Introduction to Measure Theory

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

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1. Extension of Measures
   - Properties of Induced Measures
   - Extension, Completion and Approximation
   - Inner Measures
   - Lebesgue Measure
   - Non-Measurable Sets

Subsection 1

Properties of Induced Measures

The Framework

- Suppose we start with a measure μ.
- We form the induced outer measure μ^*.
- We then form the measure $\overline{\mu}$.
- Our goal is to study the relation between μ and $\overline{\mu}$.
- Throughout, we assume that:
  - μ is a measure on a ring $\boldsymbol{R}$;
  - μ^* is the induced outer measure on $\boldsymbol{H}(\boldsymbol{R})$;
  - $\overline{\mu}$ is the measure induced by μ^* on the σ-ring $\overline{\boldsymbol{S}}$ of all μ^*-measurable sets.

Measurability of Sets in $S(R)$

Theorem

Every set in $\boldsymbol{S}(\boldsymbol{R})$ is μ^*-measurable.

- Suppose $E \in \boldsymbol{R}$, $A \in \boldsymbol{H}(\boldsymbol{R})$, and $\epsilon > 0$.

  By the definition of μ^*, there exists a sequence $\{E_n\}$ of sets in $\boldsymbol{R}$, such that $A \subseteq \bigcup_{n=1}^{\infty} E_n$ and

$$\begin{aligned} \mu^*(A) + \epsilon &\geq \sum_{n=1}^{\infty} \mu(E_n) \\ &= \sum_{n=1}^{\infty}(\mu(E_n \cap E) + \mu(E_n \cap E')) \\ &\geq \mu^*(A \cap E) + \mu^*(A \cap E'). \end{aligned}$$

  Since this is true for every ϵ, it follows that E is μ^*-measurable.

  I.e., we have proved that $\boldsymbol{R} \subseteq \overline{\boldsymbol{S}}$.

  It follows from the fact that $\overline{\boldsymbol{S}}$ is a σ-ring that $\boldsymbol{S}(\boldsymbol{R}) \subseteq \overline{\boldsymbol{S}}$.

The Outer Measure and the Measure it Induces

Theorem

If $E \in \boldsymbol{H}(\boldsymbol{R})$, then

$$\mu^*(E) = \inf\{\overline{\mu}(F) : E \subseteq F \in \overline{\boldsymbol{S}}\} = \inf\{\overline{\mu}(F) : E \subseteq F \in \boldsymbol{S}(\boldsymbol{R})\}.$$

- Equivalently, the outer measure induced by $\overline{\mu}$ on $\boldsymbol{S}(\boldsymbol{R})$ and the outer measure induced by $\overline{\mu}$ on $\overline{\boldsymbol{S}}$ both coincide with μ^*.
- For F in $\boldsymbol{R}$, $\mu(F) = \overline{\mu}(F)$ (by the definition of $\overline{\mu}$ and the first theorem of the preceding set).

  Hence,

$$\begin{aligned} \mu^*(E) &= \inf\{\textstyle\sum_{n=1}^{\infty} \mu(E_n) : E \subseteq \bigcup_{n=1}^{\infty} E_n, E_n \in \boldsymbol{R}, n = 1, 2, \ldots\} \\ &\geq \inf\{\textstyle\sum_{n=1}^{\infty} \overline{\mu}(E_n) : E \subseteq \bigcup_{n=1}^{\infty} E_n, E_n \in \boldsymbol{S}(\boldsymbol{R}), \\ &\qquad n = 1, 2, \ldots\}. \end{aligned}$$

The Outer Measure and the Measure it Induces (Cont'd)

- Note that the following also hold:
  - Every sequence $\{E_n\}$ of sets in $\boldsymbol{S}(\boldsymbol{R})$ for which $E \subseteq \bigcup_{n=1}^{\infty} E_n = F$ may be replaced by a disjoint sequence with the same property, without increasing the sum of the measures of the terms of the sequence;
  - $\overline{\mu}(F) = \mu^*(F)$ for F in $\overline{\boldsymbol{S}}$, by the definition of $\overline{\mu}$.

Hence, we get

$$\begin{aligned} \mu^*(E) &\geq \inf\{\overline{\mu}(F) : E \subseteq F \in \boldsymbol{S}(\boldsymbol{R})\} \\ &\geq \inf\{\overline{\mu}(F) : E \subseteq F \in \overline{\boldsymbol{S}}\} \\ &\geq \mu^*(E). \end{aligned}$$

Measurable Cover

- Suppose $E \in \boldsymbol{H}(\boldsymbol{R})$ and $F \in \boldsymbol{S}(\boldsymbol{R})$.
- We say that F is a **measurable cover** of E if:
  - $E \subseteq F$;
  - For every set G in $\boldsymbol{S}(\boldsymbol{R})$ for which $G \subseteq F - E$,
$$\overline{\mu}(G) = 0.$$
- Loosely speaking, a measurable cover of a set E in $\boldsymbol{H}(\boldsymbol{R})$ is a minimal set in $\boldsymbol{S}(\boldsymbol{R})$ which covers E.

Measurable Cover for Sets of σ-Finite Outer Measure

Theorem

If a set E in $\boldsymbol{H}(\boldsymbol{R})$ is of σ-finite outer measure, then there exists a set F in $\boldsymbol{S}(\boldsymbol{R})$, such that $\mu^*(E) = \overline{\mu}(F)$ and F is a measurable cover of E.

- If $\mu^*(E) = \infty$, and $E \subseteq F \in \boldsymbol{S}(\boldsymbol{R})$, then clearly $\overline{\mu}(F) = \infty$.

  So it is sufficient to prove the assertion $\mu^*(E) = \overline{\mu}(F)$ only in the case in which $\mu^*(E) < \infty$.

