol{E} \cap A)$. Hence $E \in \boldsymbol{C}$.

  It follows that $\boldsymbol{S}(\boldsymbol{E}) \subseteq \boldsymbol{C}$.

  Thus, $\boldsymbol{S}(\boldsymbol{E}) \cap A \subseteq \boldsymbol{C} \cap A$.

  But, obviously, $\boldsymbol{C} \cap A = \boldsymbol{S}(\boldsymbol{E} \cap A)$.

  Thus, $\boldsymbol{S}(\boldsymbol{E}) \cap A \subseteq \boldsymbol{S}(\boldsymbol{E} \cap A)$.

  On the other hand:
  - $\boldsymbol{S}(\boldsymbol{E}) \cap A$ is a σ-ring;
  - $\boldsymbol{E} \cap A \subseteq \boldsymbol{S}(\boldsymbol{E}) \cap A$.

  These give the reverse inequality, $\boldsymbol{S}(\boldsymbol{E} \cap A) \subseteq \boldsymbol{S}(\boldsymbol{E}) \cap A$.

Subsection 6

Monotone Classes

Monotone Class

- It is impossible to give a constructive process for obtaining the σ-ring generated by a class of sets.
- By studying another type of class, less restricted than a σ-ring, it is possible to obtain a technically very helpful theorem concerning the structure of generated σ-rings.
- A non empty class $\boldsymbol{M}$ of sets is **monotone** if, for every monotone sequence $\{E_n\}$ of sets in $\boldsymbol{M}$, we have

$$\lim_n E_n \in \boldsymbol{M}.$$

Monotone Class Generated by a Class of Sets

- Recall that:
  - The class of all subsets of X is a (σ-)ring;
  - The intersection of any collection of (σ-)rings is a (σ-)ring.

  These facts enabled the definition of a (σ-)*ring generated by* a class of sets.
- It is also true for monotone classes that:
  - The class of all subsets of X is a monotone class;
  - The intersection of any collection of monotone classes is a monotone class.
- Thus, we may define the **monotone class $\boldsymbol{M}(\boldsymbol{E})$ generated by** any class $\boldsymbol{E}$ of sets as the smallest monotone class containing $\boldsymbol{E}$.

Monotone Rings and σ-Rings

Theorem

- A σ-ring is a monotone class.
- A monotone ring is a σ-ring.

- The first assertion is obvious, since:
  - For an increasing sequence $\{E_n\}$, $\lim_n E_n = \bigcup_{n=1}^{\infty} E_n$;
  - For a decreasing sequence $\{E_n\}$, $\lim_n E_n = \bigcap_{n=1}^{\infty} E_n$.
- To prove the second assertion we must show that a monotone ring is closed under the formation of countable unions.

  Suppose $\boldsymbol{M}$ be a monotone ring.

  Let $E_i \in \boldsymbol{M}$, $i = 1, 2, \ldots$.

  Since $\boldsymbol{M}$ is a ring, $\bigcup_{i=1}^{n} E_i \in \boldsymbol{M}$, $n = 1, 2, \ldots$.

  But $\{\bigcup_{i=1}^{n} E_i\}$ is an increasing sequence whose union is $\bigcup_{i=1}^{\infty} E_i$.

  Hence, since $\boldsymbol{M}$ is a monotone class, $\bigcup_{i=1}^{\infty} E_i \in \boldsymbol{M}$.

Monotone Class and σ-Ring Generated by a Ring

Theorem

If $\boldsymbol{R}$ is a ring, then $\boldsymbol{M}(\boldsymbol{R}) = \boldsymbol{S}(\boldsymbol{R})$. Hence, if a monotone class contains a ring $\boldsymbol{R}$, then it contains $\boldsymbol{S}(\boldsymbol{R})$.

- Since a σ-ring is a monotone class and $\boldsymbol{R} \subseteq \boldsymbol{S}(\boldsymbol{R})$, it follows that $\boldsymbol{M}(\boldsymbol{R}) \subseteq \boldsymbol{S}(\boldsymbol{R})$. The proof will be completed by showing that $\boldsymbol{M}(\boldsymbol{R})$ is a σ-ring. Since $\boldsymbol{R} \subseteq \boldsymbol{M}(\boldsymbol{R})$, it will then follow that $\boldsymbol{S}(\boldsymbol{R}) \subseteq \boldsymbol{M}(\boldsymbol{R})$.
  For any set F, let $\boldsymbol{K}(F)$ be the class of all those sets E for which:
  - $E - F \in \boldsymbol{M}(\boldsymbol{R})$;
  - $F - E \in \boldsymbol{M}(\boldsymbol{R})$;
  - $E \cup F \in \boldsymbol{M}(\boldsymbol{R})$.

  Observe that, because of the symmetric roles of E and F in the definition of $\boldsymbol{K}(F)$, the relations $E \in \boldsymbol{K}(F)$ and $F \in \boldsymbol{K}(E)$ are equivalent.

Monotone Class and σ-Ring Generated by a Ring (Cont'd)

- We show, next, that, if $\boldsymbol{K}(F)$ is not empty, then it is a monotone class.

  Suppose $\{E_n\}$ is a monotone sequence of sets in $\boldsymbol{K}(F)$.

  Then

$$\begin{aligned} \lim_n E_n - F &= \lim_n (E_n - F) \in \boldsymbol{M}(\boldsymbol{R}); \\ F - \lim_n E_n &= \lim_n (F - E_n) \in \boldsymbol{M}(\boldsymbol{R}); \\ F \cup \lim_n E_n &= \lim_n (F \cup E_n) \in \boldsymbol{M}(\boldsymbol{R}). \end{aligned}$$

  So, if $\boldsymbol{K}(F)$ is not empty, it is indeed a monotone class.

Finishing the Proof

- If $E \in \boldsymbol{R}$ and $F \in \boldsymbol{R}$, then, by the definition of a ring, $E \in \boldsymbol{K}(F)$.
  This is true for every E in $\boldsymbol{R}$.
  It follows that $\boldsymbol{R} \subseteq \boldsymbol{K}(F)$.
  Therefore, by definition of $\boldsymbol{M}(\boldsymbol{R})$, $\boldsymbol{M}(\boldsymbol{R}) \subseteq \boldsymbol{K}(F)$.
  Hence, if $E \in \boldsymbol{M}(\boldsymbol{R})$ and $F \in \boldsymbol{R}$, then $E \in \boldsymbol{K}(F)$.
  It now follows that $F \in \boldsymbol{K}(E)$.
  Since this is true, for every F in $\boldsymbol{R}$, we get, as before, $\boldsymbol{M}(\boldsymbol{R}) \subseteq \boldsymbol{K}(E)$.
  The validity of this relation, for every E in $\boldsymbol{M}(\boldsymbol{R})$ is equivalent to the assertion that $\boldsymbol{M}(\boldsymbol{R})$ is a ring.
  The desired conclusion follows from the preceding theorem.
- The theorem does not tell us, given a ring $\boldsymbol{R}$, how to construct the generated σ-ring.
- It tells us that, instead of studying the σ-ring generated by $\boldsymbol{R}$, it is sufficient to study the monotone class generated by $\boldsymbol{R}$.