Introduction to Measure Theory

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 422

Subsection 1

Measure Spaces

Measurable Spaces

- A **measurable space** is a set X and a σ-ring $\boldsymbol{S}$ of subsets of X with the property that $\bigcup \boldsymbol{S} = X$.
- Usually, we denote a measurable space by the same symbol as the underlying set X.

  On the occasions when attention is focused on the particular σ-ring under consideration, we write $(X, \boldsymbol{S})$ for X.
- We call a subset E of X **measurable** if and only if it belongs to the σ-ring $\boldsymbol{S}$.

  This is not meant to indicate that $\boldsymbol{S}$ is the σ-ring of all μ^*-measurable sets with respect to some outer measure μ^*, nor even that a non trivial measure is or may be defined on $\boldsymbol{S}$.

Measurable Spaces: A Comment

- In the language of measurable sets, the condition in the definition of measurable spaces may be expressed by saying that the union of all measurable sets is the entire space.

  Equivalently, every point is contained in some measurable set.
- The purpose of this restriction is to eliminate certain obvious and not at all useful pathological considerations.

Measure Spaces

- A **measure space** is a measurable space $(X, \boldsymbol{S})$ and a measure μ on $\boldsymbol{S}$.
- We also confuse a measure space whose underlying set is X with the set X.

  To focus on the particular σ-ring and measure under consideration, we write $(X, \boldsymbol{S}, \mu)$ for X.
- The measure space X is called **[totally] finite**, σ-**finite**, or **complete**, according as the measure μ is [totally] finite, σ-finite, or complete.
- For measure spaces we make use of the outer measure μ^* and (in the σ-finite case) the inner measure μ_* induced by μ on the hereditary σ-ring $\boldsymbol{H}(\boldsymbol{S})$.

Measurable Subsets of Measure Spaces

- A measurable subset X_0 of a measure space $(X, \boldsymbol{S}, \mu)$ may itself be considered as a measure space $(X_0, \boldsymbol{S}_0, \mu_0)$, where $\boldsymbol{S}_0$ is the class of all measurable subsets of X_0, and, for E in $\boldsymbol{S}_0$, $\mu_0(E) = \mu(E)$.
- Conversely, if a subset X_0 of a set X is a measure space $(X_0, \boldsymbol{S}_0, \mu_0)$, then X may be made into a measure space $(X, \boldsymbol{S}, \mu)$, where $\boldsymbol{S}$ is the class of all those subsets of X whose intersection with X_0 is in $\boldsymbol{S}_0$, and, for E in $\boldsymbol{S}$, $\mu(E) = \mu_0(E \cap X_0)$.
- A modification of this construction is frequently useful even if X is already a measure space:

  If X_0 is a measurable subset of X, a new measure μ_0 may be defined on the class of all measurable subsets E of X by the equation $\mu_0(E) = \mu(E \cap X_0)$.

  It is easy to verify that $(X, \boldsymbol{S}, \mu_0)$ is indeed a measure space.

Thick Subsets of Measure Spaces

- A subset X_0 of a measure space $(X, \boldsymbol{S}, \mu)$ is **thick** if $\mu_*(E - X_0) = 0$, for every measurable set E.
- If X itself is measurable, then X_0 is thick if and only if $\mu_*(X - X_0) = 0$.
- If μ is totally finite, then X_0 is thick if and only if $\mu^*(X_0) = \mu(X)$.

Nonmeasurable Subsets of Measure Spaces

Theorem

If X_0 is a thick subset of a measure space $(X, \boldsymbol{S}, \mu)$, if $\boldsymbol{S}_0 = \boldsymbol{S} \cap X_0$, and if, for E in $\boldsymbol{S}$, $\mu_0(E \cap X_0) = \mu(E)$, then $(X_0, \boldsymbol{S}_0, \mu_0)$ is a measure space.

- If two sets, E_1 and E_2, in $\boldsymbol{S}$ are such that $E_1 \cap X_0 = E_2 \cap X_0$, then $(E_1 \triangle E_2) \cap X_0 = \emptyset$, so $\mu(E_1 \triangle E_2) = 0$ and, hence, $\mu(E_1) = \mu(E_2)$. Thus, μ_0 is unambiguously defined on $\boldsymbol{S}_0$.