  Since a set of σ-finite outer measure is a countable disjoint union of sets of finite outer measure, it is sufficient to prove the entire theorem under the added assumption that $\mu^*(E) < \infty$.

Proof of the Theorem

- By the preceding theorem, for every $n = 1, 2, \ldots$, there exists a set F_n in $\boldsymbol{S}(\boldsymbol{R})$, such that

$$E \subseteq F_n \quad \text{and} \quad \overline{\mu}(F_n) \leq \mu^*(E) + \frac{1}{n}.$$

Write $F = \bigcap_{n=1}^{\infty} F_n$.
Then $E \subseteq F \in \boldsymbol{S}(\boldsymbol{R})$ and

$$\mu^*(E) \leq \overline{\mu}(F) \leq \overline{\mu}(F_n) \leq \mu^*(E) + \frac{1}{n}.$$

Since n is arbitrary, $\mu^*(E) = \overline{\mu}(F)$.
If $G \in \boldsymbol{S}(\boldsymbol{R})$ and $G \subseteq F - E$, then $E \subseteq F - G$.
Therefore,

$$\overline{\mu}(F) = \mu^*(E) \leq \overline{\mu}(F - G) = \overline{\mu}(F) - \overline{\mu}(G) \leq \overline{\mu}(F).$$

The fact that F is a measurable cover of E follows from the finiteness of $\overline{\mu}(F)$.

Outer Measure, Induced Measure and Measurable Covers

Theorem

If $E \in \boldsymbol{H}(\boldsymbol{R})$ and F is a measurable cover of E, then $\mu^*(E) = \overline{\mu}(F)$. If both F_1 and F_2 are measurable covers of E, then $\overline{\mu}(F_1 \triangle F_2) = 0$.

- Note $E \subseteq F_1 \cap F_2 \subseteq F_1$ implies $F_1 - (F_1 \cap F_2) \subseteq F_1 - E$.

  Since F_1 is a measurable cover of E, $\overline{\mu}(F_1 - (F_1 \cap F_2)) = 0$.

  Similarly, $\overline{\mu}(F_2 - (F_1 \cap F_2)) = 0$.

  Thus, we have $\overline{\mu}(F_1 \triangle F_2) = 0$.

  If $\mu^*(E) = \infty$, then the relation $\mu^*(E) = \overline{\mu}(F)$ is trivial.

  If $\mu^*(E) < \infty$, then it follows from the preceding theorem that there exists a measurable cover F_0 of E with $\overline{\mu}(F_0) = \mu^*(E)$.

  The result just shown implies that every two measurable covers have the same measure.

σ-Finiteness of Induced Measures

Theorem

If μ on $\boldsymbol{R}$ is σ-finite, then so are the measures $\overline{\mu}$ on $\boldsymbol{S}(\boldsymbol{R})$ and $\overline{\mu}$ on $\overline{\boldsymbol{S}}$.

- We know that, if μ is σ-finite, then so is μ^*.

  Hence, for every E in $\overline{\boldsymbol{S}}$, there exists a sequence $\{E_i\}$ of sets in $\boldsymbol{H}(\boldsymbol{R})$, such that $E \subseteq \bigcup_{i=1}^{\infty} E_i$ and $\mu^*(E_i) < \infty$, $i = 1, 2, \ldots$.

  An application of the third theorem of the set to each set E_i concludes the proof.

Induced and Regular Outer Measures

- Suppose that:
  - We start with an outer measure μ^*.
  - We form the induced measure $\overline{\mu}$.
  - Then form the outer measure $\overline{\mu}^*$ induced by $\overline{\mu}$.
- In general μ^* and $\overline{\mu}^*$ are not equal.
- If the induced outer measure $\overline{\mu}^*$ does coincide with the original outer measure μ^*, then μ^* is called **regular**.
- The assertion of the second theorem is exactly that the outer measure induced by a measure on a ring is always regular.
- The converse of this last statement is also true:

  If μ^* is regular, then $\mu^* = \overline{\mu}^*$ is induced by a measure on a ring, namely by $\overline{\mu}$ on the class of μ^*-measurable sets.

- Thus, the notions of induced outer measure and regular outer measure are coextensive.

Subsection 2

Extension, Completion and Approximation

The Extension Theorem

Theorem

If μ is a σ-finite measure on a ring $\boldsymbol{R}$, then there is a unique measure $\overline{\mu}$ on the σ-ring $\boldsymbol{S}(\boldsymbol{R})$, such that, for E in $\boldsymbol{R}$, $\overline{\mu}(E) = \mu(E)$. The measure $\overline{\mu}$ is σ-finite.

- The measure $\overline{\mu}$ is called the **extension** of μ. Unless confusion is likely, we write $\mu(E)$ instead of $\overline{\mu}(E)$ even for sets E in $\boldsymbol{S}(\boldsymbol{R})$.
- The existence of $\overline{\mu}$ (even without the restriction of σ-finiteness) has been established.

  To prove uniqueness, suppose that μ_1 and μ_2 are two measures on $\boldsymbol{S}(\boldsymbol{R})$, such that $\mu_1(E) = \mu_2(E)$, whenever $E \in \boldsymbol{R}$. Let $\boldsymbol{M}$ be the class of all sets E in $\boldsymbol{S}(\boldsymbol{R})$ for which $\mu_1(E) = \mu_2(E)$.
  - If one of the two measures is finite, and if $\{E_n\}$ is a monotone sequence of sets in $\boldsymbol{M}$, then, since $\mu_i(\lim_n E_n) = \lim_n \mu_i(E_n)$, $i = 1, 2$, we have $\lim_n E_n \in \boldsymbol{M}$. Therefore, $\boldsymbol{M}$ is a monotone class. But $\boldsymbol{M}$ contains $\boldsymbol{R}$, whence $\boldsymbol{M}$ contains $\boldsymbol{S}(\boldsymbol{R})$.

