Introduction to Measure Theory

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Subsection 1

Measure on Rings

Finitely Additive and Countably Additive Set Functions

- A **set function** is a function whose domain is a class of sets.
- An extended real valued set function μ defined on a class $\boldsymbol{E}$ of sets is **additive** if, whenever $E \in \boldsymbol{E}$, $F \in \boldsymbol{E}$, $E \cup F \in \boldsymbol{E}$ and $E \cap F = \emptyset$, then $\mu(E \cup F) = \mu(E) + \mu(F)$.
- An extended real valued set function μ defined on a class $\boldsymbol{E}$ is **finitely additive** if, for every finite, disjoint class $\{E_1, \ldots, E_n\}$ of sets in $\boldsymbol{E}$ whose union is also in $\boldsymbol{E}$, we have
$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i).$$
- An extended real valued set function μ defined on a class $\boldsymbol{E}$ is **countably additive** if, for every disjoint sequence $\{E_n\}$ of sets in $\boldsymbol{E}$, whose union is also in $\boldsymbol{E}$, we have
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Measures

- A **measure** is an extended real valued, non negative, and countably additive set function μ, defined on a ring $\boldsymbol{R}$, and such that $\mu(\emptyset) = 0$.
- Rephrasing, a **measure** on a ring $\boldsymbol{R}$ is a function

$$\mu : \boldsymbol{R} \to [0, \infty],$$

  such that:
  - $\mu(\emptyset) = 0$;
  - μ is countably additive.
- In view of the identity

$$\bigcup_{i=1}^{n} E_i = E_1 \cup \cdots \cup E_n \cup \emptyset \cup \cdots ,$$

  a measure is always finitely additive.

An Example of a Measure

- A (rather trivial) measure may be obtained as follows:

  Let f be an extended real valued, non negative function defined on X:

$$f : X \to [0, \infty].$$

  Let the ring $\boldsymbol{R}$ consist of all finite subsets of X.

  Define $\mu : \boldsymbol{R} \to [0, \infty]$ by:

$$\begin{array}{rcl} \mu(\emptyset) & = & 0; \\ \mu(\{x_1, \ldots, x_n\}) & = & \sum_{i=1}^n f(x_i). \end{array}$$

Types of Measures

- If μ is a measure on a ring $\boldsymbol{R}$, a set E in $\boldsymbol{R}$ is said to **have finite measure** if $\mu(E) < \infty$.
- The measure of E is σ-**finite** if there exists a sequence $\{E_n\}$ of sets in $\boldsymbol{R}$ such that $E \subseteq \bigcup_{n=1}^{\infty} E_n$ and $\mu(E_n) < \infty$, $n = 1, 2, \ldots$.
- If the measure of every set E in $\boldsymbol{R}$ is finite or σ-finite, the measure μ is called **finite** or σ-**finite**, respectively, on $\boldsymbol{R}$.
- If $X \in \boldsymbol{R}$ (i.e., if $\boldsymbol{R}$ is an algebra) and $\mu(X)$ is finite or σ-finite, then μ is called **totally finite** or **totally** σ-**finite**, respectively.
- The measure μ is called **complete** if the conditions $E \in \boldsymbol{R}$, $F \subseteq E$ and $\mu(E) = 0$ imply that $F \in \boldsymbol{R}$.

Subsection 2

Measure on Intervals

Semi-Closed Internals of Real Numbers

- In this section the underlying space X is to be the real line.
- We denote by $\boldsymbol{P}$ the class of all bounded, left closed, and right open intervals, i.e. the class of all sets of the form

$$\{x : -\infty < a \leq x < b < \infty\}.$$

- We denote by $\boldsymbol{R}$ the class of all finite, disjoint unions of sets of $\boldsymbol{P}$, i.e., the class of all sets of the form

$$\bigcup_{i=1}^{n}\{x : -\infty < a_i \leq x < b_i < \infty\}.$$

  A union of this form may be written as a disjoint union of the same form.
- For simplicity of language, we shall always use the expression "semi-closed interval" instead of "bounded, left closed, and right open interval".

Why Semi-Closed Internals

- The consideration of semi-closed intervals, instead of open intervals or closed intervals is done for convenience:
    - For instance, if a, b, c and d are real numbers, $-\infty < a < b < c < d < \infty$, then the difference between the open intervals $\{x : a < x < d\}$ and $\{x : b < x < c\}$ is neither an open interval nor a finite union of open intervals.
    - The same negative statement holds for the closed intervals.
    - The fact that semi-closed intervals are better behaved in this respect is what makes them desirable.
- We write, for $a \leq b$:
    - $[a, b]$ for the closed interval, $[a, b] = \{x : a \leq x \leq b\}$;
    - $[a, b)$ for the semiclosed interval, $[a, b) = \{x : a \leq x < b\}$;
    - (a, b) for the open interval, $(a, b) = \{x : a < x < b\}$.