  Suppose next that $\{F_n\}$ is a disjoint sequence of sets in $\boldsymbol{S}_0$, and let E_n be a set in $\boldsymbol{S}$, such that $F_n = E_n \cap X_0$, $n = 1, 2, \ldots$.

  If $\tilde{E}_n = E_n - \bigcup\{E_i : 1 \leq i < n\}$, $n = 1, 2, \ldots$, then $(\tilde{E}_n \triangle E_n) \cap X_0 = (F_n - \bigcup\{F_i : 1 \leq i < n\}) \triangle F_n = F_n \triangle F_n = \emptyset$, so that $\mu(\tilde{E}_n \triangle E_n) = 0$, and, therefore, $\sum_{n=1}^{\infty} \mu_0(F_n) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \mu(\tilde{E}_n) = \mu(\bigcup_{n=1}^{\infty} \tilde{E}_n) = \mu(\bigcup_{n=1}^{\infty} E_n) = \mu_0(\bigcup_{n=1}^{\infty} F_n)$, i.e., μ_0 is indeed a measure.

Subsection 2

Measurable Functions

The Inverse Image of a Set Under a Function

- Suppose that f is a real valued function on a set X and let M be any subset of the real line.

  We write
  $$f^{-1}(M) = \{x : f(x) \in M\}.$$
  i.e., $f^{-1}(M)$ is the set of all those points of X which are mapped into M by f.
- The set $f^{-1}(M)$ is called the **inverse image** (under f, or with respect to f) of the set M.

  Example: If f is the characteristic function of a set E in X, then
  $$f^{-1}(\{1\}) = E \quad \text{and} \quad f^{-1}(\{0\}) = E'.$$
  More generally, $f^{-1}(M) = \emptyset, E, E'$ or X, according as M contains neither 0 nor 1, 1 but not 0, 0 but not 1, or both 0 and 1.

Properties of the Inverse Image Mapping

- It is easy to verify that, for every f:
  - $f^{-1}(\bigcup_{n=1}^{\infty} M_n) = \bigcup_{n=1}^{\infty} f^{-1}(M_n)$;
  - $f^{-1}(M - N) = f^{-1}(M) - f^{-1}(N)$,

  i.e., the mapping f^{-1} from subsets of the line to subsets of X preserves all set operations.
- It follows that if $\boldsymbol{E}$ is a class of subsets of the line (such as a ring or a σ-ring) with certain algebraic properties, then $f^{-1}(\boldsymbol{E})$ (= the class of all those subsets of X which have the form $f^{-1}(M)$, for some M in $\boldsymbol{E}$) is a class with the same algebraic properties.
- Of particular interest for later applications is the case in which $\boldsymbol{E}$ is the class of all Borel sets on the line.

Measurable Functions

- Suppose that in addition to the set X we are given also a σ-ring $\boldsymbol{S}$ of subsets of X so that $(X, \boldsymbol{S})$ is a measurable space.
- For every real valued (and also for every extended real valued) function f on X, we shall write $N(f) = \{x : f(x) \neq 0\}$.
- If a real valued function f is such that, for every Borel subset M of the real line the set $N(f) \cap f^{-1}(M)$ is measurable, then f is called a **measurable function**.
- The special role played by the value 0 should be emphasized:
  - The reason for singling out 0 lies in the fact that it is the identity element of the additive group of real numbers.
  - In the next chapter we shall introduce the concept of integral, defined for certain measurable functions; the fact that integration (the most important concept in measure theory) may be viewed as generalized addition necessitates treating 0 differently from other real numbers.

Measurability and Preservation of Sets

- If f is a measurable function on X, then $N(f)$ is a measurable set.
  Simply take for M the entire real line (which is a Borel set).
  Then $N(f) \cap f^{-1}(\mathbb{R}) = N(f)$ is measurable.
- If E is a measurable subset of X and if M is a Borel subset of the real line, then $E \cap f^{-1}(M)$ is measurable.
  Note that
  $$E \cap f^{-1}(M) = [E \cap N(f) \cap f^{-1}(M)] \cup [(E - N(f)) \cap f^{-1}(M)],$$
  where the second term in the union is either empty or else equal to $E - N(f))$.
- Suppose we say that a real valued function f defined on a measurable set E is to be called **measurable on** E whenever $E \cap f^{-1}(M)$ is measurable, for every Borel set M.
- Then we have proved that a measurable function is measurable on every measurable set.