Proof of the Extension Theorem

- For the general, not necessarily finite, case, let A be any fixed set in $\boldsymbol{R}$, of finite measure with respect to one of the two measures μ_1 and μ_2. Since $\boldsymbol{R} \cap A$ is a ring and $\boldsymbol{S}(\boldsymbol{R}) \cap A$ is the σ-ring it generates, it follows that the reasoning of the preceding paragraph applies to $\boldsymbol{R} \cap A$ and $\boldsymbol{S}(\boldsymbol{R}) \cap A$, and proves that if $E \in \boldsymbol{S}(\boldsymbol{R}) \cap A$, then $\mu_1(E) = \mu_2(E)$. Since every E in $\boldsymbol{S}(\boldsymbol{R})$ may be covered by a countable, disjoint union of sets of finite measure in $\boldsymbol{R}$ (with respect to either of the measures μ_1 and μ_2), the proof is complete.

- The extension procedure employed in the proofs of the preceding subsection yields slightly more than what the theorem states:

  The given measure μ can actually be extended to a class (the class of all μ^*-measurable sets) which is in general larger than the generated σ-ring.
- It turns out it is not necessary to make use of the theory of outer measures in order to obtain this slight enlargement of the domain of μ.

Second Extension Theorem

Theorem

If μ is a measure on a σ-ring $\boldsymbol{S}$, then the class $\overline{\boldsymbol{S}}$ of all sets of the form $E \triangle N$, where $E \in \boldsymbol{S}$ and N is a subset of a set of measure zero in $\boldsymbol{S}$, is a σ-ring, and the set function $\overline{\mu}$, defined by $\overline{\mu}(E \triangle N) = \mu(E)$ is a complete measure on $\overline{\boldsymbol{S}}$.

- The measure $\overline{\mu}$ is called the **completion** of μ.
- If $E \in \boldsymbol{S}$, $N \subseteq A \in \boldsymbol{S}$, and $\mu(A) = 0$, then
  - $E \cup N = (E - A) \triangle [A \cap (E \cup N)]$;
  - $E \triangle N = (E - A) \cup [A \cap (E \triangle N)]$.

  Thus, $\overline{\boldsymbol{S}}$ may also be described as the class of all sets of the form $E \cup N$, where $E \in \boldsymbol{S}$ and N is a subset of a set of measure zero in $\boldsymbol{S}$. Since this implies that the class $\overline{\boldsymbol{S}}$, which is obviously closed under the formation of symmetric differences, is closed also under the formation of countable unions, it follows that $\overline{\boldsymbol{S}}$ is a σ-ring.

Proof of the Second Extension Theorem

- If $E_1 \triangle N_1 = E_2 \triangle N_2$, where $E_i \in \boldsymbol{S}$ and N_i is a subset of a set of measure zero in $\boldsymbol{S}$, $i = 1, 2$, then $E_1 \triangle E_2 = N_1 \triangle N_2$. Therefore $\mu(E_1 \triangle E_2) = 0$. It follows that $\mu(E_1) = \mu(E_2)$. Hence, $\overline{\mu}$ is unambiguously defined by the relations

$$\overline{\mu}(E \triangle N) = \overline{\mu}(E \cup N) = \mu(E).$$

Using the union (instead of the symmetric difference) representation of sets in $\overline{\boldsymbol{S}}$, it is easy to verify that $\overline{\mu}$ is a measure.

The completeness of $\overline{\mu}$ is an immediate consequence of the fact that $\overline{\boldsymbol{S}}$ contains all subsets of sets of measure zero in $\boldsymbol{S}$.

Completion and Complete Extension

Theorem

If μ is a σ-finite measure on a ring $\boldsymbol{R}$, and if μ^* is the outer measure induced by μ, then the completion of the extension of μ to $\boldsymbol{S}(\boldsymbol{R})$ is identical with μ^* on the class of all μ^*-measurable sets.

- Let us denote the class of all μ^*-measurable sets by $\boldsymbol{S}^*$ and the domain of the completion $\overline{\mu}$ of μ by $\overline{\boldsymbol{S}}$. Since μ^* on $\boldsymbol{S}^*$ is a complete measure, it follows that $\overline{\boldsymbol{S}}$ is contained in $\boldsymbol{S}^*$ and that $\overline{\mu}$ and μ^* coincide on $\overline{\boldsymbol{S}}$.

  It suffces to show that $\boldsymbol{S}^*$ is contained in $\overline{\boldsymbol{S}}$. In view of the σ-finiteness of μ^* on $\boldsymbol{S}^*$, it is sufficient to prove that if $E \in \boldsymbol{S}^*$ and $\mu^*(E) < \infty$, then $E \in \overline{\boldsymbol{S}}$.

Completion and Complete Extension (Cont'd)

- If $E \in \boldsymbol{S}^*$ and $\mu^*(E) < \infty$, then $E \in \overline{\boldsymbol{S}}$.