A Set Function on Semi-Closed Intervals

- On the class $\boldsymbol{P}$ of semi-closed intervals we define a set function

$$\mu([a,b)) = b - a.$$

- When $a = b$, the interval reduces to the empty set:

$$\mu(\emptyset) = 0.$$

A Property of μ

Theorem

If $\{E_1, \ldots, E_n\}$ is a finite, disjoint class of sets in $\boldsymbol{P}$, each contained in a given set E_0 in $\boldsymbol{P}$, then

$$\sum_{i=1}^{n} \mu(E_i) \leq \mu(E_0).$$

- Write $E_i = [a_i, b_i)$, $i = 0, 1, \ldots, n$.
  Without loss of generality, assume $a_1 \leq a_2 \leq \cdots \leq a_n$.
  It follows from the assumption on $\{E_1, \ldots, E_n\}$ that $a_0 \leq a_1 \leq b_1 \leq \cdots \leq a_n \leq b_n \leq b_0$.
  Thus,

$$\begin{array}{rcl} \sum_{i=1}^{n} \mu(E_i) & = & \sum_{i=1}^{n} (b_i - a_i) \\ & \leq & \sum_{i=1}^{n} (b_i - a_i) + \sum_{i=1}^{n-1} (a_{i+1} - b_i) \\ & = & b_n - a_1 \leq b_0 - a_0 = \mu(E_0). \end{array}$$

A Closed Interval in the Union of Open Intervals

Theorem

If a closed interval F_0, $F_0 = [a_0, b_0]$, is contained in the union of a finite number of bounded, open intervals, $U_1, \ldots, U_n$, $U_i = (a_i, b_i)$, $i = 1, \ldots, n$, then

$$b_0 - a_0 < \sum_{i=1}^{n} (b_i - a_i).$$

- Let k_1 be such that $a_0 \in U_{k_1}$.
  If $b_{k_1} \leq b_0$, then let k_2 be such that $b_{k_1} \in U_{k_2}$.
  If $b_{k_2} \leq b_0$, then let k_3 be such that $b_{k_2} \in U_{k_3}$.
  Continue in the same way, by induction.
  The process stops with k_m if $b_{k_m} > b_0$.
  Without loss of generality, assume $m = n$ and $U_{k_i} = U_i$, $i = 1, \ldots, n$.
  This state of affairs may be achieved merely by omitting superfluous U_i's and changing the notation.

A Closed Interval in the Union of Open Intervals (Cont'd)

- In other words we may (and do) assume that:
  - $a_1 < a_0 < b_1$;
  - $a_n < b_0 < b_n$;
  - $a_{i+1} < b_i < b_{i+1}$ for $i = 1, \ldots, n-1$, $n > 1$.

  It follows that

$$\begin{array}{rcl} b_0 - a_0 & < & b_n - a_1 \\ & = & (b_1 - a_1) + \sum_{1 \leq i \leq n-1} (b_{i+1} - b_i) \\ & \leq & \sum_{i=1}^{n} (b_i - a_i). \end{array}$$

Domination of the Sum of a Covering

Theorem

If $\{E_0, E_1, E_2, \ldots\}$ is a sequence of sets in $\boldsymbol{P}$, such that $E_0 \subseteq \bigcup_{i=1}^{\infty} E_i$, then

$$\mu(E_0) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

- We write $E_i = [a_i, b_i)$, $i = 0, 1, 2, \ldots$. If $a_0 = b_0$, the theorem is trivial. Otherwise let ϵ be a positive number such that $\epsilon < b_0 - a_0$. For any $\delta > 0$, set $F_0 = [a_0, b_0 - \epsilon]$ and $U_i = (a_i - \frac{\delta}{2^i}, b_i)$, $i = 1, 2, \ldots$. Then we get $F_0 \subseteq \bigcup_{i=1}^{\infty} U_i$. By the Heine-Borel Theorem, there is a positive integer n, such that $F_0 \subseteq \bigcup_{i=1}^{n} U_i$.
  By the preceding theorem,
  $$\mu(E_0) - \epsilon = (b_0 - a_0) - \epsilon < \sum_{i=1}^{n} (b_i - a_i + \frac{\delta}{2^i}) \leq \sum_{i=1}^{\infty} \mu(E_i) + \delta.$$
  Since ϵ and δ are arbitrary, the conclusion follows.

