Introduction to Measure Theory

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Subsection 1

Integrable Simple Functions

Integrable Simple Functions

- A simple function $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ on a measure space $(X, \boldsymbol{S}, \mu)$ is **integrable** if $\mu(E_i) < \infty$, for every index i for which $\alpha_i \neq 0$.
- The integral of f, in symbols $\int f(x) d\mu(x)$ or $\int f d\mu$ is defined by
$$\int f d\mu = \sum_{i=1}^{\infty} \alpha_i \mu(E_i).$$
- It follows easily from the additivity of μ that if f is also equal to $\sum_{j=1}^{m} \beta_j \chi_{F_j}$, then $\int f d\mu = \sum_{j=1}^{m} \beta_j \mu(F_j)$, i.e., that the value of the integral is independent of the representation of f and is, therefore, unambiguously defined.

Observations

- We observe that:
  - the absolute value of an integrable simple function,
  - a finite, constant multiple of an integrable simple function,
  - the sum of two integrable simple functions

  are integrable simple functions.

  Indeed, notice that, if $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, $g = \sum_{j=1}^{m} \beta_j \chi_{F_j}$,

$$\begin{array}{rcl} |f| & = & \sum_{i=1}^{n} |\alpha_i| \chi_{E_i}; \\ \alpha f & = & \sum_{i=1}^{n} (\alpha \alpha_i) \chi_{E_i}; \\ f + g & = & \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i + \beta_j) \chi_{E_i \cap F_j}. \end{array}$$

Integral of a Simple Function Over a Measurable Set

- If E is a measurable set and f is an integrable simple function, then the function $\chi_E f$ is an integrable simple function also.
  If $f = \sum_{i=1}^n \alpha_i \chi_{E_i}$, then $\chi_E f = \sum_{i=1}^n \alpha_i \chi_{E \cap E_i}$.
- We define the **integral of f over E** by

$$\int_E f d\mu = \int \chi_E f d\mu.$$

- So, using the notation above,

$$\begin{aligned} \int_E f d\mu &= \int \chi_E f d\mu \\ &= \int \sum_{i=1}^n \alpha_i \chi_{E \cap E_i} d\mu \\ &= \sum_{i=1}^n \alpha_i \mu(E \cap E_i). \end{aligned}$$

Example

- The simplest example of an integrable simple function is the characteristic function of a measurable set E of finite measure.

$$\int_E d\mu = \int \chi_E d\mu = \mu(E).$$

- In this subsection, we use the word "function" as an abbreviation for "simple function."
- All our definitions and theorems will make sense not only for simple functions but also for the wider class of functions we shall consider in later subsections.

Linearity

Theorem

If f and g are integrable functions and α and β are real numbers, then

$$\int(\alpha f+\beta g)d\mu=\alpha\int fd\mu+\beta\int gd\mu.$$

- Suppose $f=\sum_{i=1}^n\alpha_i\chi_{E_i}$ and $g=\sum_{j=1}^m\beta_j\chi_{F_j}$.
  Then, we have

$$\begin{aligned}\int(\alpha f+\beta g)d\mu &= \int\sum_{i=1}^n\sum_{j=1}^m(\alpha\alpha_i+\beta\beta_j)\chi_{E_i\cap F_j}d\mu\\ &= \sum_{i=1}^n\sum_{j=1}^m(\alpha\alpha_i+\beta\beta_j)\mu(E_i\cap F_j)\\ &= \sum_{i=1}^n\sum_{j=1}^m\alpha\alpha_i\mu(E_i\cap F_j)+\\ &\qquad \sum_{i=1}^n\sum_{j=1}^m\beta\beta_j\mu(E_i\cap F_j)\\ &= \alpha\sum_{i=1}^n\alpha_i\sum_{j=1}^m\mu(E_i\cap F_j)+\\ &\qquad \beta\sum_{j=1}^m\beta_j\sum_{i=1}^n\mu(E_i\cap F_j)\\ &= \alpha\sum_{i=1}^n\alpha_i\mu(E_i)+\beta\sum_{j=1}^m\beta_j\mu(F_j)\\ &= \alpha\int fd\mu+\beta\int gd\mu.\end{aligned}$$

Positivity

Theorem

If an integrable function f is non negative a.e., then $\int f d\mu \geq 0$.