  By the third theorem of the preceding subsection, E has a measurable cover F. Since $\mu^*(F) = \mu(F) = \mu^*(E)$, it follows from the finiteness of $\mu^*(E)$ and the fact that μ^* is a measure on $\boldsymbol{S}^*$, that $\mu^*(F - E) = 0$. But $F - E$ also has a measurable cover G, and $\mu(G) = \mu^*(F - E) = 0$. Thus, the relation $E = (F - G) \cup (E \cap G)$ exhibits E as a union of a set in $\boldsymbol{S}(\boldsymbol{R})$ and a set which is a subset of a set of measure zero in $\boldsymbol{S}(\boldsymbol{R})$. This shows that $E \in \overline{\boldsymbol{S}}$, and thus completes the proof.
- Loosely speaking, the theorem says that in the σ-finite case the σ-ring of all μ^*-measurable sets and the generated σ-ring $\boldsymbol{S}(\boldsymbol{R})$ are not very different: Every μ^*-measurable set suitably modified by a set of measure zero belongs to $\boldsymbol{S}(\boldsymbol{R})$.

Measure and Extension to the Generated σ-Ring

Theorem

If μ is a σ-finite measure on a ring $\boldsymbol{R}$, then, for every set E of finite measure in $\boldsymbol{S}(\boldsymbol{R})$ and for every positive number ϵ, there exists a set E_0 in $\boldsymbol{R}$ such that $\mu(E \triangle E_0) \leq \epsilon$.

- Results of the preceding three subsections, together with the first theorem, imply that
$$\mu(E) = \inf\left\{\sum_{i=1}^{\infty}\mu(E_i) : E \subseteq \bigcup_{i=1}^{\infty}E_i, E_i \in \boldsymbol{R}, i = 1, 2, \ldots\right\}.$$
Consequently there exists a sequence $\{E_i\}$ of sets in $\boldsymbol{R}$, such that $E \subseteq \bigcup_{i=1}^{\infty} E_i$ and $\mu(\bigcup_{i=1}^{\infty} E_i) \leq \mu(E) + \frac{\epsilon}{2}$. Since $\lim_n \mu(\bigcup_{i=1}^{n} E_i) = \mu(\bigcup_{i=1}^{\infty} E_i)$, there exists a positive integer n, such that if $E_0 = \bigcup_{i=1}^{n} E_i$, then $\mu(\bigcup_{i=1}^{\infty} E_i) \leq \mu(E_0) + \frac{\epsilon}{2}$. Clearly $E_0 \in \boldsymbol{R}$. Since $\mu(E - E_0) \leq \mu(\bigcup_{i=1}^{\infty} E_i - E_0) = \mu(\bigcup_{i=1}^{\infty} E_i) - \mu(E_0) \leq \frac{\epsilon}{2}$ and $\mu(E_0 - E) \leq \mu(\bigcup_{i=1}^{\infty} E_i - E) = \mu(\bigcup_{i=1}^{\infty} E_i) - \mu(E) \leq \frac{\epsilon}{2}$, the proof of is complete.

Subsection 3

Inner Measures

The Inner Measure Induced by a Measure

- We have seen that if μ is a measure on a σ-ring $\boldsymbol{S}$, then the set function μ^* defined for every E in the hereditary σ-ring $\boldsymbol{H}(\boldsymbol{S})$ by $\mu^*(E) = \inf\{\mu(F) : E \subseteq F \in \boldsymbol{S}\}$ is an outer measure.

  In the σ-finite case the induced measure $\overline{\mu}$ on the σ-ring $\overline{\boldsymbol{S}}$ of all μ^*-measurable sets is the completion of μ.
- Analogously we now define the **inner measure** μ_* **induced by** μ: For every E in $\boldsymbol{H}(\boldsymbol{S})$, we write

$$\mu_*(E) = \sup\{\mu(F) : E \supseteq F \in \boldsymbol{S}\}.$$

- We study μ_* and its relation to μ^*.

  We show that the properties of μ_* are in a very legitimate sense the duals of those of μ^*.
- It is very easy to see that the set function μ_* is non negative, monotone, and such that $\mu_*(\emptyset) = 0$.

Notation and Characterization of the Inner Measure

- We adopt the following general assumptions:

  μ is a σ-finite measure on a σ-ring $\boldsymbol{S}$, μ^* and μ_* are the outer measure and the inner measure induced by μ, respectively, and $\overline{\mu}$ on $\overline{\boldsymbol{S}}$ is the completion of μ.

- Recall that $\overline{\mu}$ on $\overline{\boldsymbol{S}}$ coincides with μ^* on the class of all μ^*-measurable sets.

Theorem

If $E \in \boldsymbol{H}(\boldsymbol{S})$, then $\mu_*(E) = \sup\{\overline{\mu}(F) : E \supseteq F \in \overline{\boldsymbol{S}}\}$.

- Since $\boldsymbol{S} \subseteq \overline{\boldsymbol{S}}$, it is clear from the definition of μ_* that $\mu_*(E) \leq \sup\{\mu(F) : E \supseteq F \in \overline{\boldsymbol{S}}\}$.

  On the other hand the second theorem of the preceding subsection implies that, for every F in $\overline{\boldsymbol{S}}$, there is a G in $\boldsymbol{S}$, with $G \subseteq F$ and $\overline{\mu}(F) = \mu(G)$. Since this means that every value of $\overline{\mu}$ on subsets of E in $\overline{\boldsymbol{S}}$ is also attained by μ on subsets of E in $\boldsymbol{S}$, the proof is complete.

Measurable Kernels

- If $E \in \boldsymbol{H}(\boldsymbol{S})$ and $F \in \boldsymbol{S}$, we say that F is a **measurable kernel** of E if $F \subseteq E$ and if, for every set G in $\boldsymbol{S}$ for which $G \subseteq E - F$, we have $\mu(G) = 0$.
- Loosely speaking a measurable kernel of a set E in $\boldsymbol{H}(\boldsymbol{S})$ is a maximal set in $\boldsymbol{S}$ which is contained in E.

Theorem

Every set E in $\boldsymbol{H}(\boldsymbol{S})$ has a measurable kernel.