Countable Additivity of the Measure on $\boldsymbol{P}$

Theorem

The set function μ is countably additive on $\boldsymbol{P}$.

- Let $\{E_i\}$ be a disjoint sequence of sets in $\boldsymbol{P}$ whose union, E, is also in $\boldsymbol{P}$.

  By a preceding theorem, we have

  $$\sum_{i=1}^{n} \mu(E_i) \leq \mu(E), \quad n = 1, 2, \ldots .$$

  It follows that $\sum_{i=1}^{\infty} \mu(E_i) \leq \mu(E)$.

  But, by the preceding theorem $\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$.

  Therefore, we get equality.

The Measure $\overline{\mu}$ on $\boldsymbol{R}$

Theorem

There exists a unique, finite measure $\overline{\mu}$ on the ring $\boldsymbol{R}$, such that,

$$\overline{\mu}(E) = \mu(E), \quad \text{for all } E \in \boldsymbol{P}.$$

- We know that every set E in $\boldsymbol{R}$ may be represented as a finite, disjoint union of sets in $\boldsymbol{P}$. Suppose that

$$E = \bigcup_{i=1}^{n} E_i \quad \text{and} \quad E = \bigcup_{j=1}^{m} F_j$$

are two such representations of the same set E.
Then, for each $i = 1, \ldots, n$,

$$E_i = \bigcup_{j=1}^{m} (E_i \cap F_j)$$

is a representation of $E_i \in \boldsymbol{P}$ as a finite, disjoint union of sets in $\boldsymbol{P}$.

The Measure $\overline{\mu}$ on $\boldsymbol{R}$ (Cont'd)

- Therefore, since μ is finitely additive,

$$\sum_{i=1}^{n} \mu(E_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mu(E_i \cap F_j).$$

Similarly,

$$\sum_{j=1}^{m} \mu(F_j) = \sum_{j=1}^{m} \sum_{i=1}^{n} \mu(E_i \cap F_j).$$

It follows that, for every E in $\boldsymbol{R}$, the function $\overline{\mu}$ is unambiguously defined by the equation

$$\overline{\mu}(E) = \sum_{i=1}^{n} \mu(E_i),$$

where $\{E_1, \ldots, E_n\}$ is a finite, disjoint class of sets in $\boldsymbol{P}$ whose union is E.

The Measure $\overline{\mu}$ on $\boldsymbol{R}$ (Uniqueness)

- Clearly, we have:
  - The function $\overline{\mu}$ coincides with μ on $\boldsymbol{P}$;
  - The function $\overline{\mu}$ is finitely additive.

  Any function satisfying these conditions must be finitely additive when the terms of the union are in $\boldsymbol{P}$.

  It follows that $\overline{\mu}$ is unique.

The Measure $\overline{\mu}$ on $\boldsymbol{R}$ (Countable Additivity)

- We are left with showing that $\overline{\mu}$ is countably additive.

  Let $\{E_i\}$ be a disjoint sequence of sets in $\boldsymbol{R}$ whose union E is in $\boldsymbol{R}$.

  Then, each E_i is a finite, disjoint union of sets in $\boldsymbol{P}$, $E_i = \bigcup_j E_{ij}$ and $\overline{\mu}(E_i) = \sum_j \mu(E_{ij})$.

  - If $E \in \boldsymbol{P}$, then, since the class of all E_n is countable and disjoint, it follows from the countable additivity of μ that

    $$\overline{\mu}(E) = \mu(E) = \sum_i \sum_j \mu(E_{ij}) = \sum_i \overline{\mu}(E_i).$$

  - In the general case, E is a finite, disjoint union of sets in $\boldsymbol{P}$, $E = \bigcup_k F_k$. Using the result just obtained, we have

$$\overline{\mu}(E) = \sum_k \overline{\mu}(F_k) = \sum_k \sum_i \overline{\mu}(E_i \cap F_k) = \sum_i \sum_k \overline{\mu}(E_i \cap F_k) = \sum_i \overline{\mu}(E_i).$$

- We may now, without any possibility of confusion, write $\mu(E)$ instead of $\overline{\mu}(E)$ even for sets E which are in $\boldsymbol{R}$ but not in $\boldsymbol{P}$.