- If f is a simple function, such that $f \geq 0$ a.e., then

$$f = \sum_{i=1}^{n} \alpha_i \chi_{E_i} + \sum_{j=1}^{m} \beta_j \chi_{F_j},$$

where $\alpha_i \geq 0$, $\beta_j < 0$ and $\mu(F_j) = 0$, $j = 1, \ldots, m$.

Therefore, we get

$$\begin{array}{rcl} \int f d\mu & = & \sum_{i=1}^{n} \alpha_i \mu(E_i) + \sum_{j=1}^{m} \beta_j \mu(F_j) \\ & = & \sum_{i=1}^{n} \alpha_i \mu(E_i) + 0 \\ & \geq & 0. \end{array}$$

Comparison

Theorem

If f and g are integrable functions such that $f \geq g$ a.e., then

$$\int f d\mu \geq \int g d\mu.$$

- We get

$$\begin{array}{rcl} f \geq g \text{ a.e} & \text{iff} & f - g \geq 0 \text{ a.e.} \\ & \text{implies} & \int (f-g) d\mu \geq 0 \\ & \text{iff} & \int f d\mu - \int g d\mu \geq 0 \\ & \text{iff} & \int f d\mu \geq \int g d\mu. \end{array}$$

Absolute Values

Theorem

If f and g are integrable functions, then

$$\int |f+g| d\mu \leq \int |f| d\mu + \int |g| d\mu.$$

- We have $\int |f+g| d\mu \leq \int (|f|+|g|) d\mu = \int |f| d\mu + \int |g| d\mu$.

Theorem

If f is an integrable function, then

$$\left| \int f d\mu \right| \leq \int |f| d\mu.$$

- Again, $\int f d\mu \leq \int |f| d\mu$ and $\int (-f) d\mu \leq \int |f| d\mu$. Combining, we get $-\int |f| d\mu \leq \int f d\mu \leq \int |f| d\mu$, i.e., $|\int f d\mu| \leq \int |f| d\mu$.

Boundedness

Theorem

If f is an integrable function, α and β are real numbers, and E is a measurable set, such that, for x in E, $\alpha \leq f(x) \leq \beta$, then

$$\alpha\mu(E) \leq \int_E f d\mu \leq \beta\mu(E).$$

- The assumption is equivalent to the relation

$$\alpha\chi_E \leq \chi_E f \leq \beta\chi_E.$$

So the result follows from the third theorem if $\mu(E) < \infty$.

The case in which $\mu(E) = \infty$ is easily treated by direct application of the definition of integrability.

Indefinite Integral

- The **indefinite integral** of an integrable function f is the set function ν, defined, for every measurable set E, by

$$\nu(E) = \int_E f d\mu.$$

Theorem

If an integrable function f is non negative a.e., then its indefinite integral is monotone.

- If E and F are measurable sets, such that $E \subseteq F$, then $\chi_E f \leq \chi_F f$ a.e.. The desired result follows from our third theorem.

Absolute Continuity

- A finite valued set function ν, defined on the class of all measurable sets of a measure space $(X, \boldsymbol{S}, \mu)$, is **absolutely continuous** if, for every positive number ϵ, there exists a positive number δ, such that, for every measurable set E,

$$\mu(E) < \delta \quad \text{implies} \quad |\nu(E)| < \epsilon.$$

Theorem

The indefinite integral of an integrable function is absolutely continuous.

- If c is any positive number greater than all the values of $|f|$, then, for every measurable set E, we have $|\int_E f d\mu| \leq c\mu(E)$.

  So, given $\epsilon > 0$, take $\delta = \frac{\epsilon}{c}$.

  Then, $\mu(E) < \frac{\epsilon}{c}$ implies $|\nu(E)| = |\int_E f d\mu| < c\frac{\epsilon}{c} = \epsilon$.

Countable Additivity of Indefinite Integral

Lemma

The indefinite integral of a characteristic function of a measurable set E is countably additive.