Measurability and Preservation of Sets (Cont'd)

- If, in particular, the entire space X happens to be measurable, then the requirement of measurability on f is simply that $f^{-1}(M)$ be measurable for every Borel subset M of the real line.
- I.e., in case X is measurable, a measurable function is one whose inverse maps the sets of one prescribed σ-ring (namely the Borel sets on the line) into the sets of another prescribed σ-ring (namely $\boldsymbol{S}$).

Measurability With Respect to a σ-Ring

- The concept of measurability for a function depends on the σ-ring $\boldsymbol{S}$.

  So we may say that a function is **measurable with respect to $\boldsymbol{S}$**, or, more concisely, that it is **measurable ($\boldsymbol{S}$)**.
- Suppose X is the real line, and $\boldsymbol{S}$ and $\overline{\boldsymbol{S}}$ are the class of Borel sets and the class of Lebesgue measurable sets, respectively.
  - A **Borel measurable function** is a function measurable with respect to $\boldsymbol{S}$.
  - A **Lebesgue measurable function** is a function measurable with respect to $\overline{\boldsymbol{S}}$.
- It is important to emphasize also that the concept of measurability for functions does not depend on the numerical values of a prescribed measure μ, but merely on the prescribed σ-ring $\boldsymbol{S}$.

Measurability for Extended Real Functions

- We shall need the concept of measurability for extended real functions also.
- We make the convention that the one-point sets $\{\infty\}$ and $\{-\infty\}$ of the extended real line are to be regarded as Borel sets.
- Then the definition given before for real valued functions is repeated:

  A possibly infinite valued function f is **measurable**, if, for every Borel set M of real numbers, each of the three sets $f^{-1}(\{\infty\})$, $f^{-1}(\{-\infty\})$ and $N(f) \cap f^{-1}(M)$ is measurable.
- For the extended concept of Borel set, it is no longer true that the class of Borel sets is the σ-ring generated by semi-closed intervals.

Characterization of Measurability of Real Functions

Theorem

A real function f on a measurable space $(X, \boldsymbol{S})$ is measurable if and only if, for every real number c, the set $N(f) \cap \{x : f(x) < c\}$ is measurable.

- If $M = \{t : t < c\}$, then M is Borel and $f^{-1}(M) = \{x : f(x) < c\}$. Therefore, the stated condition is necessary.

  Suppose next that the condition is satisfied. If c_1 and c_2 are real numbers, $c_1 \leq c_2$, then $\{x : f(x) < c_2\} - \{x : f(x) < c_1\} = \{x : c_1 \leq f(x) < c_2\}$. I.e., if M is any semiclosed interval, then $N(f) \cap f^{-1}(M)$ is the difference of two measurable sets and is therefore measurable. Let $\boldsymbol{E}$ be the class of all those subsets M of the extended real line for which $N(f) \cap f^{-1}(M)$ is measurable. $\boldsymbol{E}$ is a σ-ring and it contains all semiclosed intervals. Therefore, it contains also all Borel sets.

Subsection 3

Combinations of Measurable Functions

Measurability of Some Intersections

Theorem

If f and g are extended real valued measurable functions on a measurable space $(X, \boldsymbol{S})$, and if c is any real number, then each of the three sets

- $A = \{x : f(x) < g(x) + c\}$;
- $B = \{x : f(x) \leq g(x) + c\}$;
- $C = \{x : f(x) = g(x) + c\}$

has a measurable intersection with every measurable set.

- Let M be the set of rational numbers on the line. Note that
$$A = \bigcup_{r \in M} (\{x : f(x) < r\} \cap \{x : r - c < g(x)\}).$$
It follows that A has the desired property. The conclusions for B and C are consequences, respectively, of the relations
$$B = X - \{x : g(x) < f(x) - c\} \quad \text{and} \quad C = B - A.$$

Measurability of Composites

Theorem

If ϕ is an extended real valued Borel measurable function on the extended real line such that $\phi(0) = 0$, and if f is an extended real valued measurable function on a measurable space X, then the function $\tilde{f}$, defined by $\tilde{f}(x) = \phi(f(x))$, is a measurable function on X.