- Let $\hat{E}$ be a measurable cover of E, let N be a measurable cover of $\hat{E} - E$ and write $F = \hat{E} - N$. We have
$$F = \hat{E} - N \subseteq \hat{E} - (\hat{E} - E) = E.$$
Moreover, if $G \subseteq E - F$,
$$G \subseteq E - (\hat{E} - N) = E \cap N \subseteq N - (\hat{E} - E).$$
It follows (since N is a measurable cover of $\hat{E} - E$), that F is a measurable kernel of E.

Measure and Measurable Kernels

Theorem

If $E \in \boldsymbol{H}(\boldsymbol{S})$, and F is a measurable kernel of E, then $\mu(F) = \mu_*(E)$. If both F_1 and F_2 are measurable kernels of E, then $\mu(F_1 \triangle F_2) = 0$.

- Since $F \subseteq E$, it is clear that $\mu(F) \leq \mu_*(E)$. If $\mu(F) < \mu_*(E)$, then $\mu(F)$ is finite and, by the definition of $\mu_*(E)$, there exists a set F_0 in $\boldsymbol{S}$, such that $F_0 \subseteq E$ and $\mu(F_0) > \mu(F)$. Since $F_0 - F \subseteq E - F$ and $\mu(F_0 - F) \geq \mu(F_0) - \mu(F) > 0$, this contradicts the hypothesis. Thus, $\mu(F) = \mu_*(E)$.

  Since the relation $F_1 \subseteq F_1 \cup F_2 \subseteq E$ implies that $(F_1 \cup F_2) - F_1 \subseteq E - F_1$, it follows from the fact that F_1 is a measurable kernel of E that $\mu((F_1 \cup F_2) - F_1) = 0$. Similarly, $\mu((F_1 \cup F_2) - F_2) = 0$. Therefore, $\mu(F_1 \triangle F_2) = 0$.

The Outer Measure on the Hereditary Ring

Theorem

If $\{E_n\}$ is a disjoint sequence of sets in $\boldsymbol{H}(\boldsymbol{S})$, then $\mu_*(\bigcup_{n=1}^{\infty} E_n) \geq \sum_{n=1}^{\infty} \mu_*(E_n)$.

- If F_n is a measurable kernel of E_n, $n = 1, 2, \ldots$, the countable additivity of μ implies that $\sum_{n=1}^{\infty} \mu_*(E_n) = \sum_{n=1}^{\infty} \mu(F_n) = \mu(\bigcup_{n=1}^{\infty} F_n) \leq \mu_*(\bigcup_{n=1}^{\infty} E_n)$.

Theorem

If $A \in \boldsymbol{H}(\boldsymbol{S})$ and if $\{E_n\}$ is a disjoint sequence of sets in $\overline{\boldsymbol{S}}$, with $\bigcup_{n=1}^{\infty} E_n = E$, then $\mu_*(A \cap E) = \sum_{n=1}^{\infty} \mu_*(A \cap E_n)$.

- If F is a measurable kernel of $A \cap E$, then $\mu_*(A \cap E) = \mu(F) = \sum_{n=1}^{\infty} \overline{\mu}(F \cap E_n) \leq \sum_{n=1}^{\infty} \mu_*(A \cap E_n)$. The preceding theorem concludes the proof.

Measure and Inner and Outer Measure

Theorem

If $E \in \overline{\boldsymbol{S}}$, then $\mu^*(E) = \mu_*(E) = \overline{\mu}(E)$. Conversely, if $E \in \boldsymbol{H}(\boldsymbol{S})$ and $\mu^*(E) = \mu_*(E) < \infty$, then $E \in \overline{\boldsymbol{S}}$.

- If $E \in \overline{\boldsymbol{S}}$, then both the supremum in the first theorem and the infimum seen previously are attained by $\overline{\mu}(E)$.

  To prove the converse, let A and B be a measurable kernel and a measurable cover of E, respectively. Since $\mu(A) = \mu_*(E) < \infty$, we have $\mu(B - A) = \mu(B) - \mu(A) = \mu^*(E) - \mu_*(E) = 0$. The desired conclusion follows from the completeness of $\overline{\mu}$ on $\overline{\boldsymbol{S}}$.

Relation Between Inner and Outer Measures

Theorem

If E and F are disjoint sets in $\boldsymbol{H}(\boldsymbol{S})$, then

$$\mu_*(E \cup F) \leq \mu_*(E) + \mu^*(F) \leq \mu^*(E \cup F).$$

- Let A be a measurable cover of F and let B be a measurable kernel of $E \cup F$. Since $B - A \subseteq E$, it follows that
  $\mu_*(E \cup F) = \mu(B) \leq \mu(B - A) + \mu(A) \leq \mu_*(E) + \mu^*(F)$.
  Dually, let A be a measurable kernel of E and let B be a measurable cover of $E \cup F$. Since $B - A \supseteq F$, it follows that
  $\mu^*(E \cup F) = \mu(B) = \mu(A) + \mu(B - A) \geq \mu_*(E) + \mu^*(F)$.

Theorem

If $E \in \overline{\boldsymbol{S}}$, then, for every subset A of X,

$$\mu_*(A \cap E) + \mu^*(A' \cap E) = \overline{\mu}(E).$$

- Applying the preceding theorem to $A \cap E$ and $A' \cap E$, we obtain $\mu_*(E) \leq \mu_*(A \cap E) + \mu^*(A' \cap E) \leq \mu^*(E)$. Since $E \in \overline{\boldsymbol{S}}$, we have, by the pre-preceding theorem, $\mu_*(E) = \mu^*(E) = \overline{\mu}(E)$.