- Let $\{E_i\}_{i=1}^{\infty}$ be a collection of disjoint measurable sets. Then, we have

$$\begin{aligned}\nu(\bigcup_i E_i) &= \int_{\bigcup_i E_i} \chi_E d\mu = \mu(E \cap \bigcup_i E_i) \\ &= \mu(\bigcup_i (E \cap E_i)) = \sum_i \mu(E \cap E_i) \\ &= \sum_i \int_{E_i} \chi_E d\mu = \sum_i \nu(E_i).\end{aligned}$$

Countable Additivity of Indefinite Integral (Cont'd)

Theorem

The indefinite integral of an integrable function is countably additive.

- Let $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ be an integrable function, ν its indefinite integral and $\{F_j\}_{j=1}^{\infty}$ a collection of disjoint measurable sets.

  Then, taking into account the Lemma, we have

$$\begin{array}{rcl} \nu(\bigcup_j F_j) & = & \int_{\cup_j F_j} (\sum_{i=1}^{n} \alpha_i \chi_{E_i}) d\mu = \sum_i \alpha_i \int_{\bigcup_j F_j} \chi_{E_i} d\mu \\ & = & \sum_i \alpha_i \sum_j \int_{F_j} \chi_{E_i} d\mu = \sum_j \sum_i \alpha_i \int_{F_j} \chi_{E_i} d\mu \\ & = & \sum_j \int_{F_j} (\sum_i \alpha_i \chi_{E_i}) d\mu = \sum_j \int_{F_j} f d\mu \\ & = & \sum_j \nu(F_j). \end{array}$$

Distance

- If f and g are integrable functions, we define the **distance** $\rho(f, g)$ between them by the equation

$$\rho(f, g) = \int |f - g| d\mu.$$

The function ρ deserves the name "distance" in every respect but one:

- It is true and trivial that:
  - $\rho(f, f) = 0$;
  - $\rho(f, g) = \rho(g, f)$;
  - $\rho(f, g) \leq \rho(g, h) + \rho(h, f)$;
- It is not the case that, if $\rho(f, g) = 0$, then $f = g$.
  The distance between two integrable functions can, for instance, vanish if they are equal almost everywhere, but not necessarily everywhere.

Subsection 2

Sequences of Integrable Simple Functions

Mean Fundamental Sequences

- We continue working with a fixed measure space $(X, \boldsymbol{S}, \mu)$ and abbreviating "simple function" to "function".
- A sequence $\{f_n\}$ of integrable functions is **fundamental in the mean**, or **mean fundamental**, if

$$\rho(f_n, f_m) \overset{n,m\to\infty}{\longrightarrow} 0.$$

Theorem

A mean fundamental sequence $\{f_n\}$ of integrable functions is fundamental in measure.

- For fixed $\epsilon > 0$, set $E_{nm} = \{x : |f_n(x) - f_m(x)| \geq \epsilon\}$. Then

$$\rho(f_n, f_m) = \int |f_n - f_m| d\mu \geq \int_{E_{nm}} |f_n - f_m| d\mu \geq \epsilon\mu(E_{nm}),$$

so that $\mu(E_{mn}) \to 0$ as $n, m \to \infty$.

The Limit of Indefinite Integrals

Theorem

If $\{f_n\}$ is a mean fundamental sequence of integrable functions, and if the indefinite integral of f_n is ν_n, $n = 1, 2, \ldots$, then $\nu(E) = \lim_n \nu_n(E)$ exists for every measurable set E, and the set function ν is finite valued and countably additive.

- Since $|\nu_n(E) - \nu_m(E)| \leq \int |f_n - f_m| d\mu \overset{n,m\to\infty}{\to} 0$, the existence, finiteness, and uniformity of the limit are clear. It follows, by finite additivity of limits, that ν is finitely additive. If $\{E_n\}$ is a disjoint sequence of measurable sets whose union is E, then, for positive n, k,
$$|\nu(E) - \sum_{i=1}^k \nu(E_i)| \leq |\nu(E) - \nu_n(E)| + |\nu_n(E) - \sum_{i=1}^k \nu_n(E_i)| + |\nu_n(\bigcup_{i=1}^k E_i) - \nu(\bigcup_{i=1}^k E_i)|.$$
The first and third terms of the right may be made arbitrarily small by choosing n sufficiently large. For fixed n, the middle term may be made arbitrarily small by choosing k sufficiently large. This proves that $\nu(E) = \lim_k \sum_{i=1}^k \nu(E_i) = \sum_{i=1}^\infty \nu(E_i)$.