- It is convenient to use here the definition of measurability.
  If M is any Borel set on the extended real line, then

$$\begin{array}{rcl} N(\tilde{f}) \cap \tilde{f}^{-1}(M) & = & \{x : \phi(f(x)) \in M - \{0\}\} \\ & = & \{x : f(x) \in \phi^{-1}(M - \{0\})\}. \end{array}$$

  Since $\phi(0) = 0$, we have $\phi^{-1}(M - \{0\}) = \phi^{-1}(M - \{0\}) - \{0\}$.
  Since ϕ is Borel measurable, $\phi^{-1}(M - \{0\})$ is a Borel set.
  So the measurability of $N(\tilde{f}) \cap \tilde{f}^{-1}(M) = N(f) \cap f^{-1}(\phi^{-1}(M - \{0\}))$ follows from the measurability of f.

Measurability of Composites: A Consequence

- For any fixed real number α,

$$t \mapsto |t|^\alpha \quad \text{and} \quad t \mapsto \alpha t$$

  are Borel measurable.
- It follows that

  if f is measurable, then so are $|f|^\alpha$ and αf.

Sums and Products of Measurable Functions

Theorem

If f and g are extended real valued measurable functions on a measurable space X, then so also are $f+g$ and fg.

- We restrict our attention to finite valued functions.

  Since, if f and g are finite and if c is a real number, then

$$\{x : f(x) + g(x) < c\} = \{x : f(x) < c - g(x)\},$$

  the measurability of $f+g$ follows from the first theorem (with $-g$ in place of g).

  The measurability of fg is a consequence of the identity.

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right].$$

Positive and Negative Parts of a Function

- Since if f and g are finite we have
$$f \cup g = \frac{1}{2}(f + g + |f - g|) \quad \text{and} \quad f \cap g = \frac{1}{2}(f + g - |f - g|),$$
the second and third theorems show that the measurability of f and g implies that of $f \cup g$ and $f \cap g$.
- If for every extended real valued function f we write
$$f^+ = f \cup 0 \quad \text{and} \quad f^- = -(f \cap 0),$$
then
$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^-.$$
- The functions f^+ and f^- are called the **positive part** and the **negative part** of f, respectively.
- The positive and negative parts of a measurable function are both measurable and, conversely, a function with measurable positive and negative parts is itself measurable.

Subsection 4

Sequences of Measurable Functions

Measurability of Suprema and Infima

Theorem

If $\{f_n\}$ is a sequence of extended real valued, measurable functions on a measurable space X, then each of the four functions h, g, f^* and f_*, defined by

$$\begin{aligned} h(x) &= \sup\{f_n(x) : n = 1, 2, \ldots\}, \\ g(x) &= \inf\{f_n(x) : n = 1, 2, \ldots\}, \\ f^*(x) &= \limsup_n f_n(x), \\ f_*(x) &= \liminf_n f_n(x), \end{aligned}$$

is measurable.

- We can reduce the general case to that of finite valued functions.
  - The equation $\{x : g(x) < c\} = \bigcup_{n=1}^{\infty}\{x : f_n(x) < c\}$ implies the measurability of g.
  - For h, note that $h(x) = -\inf\{-f_n(x) : n = 1, 2, \ldots\}$.
  - The measurability of f^* and f_* is a consequence of the relations $f^*(x) = \inf_{n \geq 1} \sup_{m \geq n} f_m(x)$ and $f_*(x) = \sup_{n \geq 1} \inf_{m \geq n} f_m(x)$.

Limit of a Sequence of Measurable Functions

- The theorem implies that the set of points of convergence of a sequence $\{f_n\}$ of measurable functions, i.e., the set

$$\{x : \limsup_n f_n(x) = \liminf_n f_n(x)\},$$

has a measurable intersection with every measurable set.

Thus, the function f, defined by

$$f(x) = \lim_n f_n(x)$$

at every x for which the limit exists, is a measurable function.