Alternative Approach to the Extension Theorem

- If μ is a σ-finite measure on a ring $\boldsymbol{R}$, and if μ^* is the induced outer measure on $\boldsymbol{H}(\boldsymbol{R})$, then, for every set E in $\boldsymbol{R}$, with $\mu(E) < \infty$, and for every A in $\boldsymbol{H}(\boldsymbol{R})$, we have

$$\mu_*(A \cap E) = \mu(E) - \mu^*(A' \cap E).$$

- If we prove that whenever E and F are two sets of finite measure in $\boldsymbol{R}$,

$$A \cap E = A \cap F \quad \text{implies} \quad \mu(E) - \mu^*(A' \cap E) = \mu(F) - \mu^*(A' \cap F),$$

  then we may use the equation for $\mu_*(A \cap E)$ as a definition of inner measure.
- We may then define a set E in $\boldsymbol{H}(\boldsymbol{R})$ of finite outer measure to be μ^*-measurable if and only if $\mu^*(E) = \mu_*(E)$.

Subsection 4

Lebesgue Measure

Borel Sets

- The purpose of this section is to apply the general extension theory to the special measure based on the ring of the semi-closed intervals.
- We assume that:
  - X is the real line;
  - $\boldsymbol{P}$ is the class of all bounded, semi-closed intervals of the form $[a, b)$;
  - $\boldsymbol{S}$ is the σ-ring generated by $\boldsymbol{P}$;
  - μ is the set function on $\boldsymbol{P}$ defined by $\mu([a, b)) = b - a$.
- The sets of the σ-ring $\boldsymbol{S}$ are called the **Borel sets** of the line.
- According to the extension theorems, we may assume that μ is defined for all Borel sets.

Lebesgue Measure

- We assume that μ is defined for all Borel sets.
- If $\overline{\mu}$ on $\overline{\boldsymbol{S}}$ is the completion of μ on $\boldsymbol{S}$, the sets of $\overline{\boldsymbol{S}}$ are the **Lebesgue measurable sets** of the line.
- The measure $\overline{\mu}$ is the **Lebesgue measure**.
- The incomplete measure μ^* on the class $\boldsymbol{S}$ of all Borel sets is usually called **Lebesgue measure** also.
- Since the entire line X is the union of countably many sets in $\boldsymbol{P}$, we see that $X \in \boldsymbol{S}$. So the σ-rings $\boldsymbol{S}$ and $\overline{\boldsymbol{S}}$ are σ-algebras.
- Since $\mu(X) = \infty$, μ is not finite on $\boldsymbol{S}$.

  But μ^* is finite on $\boldsymbol{P}$.

  So both μ on $\boldsymbol{S}$ and $\overline{\mu}$ on $\overline{\boldsymbol{S}}$ are totally σ-finite.

Countable Borel Sets are of Measure Zero

Theorem

Every countable set is a Borel set of measure zero.

- For any a, $-\infty < a < \infty$, we have

$$\{a\} = \{x : x = a\} = \bigcap_{n=1}^{\infty} \{x : a \leq x < a + \frac{1}{n}\}.$$

Therefore,

$$\mu(\{a\}) = \lim_n \mu([a, a + \frac{1}{n})) = \lim_n \frac{1}{n} = 0.$$

Thus, every one-point set is a Borel set of measure zero.

Since the Borel sets form a σ-ring and since μ is countably additive, the theorem follows.

Alternative Characterization of Borel Sets

Theorem

The class $\boldsymbol{S}$ of all Borel sets coincides with the σ-ring generated by the class $\boldsymbol{U}$ of all open sets.

- Since, for every real number a, the set $\{a\}$ is a Borel set, it follows from the relation $(a, b) = [a, b) - \{a\}$, that every bounded open interval is a Borel set.

  Since every open set on the line is a countable union of bounded open intervals, it follows that $\boldsymbol{S} \supseteq \boldsymbol{U}$. Consequently, $\boldsymbol{S} \supseteq \boldsymbol{S}(\boldsymbol{U})$.

  To prove the reverse inequality, we observe that, for every real number a, $\{a\} = \bigcap_{n=1}^{\infty}(a - \frac{1}{n}, a + \frac{1}{n})$, so that $\{a\} \in \boldsymbol{S}(\boldsymbol{U})$. It follows from the relation $[a, b) = (a, b) \cup \{a\}$ that $\boldsymbol{P} \subseteq \boldsymbol{S}(\boldsymbol{U})$. Consequently, $\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{P}) \subseteq \boldsymbol{S}(\boldsymbol{U})$.

Calculation of Outer Measure Based on Open Sets

Theorem

If $\boldsymbol{U}$ is the class of all open sets, then, for every E in X,

$$\mu^*(E) = \inf\{\mu(U) : E \subseteq U \in \boldsymbol{U}\}.$$

- Since $\mu^*(E) = \inf\{\mu(F) : E \subseteq F \in \boldsymbol{S}\}$, it follows from the fact that $\boldsymbol{U} \subseteq \boldsymbol{S}$ that $\mu^*(E) \leq \inf\{\mu(U) : E \subseteq U \in \boldsymbol{U}\}$.

  If, on the other hand, ϵ is any positive number, then it follows from the definition of μ^* that there exists a sequence $\{[a_n, b_n)\}$ of sets in $\boldsymbol{P}$, such that $E \subseteq \bigcup_{n=1}^{\infty}[a_n, b_n)$ and $\sum_{n=1}^{\infty}(b_n - a_n) \leq \mu^*(E) + \frac{\epsilon}{2}$. Consequently $E \subseteq \bigcup_{n=1}^{\infty}(a_n - \frac{\epsilon}{2^{n+1}}, b_n) = U \in \boldsymbol{U}$ and $\mu(U) \leq \sum_{n=1}^{\infty}(b_n - a_n) + \frac{\epsilon}{2} \leq \mu^*(E) + \epsilon$.