Uniform Absolute Continuity

- If $\{\nu_n\}$ is a sequence of finite valued set functions defined for all measurable sets, we say that the terms of the sequence are **uniformly absolutely continuous** if, for every positive number ϵ, there exists a positive number δ, such that, for every measurable set E and for every positive integer n,

$$\mu(E) < \delta \quad \text{implies} \quad |\nu_n(E)| < \epsilon.$$

Mean Fundamentality and Uniform Absolute Continuity

Theorem

If $\{f_n\}$ is a mean fundamental sequence of integrable functions, and if the indefinite integral of f_n is ν_n, $n = 1, 2, \ldots$, then the set functions ν_n are uniformly absolutely continuous.

- Let $\epsilon > 0$.
  Since $\{f_n\}$ is mean fundamental, there exists a positive integer n_0, such that, for all $n, m > n_0$, $\int |f_n - f_m| d\mu < \frac{\epsilon}{2}$.
  Moreover, there exists $\delta > 0$, such that, for all $n = 1, \ldots, n_0$ and all measurable E, $\mu(E) < \delta$ implies $\int_E |f_n| d\mu < \frac{\epsilon}{2}$.
  Now, suppose E is measurable, such that $\mu(E) < \delta$.
  - If $n \leq n_0$, then $|\nu_n(E)| \leq \int_E |f_n| d\mu < \frac{\epsilon}{2} < \epsilon$;
  - If $n > n_0$, then

$$|\nu_n(E)| \leq \int_E |f_n| d\mu \leq \int_E |f_n - f_{n_0}| d\mu + \int_E |f_{n_0}| d\mu < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Integrals of Co-convergent Sequences of Simple Functions

Theorem

If $\{f_n\}$ and $\{g_n\}$ are mean fundamental sequences of integrable simple functions which converge in measure to the same measurable function f, if the indefinite integrals of f_n and g_n are ν_n and λ_n, respectively, and if, for every measurable set E,

$$\nu(E) = \lim_n \nu_n(E) \quad \text{and} \quad \lambda(E) = \lim_n \lambda_n(E),$$

then the set functions ν and λ are identical.

- Since, for every $\epsilon > 0$,

$$\begin{aligned} E_n &= \{x : |f_n(x) - g_n(x)| \geq \epsilon\} \\ &\subseteq \{x : |f_n(x) - f(x)| \geq \tfrac{\epsilon}{2}\} \cup \{x : |f(x) - g_n(x)| \geq \tfrac{\epsilon}{2}\}, \end{aligned}$$

it follows that $\lim_n \mu(E_n) = 0$.

Proof (Cont'd)

For E measurable of finite measure, consider

$$\int_E |f_n - g_n| d\mu \leq \int_{E-E_n} |f_n - g_n| d\mu + \int_{E\cap E_n} |f_n| d\mu + \int_{E\cap E_n} |g_n| d\mu.$$

- The first term on the right is dominated by $\epsilon\mu(E)$.
- The last two terms can be made arbitrarily small by choosing n sufficiently large, by uniform absolute continuity.

So $\lim_n |\nu_n(E) - \lambda_n(E)| = 0$, and, hence, $\nu(E) = \lambda(E)$.
Since ν and λ are both countably additive, it follows that $\nu(E) = \lambda(E)$, for every measurable set E of σ-finite measure.
Since the f_n and g_n are simple functions, each of them is defined in terms of a finite class of measurable sets of finite measure. If E_0 is the union of all sets in all these finite classes, then E_0 is a measurable set of σ-finite measure. We have, for every measurable set E, $\nu_n(E - E_0) = \lambda_n(E - E_0) = 0$, whence $\nu(E - E_0) = \lambda(E - E_0) = 0$.
This implies that $\nu(E) = \nu(E \cap E_0)$ and $\lambda(E) = \lambda(E \cap E_0)$.