Simple Functions

- A function f, defined on a measurable space X, is called **simple** if there is a finite, disjoint class $\{E_1, \ldots, E_n\}$ of measurable sets and a finite set $\{\alpha_1, \ldots, \alpha_n\}$ of real numbers such that

$$f(x) = \begin{cases} \alpha_i, & \text{if } x \in E_i,\ i = 1, 2, \ldots \\ 0, & \text{if } x \notin E_1 \cup \cdots \cup E_n \end{cases}$$

- In other words a simple function takes on only a finite number of values different from zero, each on a measurable set.

  Example: The characteristic function χ_E of a measurable set E,

$$\chi_E = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases},$$

  is a simple function.

Elementary Properties of Simple Functions

- A simple function

$$f(x) = \begin{cases} \alpha_i, & \text{if } x \in E_i,\ i = 1, 2, \ldots \\ 0, & \text{if } x \notin E_1 \cup \cdots \cup E_n \end{cases}$$

  is always measurable.

  In fact we have

$$f(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(x).$$

- The product of two simple functions, and any finite linear combination of simple functions, are again simple functions.

Measurable Functions as Limits of Simple Functions

Theorem

Every extended real valued measurable function f is the limit of a sequence $\{f_n\}$ of simple functions. If f is non negative, then each f_n may be taken non negative and the sequence $\{f_n\}$ may be assumed increasing.

- Suppose first that $f \geq 0$. For every $n = 1, 2, \ldots$, and for every x in X, we write

$$f_n(x) = \begin{cases} \frac{i-1}{2^n}, & \text{if } \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n},\ i = 1, \ldots, 2^n n \\ n, & \text{if } f(x) \geq n \end{cases}$$

Clearly, f_n is a non negative simple function and the sequence $\{f_n\}$ is increasing. If $f(x) < \infty$, then, for some n, $0 \leq f(x) - f_n(x) \leq \frac{1}{2^n}$. If $f(x) = \infty$, then $f_n(x) = n$, for every n. This proves the second half of the theorem.

The General Case

- For f arbitrary, recall that

$$f = f^+ - f^-,$$

where $f^+ \geq 0$ and $f^- \geq 0$.

Apply the result for positive functions to obtain increasing sequences of simple functions $\{f_n^+\}$ and $\{f_n^-\}$, such that

$$\lim_n f_n^+ = f^+ \quad \text{and} \quad \lim_n f_n^- = f^-.$$

Then, since the difference of two simple functions is a simple function, $\{f_n^+ - f_n^-\}$ is a sequence of simple functions, such that

$$f = \lim_n (f_n^+ - f_n^-).$$

Subsection 5

Pointwise Convergence

Almost Everywhere

- We developed the theory of measurable functions as far as it is convenient to do so without mentioning measure.
- We assume, next, that the space X is a measure space $(X, \boldsymbol{S}, \mu)$.
- We say that a proposition is true **for almost every point**, or that it is true **almost everywhere**, in a measure space if it is true for every point, with the exception of at most a set of points which form a measurable set of measure zero.
- The abbreviation **a.e.** means "almost everywhere".

  Example: A function is constant a.e. means that there exists a real number c, such that $\{x : f(x) \neq c\}$ is a set of measure zero.

Essential Boundedness

- A function f is called **essentially bounded** if it is bounded a.e., i.e., if there exists a positive, finite constant c, such that

$$\{x : |f(x)| > c\}$$

is a set of measure zero.

- The infimum of the values of c for which this statement is true is called the **essential supremum** of $|f|$, abbreviated to $\text{ess.sup}|f|$.

Convergence a.e.

- Let $\{f_n\}$ be a sequence of extended real valued functions which converges a.e. on the measure space to a limit function f.

  Thus, there exists a set E_0 of measure zero (which may be empty), such that, if $x \notin E_0$ and $\epsilon > 0$, then an integer $n_0 = n_0(x, \epsilon)$ can be found with the property that, for all $n \geq n_0$,

$$\begin{array}{ll} f_n(x) < -\frac{1}{\epsilon}, & \text{if } f(x) = -\infty, \\ |f_n(x) - f(x)| < \epsilon, & \text{if } -\infty < f(x) < \infty, \\ f_n(x) > \frac{1}{\epsilon}, & \text{if } f(x) = \infty. \end{array}$$

Fundamental a.e. Sequences

- We say that a sequence $\{f_n\}$ of real valued functions is **fundamental a.e.** if there exists a set E_0 of measure zero such that, if $x \notin E_0$ and $\epsilon > 0$, then there exists an integer $n_0 = n_0(x, \epsilon)$, such that

$$n \geq n_0 \text{ and } m \geq n_0 \text{ imply } |f_n(x) - f_m(x)| < \epsilon.$$

- In the theory of real sequences one also distinguishes between:
  - a sequence $\{a_n\}$ of extended reals converging to an extended real a;
  - a sequence $\{a_n\}$ of finite real numbers which is a fundamental sequence.