  The desired result follows from the arbitrariness of ϵ.

Lebesgue Measure and Linear Transformations

Theorem

Let T be the one to one transformation of the entire real line onto itself, defined by $T(x) = \alpha x + \beta$, where α and β are real numbers and $\alpha \neq 0$. If, for every subset E of X, $T(E)$ denotes the set of all points of the form $T(x)$, with x in E, i.e. $T(E) = \{\alpha x + \beta : x \in E\}$, then

$$\mu^*(T(E)) = |\alpha|\mu^*(E) \quad \text{and} \quad \mu_*(T(E)) = |\alpha|\mu_*(E).$$

The set $T(E)$ is a Borel set or a Lebesgue measurable set if and only if E is a Borel set or a Lebesgue measurable set, respectively.

- It is sufficient to prove the theorem for $\alpha > 0$: If $\alpha < 0$, then the transformation T is the result of composing the transformations T_1 and T_2, $T(x) = T_1(T_2(x))$, where $T_1(x) = |\alpha|x + \beta$ and $T_2(x) = -x$. The transformation T_2 sends Borel sets and Lebesgue measurable sets into Borel sets and Lebesgue measurable sets, respectively, and preserves the inner and outer measures of all sets.

Lebesgue Measure and Linear Transformations (Cont'd)

- Suppose, now, that $\alpha > 0$, and let $T(\boldsymbol{S})$ be the class of all sets of the form $T(E)$ with E in $\boldsymbol{S}$. It is clear that $T(\boldsymbol{S})$ is a σ-ring. We are to prove that $T(\boldsymbol{S}) = \boldsymbol{S}$.
  - If $E = [a, b) \in \boldsymbol{P}$, then $E = T(F)$, where $F = [\frac{a-\beta}{\alpha}, \frac{b-\beta}{\alpha}) \in \boldsymbol{P}$. Thus, $E \in T(\boldsymbol{S})$ and, therefore, $\boldsymbol{S} \subseteq T(\boldsymbol{S})$.
  - By the same reasoning applied to $T^{-1}(x) = \frac{x-\beta}{\alpha}$, we may conclude that $\boldsymbol{S} \subseteq T^{-1}(\boldsymbol{S})$. Applying the transformation T to both sides, we obtain, $T(\boldsymbol{S}) \subseteq \boldsymbol{S}$.

  Therefore, $T(\boldsymbol{S}) = \boldsymbol{S}$.

  If, for every Borel set E we write $\mu_1(E) = \mu(T(E))$ and $\mu_2(E) = \alpha\mu(E)$, then both μ_1 and μ_2 are measures on $\boldsymbol{S}$.

  If $E = [a, b) \in \boldsymbol{P}$, then $T(E) = [\alpha a + \beta, \alpha b + \beta)$, and $\mu_1(E) = \mu(T(E)) = (\alpha b+\beta)-(\alpha a+\beta) = \alpha(b-a) = \alpha\mu(E) = \mu_2(E)$. Hence, $\mu(T(E)) = \alpha\mu(E)$, for every E in $\boldsymbol{S}$.

Lebesgue Measure and Transformations (Conclusion)

- Applying the results of the preceding two paragraphs to the transformation T^{-1}, we obtain the relations

$$\begin{aligned}\mu^*(T(E)) &= \inf\{\mu(F) : T(E) \subseteq F \in \boldsymbol{S}\} \\ &= \inf\{\alpha\mu(T^{-1}(F)) : E \subseteq T^{-1}(F) \in \boldsymbol{S}\} \\ &= \alpha \inf\{\mu(G) : E \subseteq G \in \boldsymbol{S}\} \\ &= \alpha\mu^*(E),\end{aligned}$$

and, replacing inf by sup, μ^* by μ_*, and $\subseteq$ by $\supseteq$ throughout, $\mu_*(T(E)) = \alpha\mu_*(E)$, for every set E.
If E is a Lebesgue measurable set and A is any set, then

$$\begin{aligned}&\mu^*(A \cap T(E)) + \mu^*(A \cap (T(E))') \\ &\qquad = \mu^*(T(T^{-1}(A) \cap E)) + \mu^*(T(T^{-1}(A) \cap E')) \\ &\qquad = \alpha[\mu^*(T^{-1}(A) \cap E) + \mu^*(T^{-1}(A) \cap E')] \\ &\qquad = \alpha\mu^*(T^{-1}(A)) = \mu^*(A),\end{aligned}$$

so that $T(E)$ is Lebesgue measurable. This result applied to T^{-1} proves its own converse and completes the proof of the theorem.

Subsection 5

Non-Measurable Sets

Measurable Sets and Open Intervals

- If E is any subset of the real line and a is any real number, then $E + a$ denotes the set of all numbers of the form $x + a$, with x in E.
- If E and F are both subsets of the real line, then $E + F$ denotes the set of all numbers of the form $x + y$ with x in E and y in F.
- The symbol $D(E)$ will be used to denote the **difference set** of E, i.e., the set of all numbers of the form $x - y$ with x in E and y in E.

Theorem

If E is a Lebesgue measurable set of positive, finite measure, $0 \leq \alpha < 1$, then there exists an open interval U, such that $\overline{\mu}(E \cap U) \geq \alpha\mu(U)$.