Subsection 3

Properties of Measures

Monotone and Subtractive Set Functions

- An extended real valued set function μ on a class $\boldsymbol{E}$ is **monotone** if, whenever $E \in \boldsymbol{E}$, $F \in \boldsymbol{E}$,

$$E \subseteq F \quad \text{implies} \quad \mu(E) \leq \mu(F).$$

- An extended real valued set function μ on a class $\boldsymbol{E}$ is **subtractive** if, whenever $E \in \boldsymbol{E}$, $F \in \boldsymbol{E}$, such that $E \subseteq F$,

$$F - E \in \boldsymbol{E} \text{ and } |\mu(E)| < \infty \quad \text{imply} \quad \mu(F - E) = \mu(F) - \mu(E).$$

Measures are Monotone and Subtractive

Theorem

If μ is a measure on a ring $\boldsymbol{R}$, then μ is monotone and subtractive.

- Suppose $E \in \boldsymbol{R}$, $F \in \boldsymbol{R}$, and $E \subseteq F$.
  Since $\boldsymbol{R}$ is a ring, $F - E \in \boldsymbol{R}$.
  Since μ is a measure,

$$\mu(F) = \mu(E) + \mu(F - E).$$

  - By nonnegativity,

$$\mu(F) = \mu(E) + \mu(F - E) \leq \mu(E).$$

  - If $|\mu(E)| < \infty$, then

$$\mu(F) - \mu(E) = \mu(F - E).$$

    Hence, μ is subtractive.

The Measure of a Set Included in a Union

Theorem

If μ is a measure on a ring $\boldsymbol{R}$, if $E \in \boldsymbol{R}$, and if $\{E_i\}$ is a finite or infinite sequence of sets in $\boldsymbol{R}$, such that $E \subseteq \bigcup_i E_i$, then

$$\mu(E) \leq \sum_i \mu(E_i).$$

- If $\{F_i\}$ is any sequence of sets in a ring $\boldsymbol{R}$, then there exists a disjoint sequence $\{G_i\}$ of sets in $\boldsymbol{R}$, such that $G_i \subseteq F_i$ and $\bigcup_i G_i = \bigcup_i F_i$: E.g., set $G_i = F_i - \bigcup\{F_j : 1 \leq j < i\}$.
  We apply this result to the sequence $\{E \cap E_i\}$:

$$\begin{array}{rcl} \mu(E) & = & \mu(E \cap \bigcup_i E_i) = \mu(\bigcup_i (E \cap E_i)) \\ & = & \mu(\bigcup_i G_i) = \sum_i \mu(G_i) \\ & \leq & \sum_i \mu(E \cap E_i) \leq \sum_i \mu(E_i). \end{array}$$

The Measure of a Set Covering a Union

Theorem

If μ is a measure on a ring $\boldsymbol{R}$, if $E \in \boldsymbol{R}$, and if E_i is a finite or infinite disjoint sequence of sets in $\boldsymbol{R}$, such that $\bigcup_i E_i \subseteq E$, then

$$\sum_i \mu(E_i) \leq \mu(E).$$

- If the sequence $\{E_i\}$ is finite, then $\bigcup_i E_i \in \boldsymbol{R}$,
  It follows that
  $$\sum_i \mu(E_i) = \mu\left(\bigcup_i E_i\right) \leq \mu(E).$$
  The validity of the inequality for an infinite sequence of sets is a consequence of its validity for every finite subsequence.

Measure of the Limit of an Increasing Sequence

Theorem

If μ is a measure on a ring $\boldsymbol{R}$ and if $\{E_n\}$ is an increasing sequence of sets in $\boldsymbol{R}$ for which $\lim_n E_n \in \boldsymbol{R}$, then

$$\mu\left(\lim_n E_n\right) = \lim_n \mu(E_n).$$

- If we write $E_0 = \emptyset$, then

$$\begin{aligned} \mu(\lim_n E_n) &= \mu(\textstyle\bigcup_{i=1}^\infty E_i) \\ &= \mu(\textstyle\bigcup_{i=1}^\infty (E_i - E_{i-1})) \\ &= \textstyle\sum_{i=1}^\infty \mu(E_i - E_{i-1}) \\ &= \lim_n \textstyle\sum_{i=1}^n \mu(E_i - E_{i-1}) \\ &= \lim_n \mu(\textstyle\bigcup_{i=1}^n (E_i - E_{i-1})) \\ &= \lim_n \mu(E_n). \end{aligned}$$