Convergence and Fundamental Sequences

- It is clear that if a sequence converges to a finite valued limit function a.e., then it is fundamental a.e..

  Conversely, if a sequence is fundamental a.e., there always exists a finite valued limit function to which it converges a.e..
- If, moreover, the sequence converges a.e. to f and also converges a.e. to g, then $f(x) = g(x)$ a.e., i.e. the limit function is uniquely determined to within a set of measure zero.
- If we define a new kind of convergence of a sequence $\{f_n\}$ to a limit f, by specifying the sense in which f_n is to be near to f for large n, then we shall use without any further explanation the notion of a sequence which is fundamental in this sense of convergence.

  The meaning is that, for large n and m, the differences $f_n - f_m$ are to be near to 0 in the specified sense of nearness.

Uniform Convergence

- The sequence $\{f_n\}$ **converges to f uniformly a.e.** if there exists a set E_0 of measure zero, such that, for every $\epsilon > 0$, there exists an integer $n_0 = n_0(\epsilon)$, such that
$$n \geq n_0 \text{ and } x \notin E_0 \quad \text{imply} \quad |f_n(x) - f(x)| < \epsilon.$$
- In other words, a sequence converges uniformly to f a.e. if it converges uniformly to f (in the ordinary sense of that phrase) on the set $X - E_0$.
- It is true that a sequence converges uniformly a.e to some limit function if and only if it is uniformly fundamental a.e..

Egoroff's Theorem

Theorem

If E is a measurable set of finite measure, and if $\{f_n\}$ is a sequence of a.e. finite valued measurable functions which converges a.e. on E to a finite valued measurable function f, then, for every $\epsilon > 0$, there exists a measurable subset F of E such that $\mu(F) < \epsilon$ and such that the sequence $\{f_n\}$ converges to f uniformly on $E - F$.

- By omitting, if necessary, a set of measure zero from E, we may assume that the sequence $\{f_n\}$ converges to f everywhere on E.

  If $E_n^m = \bigcap_{i=n}^{\infty}\{x : |f_i(x) - f(x)| < \frac{1}{m}\}$, then $E_1^m \subseteq E_2^m \subseteq \cdots$.

  Since $\{f_n\}$ converges to f on E, $\lim_n E_n^m \supseteq E$, for every $m = 1, 2, \ldots$.

  Hence, $\lim_n \mu(E - E_n^m) = 0$. So, there exists a positive integer $n_0 = n_0(m)$, such that $\mu(E - E_{n_0(m)}^m) < \frac{\epsilon}{2^m}$.

Egoroff's Theorem (Cont'd)

- Let
$$F = \bigcup_{m=1}^{\infty} (E - E^m_{n_0(m)}).$$
  - F is a measurable set;
  - $F \subseteq E$;
  - $\mu(F) = \mu(\bigcup_{m=1}^{\infty}(E - E^m_{n_0(m)})) \leq \sum_{m=1}^{\infty} \mu(E - E^m_{n_0(m)}) < \epsilon$.

  We have $E - F = E \cap \bigcap_{m=1}^{\infty} E^m_{n_0(m)}$.

  So and for x in $E - F$, we have $x \in E^m_n$.

  It follows that, for $n \geq n_0(m)$, $|f_n(x) - f(x)| < \frac{1}{m}$.

  Therefore, $\{f_n\}$ converges to f uniformly on $E - F$.