- Let $\boldsymbol{U}$ be the class of all open sets. $\overline{\mu}(E) = \inf\{\mu(U) : E \subseteq U \in \boldsymbol{U}\}$, implies the existence of an open set U_0, such that $E \subseteq U_0$ and $\alpha\mu(U_0) \leq \overline{\mu}(E)$. If $\{U_n\}$ is the disjoint sequence of open intervals whose union is U_0, then $\alpha \sum_{n=1}^{\infty} \mu(U_n) \leq \sum_{n=1}^{\infty} \overline{\mu}(E \cap U_n)$. Consequently, we must have $\alpha\mu(U_n) \leq \overline{\mu}(E \cap U_n)$, for at least one value of n. The interval U_n may be chosen for U.

Different Sets and Open Intervals

Theorem

If E is a Lebesgue measurable set of positive measure, then there exists an open interval, containing 0, lying entirely in the difference set $D(E)$.

- If E is, or at least contains, an open interval, the result is trivial.
- In the general case we make use of the preceding theorem, which asserts that a suitable subset of E is arbitrarily close to an interval, to find a bounded open interval U, such that $\overline{\mu}(E \cap U) \geq \frac{3}{4}\mu(U)$. If $-\frac{1}{2}\mu(U) < x < \frac{1}{2}\mu(U)$, then the set $(E \cap U) \cup ((E \cap U) + x)$ is contained in an interval, namely $U \cup (U + x)$, whose length is less than $\frac{3}{2}\mu(U)$. If $E \cap U$ and $(E \cap U) + x$ were disjoint, then, since they have the same measure, we should have $\overline{\mu}((E \cap U) \cup [(E \cap U) + x]) = 2\overline{\mu}(E \cap U) \geq \frac{3}{2}\mu(U)$. Hence, at least one point of $E \cap U$ belongs also to $(E \cap U) + x$, which proves that $x \in D(E)$. I.e., the interval $(-\frac{1}{2}\mu(U), \frac{1}{2}\mu(U))$ satisfies the conditions stated in the theorem.

A Dense Set on the Real Line

Theorem

If ξ is an irrational number, then the set A of all numbers of the form $n+m\xi$, where n and m are arbitrary integers, is everywhere dense on the line. The same is true of the subset B of all numbers of the form $n+m\xi$ with n even, and the subset C of numbers of the form $n+m\xi$ with n odd.

- For every positive integer i, there exists a unique integer n_i (positive, negative, or zero), such that $0 \leq n_i + i\xi < 1$. We write $x_i = n_i + i\xi$. If U is any open interval, then there is a positive integer k, such that $\mu(U) > \frac{1}{k}$. Among the $k+1$ numbers $x_1, x_2, \ldots, x_{k+1}$ in the unit interval, there must be at least two, say x_i and x_j, such that $|x_i - x_j| < \frac{1}{k}$. It follows that some integral multiple of $x_i - x_j$, i.e., some element of A, belongs to the interval U.

  For B, we have to replace the unit interval by the interval $[0,2)$.

  For C, note that $C = B + 1$.

Existence of Nonmeasurable Sets

Theorem

There exists at least one set E_0 which is not Lebesgue measurable.

- For any two real numbers x and y, we write $x \sim y$ if $x - y \in A$, where A is the set described in the preceding theorem:

$$A = \{n + m\xi : n, m \in \mathbb{Z}\},\ \xi \text{ a fixed irrational.}$$

The relation $\sim$ is reflexive, symmetric and transitive. Therefore, the set of all real numbers is the union of a disjoint class of sets, each set consisting of all those numbers which are in the relation $\sim$ with a given number. By the axiom of choice, we may find a set E_0 containing exactly one point from each such set.

Existence of Nonmeasurable Sets

Claim

E_0 is not measurable.

Suppose that F is a Borel set, such that $F \subseteq E_0$. Since the difference set $D(F)$ cannot contain any non zero elements of the dense set A, it follows from the second theorem that F must have measure zero, so that $\mu_*(E_0) = 0$. I.e., if E_0 is Lebesgue measurable, then its measure must be zero. But, if a_1 and a_2 are two different elements of A, then the sets $E_0 + a_1$ and $E_0 + a_2$ are disjoint:

If $x_1 + a_1 = x_2 + a_2$, with x_1 in E_0 and x_2 in E_0, then $x_1 - x_2 = a_2 - a_1 \in A$.

The countable class of sets of the form $E_0 + a$, where $a \in A$, covers the entire real line, i.e., $E_0 + A = X$. Moreover, the Lebesgue measurability of E_0 would imply that each $E_0 + a$ is Lebesgue measurable and of the same measure as E_0. Hence, the Lebesgue measurability of E_0 would imply the nonsensical result $\mu(X) = 0$.

Refinement on the Existence of Nonmeasurable Sets

Theorem

There exists a subset M of the real line such that, for every Lebesgue measurable set E, $\mu_*(M \cap E) = 0$ and $\mu^*(M \cap E) = \overline{\mu}(E)$.

- Write $A = B \cup C$ (A, B, C as in the third theorem), and, if E_0 is the set constructed in the proof of the preceding theorem, write $M = E_0 + B$. If F is a Borel set, such that $F \subseteq M$, then the difference set $D(F)$ cannot contain any elements of the dense set C. It follows from the second theorem that $\mu_*(M) = 0$. The relations $M' = E_0 + C = E_0 + (B + 1) = M + 1$ imply that $\mu_*(M') = 0$. If E is any Lebesgue measurable set, then the monotone character of μ_* implies that $\mu_*(M \cap E) = \mu_*(M' \cap E) = 0$. Since, by a preceding theorem, $\mu_*(M' \cap E) + \mu^*(M \cap E) = \overline{\mu}(E)$, $\mu^*(M \cap E) = \overline{\mu}(E)$.

Impossibility of Extension

- The preceding results imply that it is impossible to extend Lebesgue measure to the class of all subsets of the real line so that the extended set function is still a measure and is invariant under translations.