Measure of the Limit of a Decreasing Sequence

Theorem

If μ is a measure on a ring $\boldsymbol{R}$, and if $\{E_n\}$ is a decreasing sequence of sets in $\boldsymbol{R}$ of which at least one has finite measure and for which $\lim_n E_n \in \boldsymbol{R}$, then

$$\mu\left(\lim_n E_n\right) = \lim_n \mu(E_n).$$

- If $\mu(E_m) < \infty$, then $\mu(E_n) \leq \mu(E_m) < \infty$, for $n \geq m$. Therefore, $\mu(\lim_n E_n) < \infty$. Note that $\{E_m - E_n\}$ is an increasing sequence:

$$\begin{aligned} \mu(E_m) - \mu(\lim_n E_n) &= \mu(E_m - \lim_n E_n) \\ &= \mu(\lim_n (E_m - E_n)) \\ &= \lim_n \mu(E_m - E_n) \\ &= \lim_n (\mu(E_m) - \mu(E_n)) \\ &= \mu(E_m) - \lim_n \mu(E_n). \end{aligned}$$

Since $\mu(E_m) < \infty$, the proof of the theorem is complete.

Continuity from Below/from Above

- We shall say that an extended real valued set function μ defined on a class $\boldsymbol{E}$ is **continuous from below** at a set E (in $\boldsymbol{E}$) if, for every increasing sequence $\{E_n\}$ of sets in $\boldsymbol{E}$ for which $\lim_n E_n = E$, we have
$$\lim_n \mu(E_n) = \mu(E).$$
- Similarly μ is **continuous from above** at E if, for every decreasing sequence $\{E_n\}$ of sets in E for which $|\mu(E_m)| < \infty$, for at least one value of m, and for which $\lim_n E_n = E$, we have
$$\lim_n \mu(E_n) = \mu(E).$$
- The preceding two theorems assert that, if μ is a measure, then μ is continuous from above and from below (at every set in the ring of definition of μ).

Continuity and Measures

Theorem

Let μ be a finite, nonnegative, and additive set function on a ring $\boldsymbol{R}$. If μ is either continuous from below at every E in $\boldsymbol{R}$, or continuous from above at $\emptyset$, then μ is a measure on $\boldsymbol{R}$.

- The additivity of μ, together with the fact that $\boldsymbol{R}$ is a ring, implies, by mathematical induction, that μ is finitely additive.

  Let $\{E_n\}$ be a disjoint sequence of sets in $\boldsymbol{R}$, whose union E is in $\boldsymbol{R}$. Write

$$F_n = \bigcup_{i=1}^{n} E_i \quad \text{and} \quad G_n = E - F_n.$$

Continuity and Measures (Cont'd)

- Suppose μ is continuous from below.
  $\{F_n\}$ is an increasing sequence of sets in $\boldsymbol{R}$ with $\lim_n F_n = E$.

$$\mu(E) = \lim_n \mu(F_n) = \lim_n \sum_{i=1}^{n} \mu(E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

- Suppose μ is continuous from above at $\emptyset$.
  $\{G_n\}$ is a decreasing sequence of sets in $\boldsymbol{R}$, with $\lim_n G_n = \emptyset$, and μ is finite.

$$\mu(E) = \left(\sum_{i=1}^{n} \mu(E_i)\right) + \mu(G_n) = \lim_n \sum_{i=1}^{n} \mu(E_i) + \lim_n \mu(G_n) = \sum_{i=1}^{\infty} \mu(E_i).$$

In either case μ is countably additive.

Subsection 4

Outer Measures

Hereditary Classes

- A non empty class $\boldsymbol{E}$ of sets is **hereditary** if,

$$E \in \boldsymbol{E} \quad \text{and} \quad F \subseteq E \quad \text{imply} \quad F \in \boldsymbol{E}.$$

  Example: The class of all subsets of some subset E of X is a typical example of a hereditary class.
- The intersection of every collection of hereditary classes is again a hereditary class.
- Thus, corresponding to any class of sets, there is a smallest hereditary class containing it.
- A hereditary class is a σ-ring if and only if it is closed under the formation of countable unions.

Hereditary σ-Ring Generated by a Class

- If $\boldsymbol{E}$ is any class of sets, the hereditary σ-ring generated by $\boldsymbol{E}$, i.e., the smallest hereditary σ-ring containing $\boldsymbol{E}$, is denoted by $\boldsymbol{H}(\boldsymbol{E})$.
- The hereditary σ-ring generated by $\boldsymbol{E}$ is, in fact, the class of all sets which can be covered by countably many sets in $\boldsymbol{E}$.
- Thus, if $\boldsymbol{E}$ is a non empty class closed under the formation of countable unions (e.g., if $\boldsymbol{E}$ is a σ-ring), then $\boldsymbol{H}(\boldsymbol{E})$ is the class of all sets which are subsets of some set in $\boldsymbol{E}$.