Almost Uniform Convergence

- Motivated by Egoroff's theorem we introduce the concept of almost uniform convergence.
- A sequence $\{f_n\}$ of a.e. finite valued measurable functions will be said to **converge** to the measurable function f **almost uniformly** if, for every $\epsilon > 0$, there exists a measurable set F, such that $\mu(F) < \epsilon$ and such that the sequence $\{f_n\}$ converges to f uniformly on F'.
- In this language Egoroff's theorem asserts that on a set of finite measure convergence a.e. implies almost uniform convergence.

Almost Uniform Convergence and Convergence a.e.

Theorem

If $\{f_n\}$ is a sequence of measurable functions which converges to f almost uniformly, then $\{f_n\}$ converges to f a.e..

- Let F_n be a measurable set such that $\mu(F_n) < \frac{1}{n}$ and such that the sequence $\{f_n\}$ converges to f uniformly on F_n', $n = 1, 2, \ldots$.

  If $F = \bigcap_{n=1}^{\infty} F_n$, then $\mu(F) \leq \mu(F_n) < \frac{1}{n}$, so that $\mu(F) = 0$.

  Moreover, for x in F', $\{f_n(x)\}$ converges to $f(x)$.
- Almost uniform convergence and *almost everywhere* uniform convergence are different concepts.

  Perhaps, "nearly uniform convergence" should have been used instead of almost uniform convergence.

Subsection 6

Convergence in Measure

Characterization of Convergence a.e.

Theorem

Suppose that f and f_n, $n = 1, 2, \ldots$, are real valued measurable functions on a set E of finite measure, and write, for every $\epsilon > 0$,

$$E_n(\epsilon) = \{x : |f_n(x) - f(x)| \geq \epsilon\}, \; n = 1, 2, \ldots.$$

The sequence $\{f_n\}$ converges to f a.e. on E if and only if, for every $\epsilon > 0$, $\lim_n \mu(E \cap \bigcup_{m=n}^{\infty} E_m(\epsilon)) = 0$.

- It follows from the definition of convergence that the sequence $\{f_n(x)\}$ of real numbers fails to converge to the real number $f(x)$ if and only if, there is a positive number ϵ, such that x belongs to $E_n(\epsilon)$ for an infinite number of values of n. In other words, if D is the set of those points x at which $\{f_n(x)\}$ does not converge to $f(x)$, then

$$D = \bigcup_{\epsilon > 0} \limsup_n E_n(\epsilon) = \bigcup_{k=1}^{\infty} \limsup_n E_n\left(\frac{1}{k}\right).$$

Characterization of Convergence a.e. (Cont'd)

- if D is the set of those points x at which $\{f_n(x)\}$ does not converge to $f(x)$, then

$$D = \bigcup_{\epsilon > 0} \limsup_n E_n(\epsilon) = \bigcup_{k=1}^{\infty} \limsup_n E_n\left(\frac{1}{k}\right).$$

Consequently, a necessary and sufficient condition that $\mu(E \cap D) = 0$, i.e., that the sequence $\{f_n\}$ converge to f a.e. on E, is that $\mu(E \cap \limsup_n E_n(\epsilon)) = 0$, for every $\epsilon > 0$. The desired conclusion follows from the relations

$$\begin{array}{rcl} \mu(E \cap \limsup_n E_n(\epsilon)) & = & \mu(E \cap \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m(\epsilon)) \\ & = & \lim_n \mu(E \cap \bigcup_{m=n}^{\infty} E_m(\epsilon)). \end{array}$$

Convergence in Measure

- A sequence $\{f_n\}$ of a.e. finite valued, measurable functions **converges in measure** to the measurable function f if

$$\lim_n \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = 0, \text{ for every } \epsilon > 0.$$

- We say that a sequence $\{f_n\}$ of a.e. finite valued, measurable functions is **fundamental in measure** if, for every $\epsilon > 0$,

$$\mu(\{x : |f_n(x) - f_m(x)| \geq \epsilon\}) \xrightarrow{n,m\to\infty} 0.$$

- It follows trivially from the preceding theorem that if a sequence of finite valued measurable functions converges a.e. to a finite limit [or is fundamental a.e.] on a set E of finite measure, then it converges in measure [or is fundamental in measure] on E.
- The following theorem is a slight strengthening of this assertion in that it makes no assumptions of finiteness.

Almost Uniform Convergence and Convergence in Measure

Theorem

Almost uniform convergence implies convergence in measure.

- Suppose $\{f_n\}$ converges to f almost uniformly. Then, for any two positive numbers ϵ and δ, there exists a measurable set F, such that:
  - $\mu(F) < \delta$;
  - $|f_n(x) - f(x)| < \epsilon$, whenever x belongs to F' and n is sufficiently large.

  Therefore, $\lim_n \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = 0$, for all $\epsilon > 0$, and, hence, $\{f_n\}$ converges to f in measure.

Convergence and Fundamentality in Measure

Theorem

If $\{f_n\}$ converges in measure to f, then $\{f_n\}$ is fundamental in measure. If also $\{f_n\}$ converges in measure to g, then $f = g$ a.e..

- The first assertion follows from

$$\{x : |f_n(x) - f_m(x)| \geq \epsilon\} \subseteq \{x : |f_n(x) - f(x)| \geq \tfrac{\epsilon}{2}\} \cup \{x : |f_m(x) - f(x)| \geq \tfrac{\epsilon}{2}\}.$$

For the second assertion, we observe that, similarly,

$$\{x : |f(x) - g(x)| \geq \epsilon\} \subseteq \{x : |f_n(x) - f(x)| \geq \tfrac{\epsilon}{2}\} \cup \{x : |f_n(x) - g(x)| \geq \tfrac{\epsilon}{2}\}.$$

Since, by proper choice of n, the measure of both sets on the right can be made arbitrarily small, we have $\mu(\{x : |f(x) - g(x)| \geq \epsilon\}) = 0$, for every $\epsilon > 0$. This implies that $f = g$ a.e..

In Measure and Almost Uniform Fundamentality

Theorem

If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure, then some subsequence $\{f_{n_k}\}$ is almost uniformly fundamental.

- For any positive integer k, we may find an integer $\overline{n}(k)$, such that, if $n \geq \overline{n}(k)$ and $m \geq \overline{n}(k)$, then $\mu(\{x : |f_n(x) - f_m(x)| \geq \frac{1}{2^k}\}) < \frac{1}{2^k}$. Write $n_1 = \overline{n}(1)$, $n_2 = (n_1 + 1) \cup \overline{n}(2)$, $n_3 = (n_2 + 1) \cup \overline{n}(3)$, ... Then $n_1 < n_2 < n_3 < \cdots$. Thus, the sequence $\{f_{n_k}\}$ is indeed an infinite subsequence of $\{f_n\}$. If $E_k = \{x : |f_{n_k}(x) - f_{n_{k+1}}(x)| \geq \frac{1}{2^k}\}$, then, for all $k \leq i \leq j$ and all $x \notin E_k \cup E_{k+1} \cup E_{k+2} \cup \cdots$,
$$|f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{m=i}^{\infty} |f_{n_m}(x) - f_{n_{m+1}}(x)| < \frac{1}{2^{i-1}}.$$
Hence, $\{f_{n_i}\}$ is uniformly fundamental on $X - (E_k \cup E_{k+1} \cup \cdots)$. Finally, note that $\mu(E_k \cup E_{k+1} \cup \cdots) \leq \sum_{m=k}^{\infty} \mu(E_m) < \frac{1}{2^{k-1}}$.

Fundamentality in Measure and Convergence in Measure

Theorem

If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure, then there exists a measurable function f, such that $\{f_n\}$ converges in measure to f.

- By the preceding theorem, we can find a subsequence $\{f_{n_k}\}$ which is almost uniformly fundamental and, therefore, fundamental a.e.. We write $f(x) = \lim_k f_{n_k}(x)$, for every x for which the limit exists. We observe that, for every $\epsilon > 0$,

$$\begin{aligned}&\{x : |f_n(x) - f(x)| \geq \epsilon\}\\ &\qquad\subseteq \{x : |f_n(x) - f_{n_k}(x)| \geq \tfrac{\epsilon}{2}\} \cup \{x : |f_{n_k}(x) - f(x)| \geq \tfrac{\epsilon}{2}\}.\end{aligned}$$

  - The measure of the first term on the right is by hypothesis arbitrarily small if n and n_k are sufficiently large.
  - The measure of the second term also approaches 0 (as $k \to \infty$), since almost uniform convergence implies convergence in measure.