Subadditivity

- An extended real valued set function μ^* defined on a class $\boldsymbol{E}$ of sets is **subadditive** if, whenever $E \in \boldsymbol{E}$, $F \in \boldsymbol{E}$, and $E \cup F \in \boldsymbol{E}$, then
$$\mu^*(E \cup F) \leq \mu^*(E) + \mu^*(F).$$
- An extended real valued set function μ^* on $\boldsymbol{E}$ is **finitely subadditive** if, for every finite class $\{E_1, \ldots, E_n\}$ of sets in $\boldsymbol{E}$ whose union is also in $\boldsymbol{E}$, we have
$$\mu^*\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mu^*(E_i).$$
- An extended real valued set function μ^* on $\boldsymbol{E}$ is **countably subadditive** if, for every sequence $\{E_i\}$ of sets in $\boldsymbol{E}$ whose union is also in $\boldsymbol{E}$, we have
$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i).$$

Outer Measures

- An **outer measure** is an extended real valued, non negative, set function μ^*, defined on a hereditary σ-ring $\boldsymbol{H}$, such that:
  - $\mu^*(\emptyset) = 0$;
  - μ^* is monotone;
  - μ^* is countably subadditive.
- An outer measure is necessarily finitely subadditive.
- The same terminology concerning [**total**] **finiteness** and σ-**finiteness** is used for outer measures as for measures.
- Outer measures arise naturally in the attempt to extend measures from rings to larger classes of sets.

Extensions of Measures

Theorem

If μ is a measure on a ring $\boldsymbol{R}$ and if, for every E in $\boldsymbol{H}(\boldsymbol{R})$,

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(E_n) : E_n \in \boldsymbol{R}, n = 1, 2, \ldots, E \subseteq \bigcup_{n=1}^{\infty} E_n\right\},$$

then μ^* is an extension of μ to an outer measure on $\boldsymbol{H}(\boldsymbol{R})$.
If μ is [totally] σ-finite, then so is μ^*.

- $\mu^*(E)$ is the lower bound of sums of the type $\sum_{n=1}^{\infty} \mu(E_n)$, where $\{E_n\}$ is a sequence of sets in $\boldsymbol{R}$ whose union contains E.
- μ^* is called the **outer measure induced by** the measure μ.

Extensions of Measures (Extension)

- Suppose $E \in \boldsymbol{R}$.
  - On the one hand, $E \subseteq E \cup \emptyset \cup \emptyset \cup \cdots$.
    Therefore, $\mu^*(E) \leq \mu(E) + \mu(\emptyset) + \mu(\emptyset) + \cdots = \mu(E)$.
  - On the other, if $E_n \in \boldsymbol{R}$, $n = 1, 2, \ldots$, and $E \subseteq \bigcup_{n=1}^{\infty} E_n$, then

$$\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

    Thus, $\mu(E) \leq \mu^*(E)$.

  This proves that μ^* is indeed an extension of μ, i.e., that, if $E \in \boldsymbol{R}$, then $\mu^*(E) = \mu(E)$.

  In particular, $\mu^*(\emptyset) = 0$.

Extensions of Measures (Countable Subadditivity)

- Suppose $E \in \boldsymbol{H}(\boldsymbol{R})$, $F \in \boldsymbol{H}(\boldsymbol{R})$, such that $E \subseteq F$.
  Let $\{E_n\}$ be a sequence of sets in $\boldsymbol{R}$ which covers F.
  Then $\{E_n\}$ also covers E. So $\mu^*(E) \leq \mu^*(F)$, i.e., μ^* is monotone.
- To prove that μ^* is countably subadditive, suppose that E and E_i are sets in $\boldsymbol{H}(\boldsymbol{R})$, such that $E \subseteq \bigcup_{i=1}^{\infty} E_i$.
  By the definition of $\mu^*(E_i)$, there exists, for every $\epsilon > 0$ and all $i = 1, 2, \ldots$, a sequence E_{ij} of sets in $\boldsymbol{R}$, such that

  $$E_i \subseteq \bigcup_{j=1}^{\infty} E_{ij} \quad \text{and} \quad \sum_{j=1}^{\infty} \mu(E_{ij}) \leq \mu^*(E_i) + \frac{\epsilon}{2^i}.$$

  Then, since the E_{ij}'s form a countable class of sets in $\boldsymbol{R}$ covering E,

  $$\mu^*(E) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu(E_{ij}) \leq \sum_{i=1}^{\infty} \mu^*(E_i) + \epsilon.$$

  ϵ arbitrary implies that $\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$.

Extensions of Measures (Finiteness)

- Suppose, finally, that μ is σ-finite.

  Let E be any set in $\boldsymbol{H}(\boldsymbol{R})$.

  By the definition of $\boldsymbol{H}(\boldsymbol{R})$, there exists a sequence $\{E_i\}$ of sets in $\boldsymbol{R}$, such that $E \subseteq \bigcup_{i=1}^{\infty} E_i$.

  Since μ is σ-finite, there exists, for each $i = 1, 2, \ldots$, a sequence $\{E_{ij}\}$ of sets in $\boldsymbol{R}$, such that

$$E_i \subseteq \bigcup_{j=1}^{\infty} E_{ij} \quad \text{and} \quad \mu(E_{ij}) < \infty.$$

  Consequently, $E \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} E_{ij}$ and $\mu^*(E_{ij}) = \mu(E_{ij}) < \infty$.

  Thus, μ^* is σ-finite.

Subsection 5

Measurable Sets

μ^*-Measurability

- Let μ^* be an outer measure on a hereditary σ-ring $\boldsymbol{H}$.

  A set E in $\boldsymbol{H}$ is μ^*-**measurable** if, for every set A in $\boldsymbol{H}$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E').$$

- The concept of μ^*-measurability is the most important one in the theory of outer measures.
- An outer measure is not necessarily a countably, nor even finitely, additive set function.
- In an attempt to satisfy the reasonable requirement of additivity, we single out those sets which split every other set additively, giving rise to the definition of μ^*-measurability.
- The greatest justification of this concept is its success as a tool in proving the important and useful extension theorem for measures.

The Ring of Measurable Sets

Theorem

If μ^* is an outer measure on a hereditary σ-ring $\boldsymbol{H}$ and if $\overline{\boldsymbol{S}}$ is the class of all μ^*-measurable sets, then $\overline{\boldsymbol{S}}$ is a ring.

- If E and F are in $\overline{\boldsymbol{S}}$ and $A \in \boldsymbol{H}$, then:
  - (a) $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E')$;
  - (b) $\mu^*(A \cap E) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F')$;
  - (c) $\mu^*(A \cap E') = \mu^*(A \cap E' \cap F) + \mu^*(A \cap E' \cap F')$.

  Substituting (b) and (c) into (a) we obtain

  - (d) $\mu^*(A) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F') + \mu^*(A \cap E' \cap F) + \mu^*(A \cap E' \cap F')$.

The Ring of Measurable Sets (Cont'd)

- We got

  (d) $\mu^*(A) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F') + \mu^*(A \cap E' \cap F) + \mu^*(A \cap E' \cap F')$.

  If in equation (d) we replace A by $A \cap (E \cup F)$, the first three terms of the right hand side remain unaltered and the last term drops out:

  (e) $\mu^*(A \cap (E \cup F)) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F') + \mu^*(A \cap E' \cap F)$.

  Since $E' \cap F' = (E \cup F)'$, substituting (e) into (d) yields

  (f) $\mu^*(A) = \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)')$.

  This proves that $E \cup F \in \overline{\boldsymbol{S}}$.

The Ring of Measurable Sets (Cont'd)

- We got

  (d) $\mu^*(A) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F') + \mu^*(A \cap E' \cap F) + \mu^*(A \cap E' \cap F')$.

  If we replace A in equation (d) by $A \cap (E - F)' = A \cap (E' \cup F)$, we get

  (g) $\mu^*(A \cap (E - F)') = \mu^*(A \cap E \cap F) + \mu^*(A \cap E' \cap F) + \mu^*(A \cap E' \cap F')$.

  Since $E \cap F' = E - F$, substituting (g) into (d) yields

  (h) $\mu^*(A) = \mu^*(A \cap (E - F)) + \mu^*(A \cap (E - F)')$.

  This proves that $E - F \in \overline{\boldsymbol{S}}$.

  Since it is clear that $E = \emptyset$ satisfies (a), it follows that $\overline{\boldsymbol{S}}$ is a ring.

Remark

- Suppose μ^* is an outer measure on a hereditary σ-ring $\boldsymbol{H}$. Let E in $\boldsymbol{H}$ be such that, for every A in $\boldsymbol{H}$,

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E').$$

  Then E is μ^*-measurable.
- For the proof, recall that

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E')$$

  is an automatic consequence of the subadditivity of μ^*.

Structure of Measurable Sets on a Hereditary σ-Ring

Theorem

If μ^* is an outer measure on a hereditary σ-ring $\boldsymbol{H}$ and if $\overline{\boldsymbol{S}}$ is the class of all μ^*-measurable sets, then $\overline{\boldsymbol{S}}$ is a σ-ring. If $A \in \boldsymbol{H}$ and if $\{E_n\}$ is a disjoint sequence of sets in $\overline{\boldsymbol{S}}$, with $\bigcup_{n=1}^{\infty} E_n = E$, then

$$\mu^*(A \cap E) = \sum_{n=1}^{\infty} \mu^*(A \cap E_n).$$

- In the preceding proof, we showed

  (e) $\mu^*(A \cap (E \cup F)) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F') + \mu^*(A \cap E' \cap F)$.

  Replacing E and F in (e) by E_1 and E_2, respectively, we get

$$\mu^*(A \cap (E_1 \cup E_2)) = \mu^*(A \cap E_1) + \mu^*(A \cap E_2).$$

The Proof

- From $\mu^*(A \cap (E_1 \cup E_2)) = \mu^*(A \cap E_1) + \mu^*(A \cap E_2)$, it follows by mathematical induction, that
$$\mu^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu^*(A \cap E_i),$$
for every positive integer n.
Write $F_n = \bigcup_{i=1}^{n} E_i$, $i = 1, 2, \ldots$.
Then, by the preceding theorem,
$$\begin{array}{rcl} \mu^*(A) & = & \mu^*(A \cap F_n) + \mu^*(A \cap F_n') \\ & \geq & \sum_{i=1}^{n} \mu^*(A \cap E_i) + \mu^*(A \cap E'). \end{array}$$
Since this is true for every n, we obtain
(i) $\mu^*(A) \geq \sum_{i=1}^{\infty} \mu(A \cap E_i) + \mu^*(A \cap E') \geq \mu^*(A \cap E) + \mu^*(A \cap E')$.
Thus $E \in \overline{\boldsymbol{S}}$.
So $\overline{S}$ is closed under the formation of disjoint countable unions.

The Proof (Cont'd)

- Since $E \in \overline{\boldsymbol{S}}$,

  (j) $\sum_{i=1}^{\infty} \mu^*(A \cap E_i) + \mu^*(A \cap E') = \mu^*(A \cap E) + \mu^*(A \cap E')$.

  Replacing A by $A \cap E$ in (j), we obtain that, if $A \in \boldsymbol{H}$ and if $\{E_n\}$ is a disjoint sequence of sets in $\overline{S}$ with $\bigcup_{i=1}^{n} E_n = E$, then

$$\mu^*(A \cap E) = \sum_{n=1}^{\infty} \mu^*(A \cap E_n).$$

  Since every countable union of sets in a ring may be written as a disjoint countable union of sets, we see also that $\overline{\boldsymbol{S}}$ is a σ-ring.

Measures Induced by Outer Measures

Theorem

If μ^* is an outer measure on a hereditary σ-ring $\boldsymbol{H}$ and if $\overline{\boldsymbol{S}}$ is the class of all μ^*-measurable sets, then:

- Every set of outer measure zero belongs to $\overline{\boldsymbol{S}}$;
- The set function $\overline{\mu}$, defined for E in $\overline{\boldsymbol{S}}$ by $\overline{\mu}(E) = \mu^*(E)$, is a complete measure on $\overline{\boldsymbol{S}}$.

- $\overline{\mu}$ is called the **measure induced by** the outer measure μ^*.
- Suppose $E \in \boldsymbol{H}$ and $\mu^*(E) = 0$. For every A in $\boldsymbol{H}$, we have
$$\mu^*(A) = \mu^*(E) + \mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E').$$
Thus, $E \in \overline{\boldsymbol{S}}$.
- Countable additivity of $\overline{\mu}$ on $\overline{\boldsymbol{S}}$ follows from (j) upon replacing A by E.
- For completeness, suppose $E \in \overline{\boldsymbol{S}}$, $F \subseteq E$ and $\overline{\mu}(E) = \mu^*(E) = 0$. Then $\mu^*(F) = 0$. So $F \in \overline{\boldsymbol{S}}$.