Subsection 3

Integrable Functions

Integrable Functions

- An a.e. finite valued, measurable function f on a measure space $(X, \boldsymbol{S}, \mu)$ is **integrable** if there exists a mean fundamental sequence $\{f_n\}$ of integrable simple functions which converges in measure to f.
- The **integral** of f, in symbols $\int f(x) d\mu(x)$ or $\int f d\mu$, is defined by

$$\int f d\mu = \lim_n \int f_n d\mu.$$

- It follows by a preceding result, that the value of the integral of f is uniquely determined by any particular such sequence.
- Moreover, the value of the integral is always finite.

Absolute Value of an Integrable Function

Proposition

The absolute value of an integrable function f is integrable.

- Since f is integrable, there exists a mean fundamental sequence $\{f_n\}$ of integrable simple functions, such that $f_n \to f$ in measure.
  Consider the sequence $\{|f_n|\}$.
  - It consists of integrable simple functions.
  - It is a mean fundamental sequence, since

$$\int ||f_n| - |f_m|| d\mu \leq \int |f_n - f_m| d\mu \stackrel{m,n\to\infty}{\longrightarrow} 0.$$

  - It converges to $|f|$ in measure, since

$$\mu(\{x : ||f_n(x)| - |f(x)|| \geq \epsilon\}) \leq \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \stackrel{n\to\infty}{\longrightarrow} 0.$$

Finite Constant Multiple of an Integrable Function

Proposition

A finite constant multiple of an integrable function f is integrable.

- Let f be integrable and α a finite constant.
  Since f is integrable, there exists a mean fundamental sequence $\{f_n\}$ of integrable simple functions, such that $f_n \to f$ in measure.
  Consider the sequence $\{\alpha f_n\}$.
  - It consists of integrable simple functions.
  - It is a mean fundamental sequence, since

$$\int |\alpha f_n - \alpha f_m| d\mu = |\alpha| \int |f_n - f_m| d\mu \overset{m,n\to\infty}{\longrightarrow} 0.$$

  - It converges to αf in measure, since

$$\mu(\{x : |\alpha f_n(x) - \alpha f(x)| \geq \epsilon\}) = \mu\left(\left\{x : |f_n(x) - f(x)| \geq \frac{\epsilon}{|\alpha|}\right\}\right) \overset{n\to\infty}{\longrightarrow} 0.$$

Sum of Integrable Functions

Proposition

The sum of two integrable functions f and g is integrable.

- Since f and g are integrable, there exist mean fundamental sequences $\{f_n\}$ and $\{g_n\}$ of integrable simple functions, such that $f_n \to f$ and $g_n \to g$ in measure.
  Consider the sequence $\{f_n + g_n\}$.
  - It consists of integrable simple functions.
  - It is a mean fundamental sequence, since

$$\int |(f_n + g_n) - (f_m + g_m)| d\mu \leq \int |f_n - f_m| d\mu + \int |g_n - g_m| d\mu \overset{m,n\to\infty}{\longrightarrow} 0.$$

  - It converges to $f + g$ in measure, since $\{x : |(f_n(x) + g_n(x)) - (f(x) + g(x))| \geq \epsilon\} \subseteq \{x : |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\} \cup \{x : |g_n(x) - g(x)| \geq \frac{\epsilon}{2}\}$, and, therefore

$$\mu(\{x : |(f_n(x) + g_n(x)) - (f(x) + g(x))| \geq \epsilon\}) \leq \mu(\{x : |f_n(x) - f(x)| \geq \tfrac{\epsilon}{2}\}) + \mu(\{x : |g_n(x) - g(x)| \geq \tfrac{\epsilon}{2}\}) \overset{n\to\infty}{\longrightarrow} 0.$$

Positive and Negative Parts of Integrable Functions

Proposition

If f is an integrable function, then f^+ and f^- are integrable.

- The results follows by the preceding results and the relations

$$f^+ = \frac{1}{2}(|f| + f) \quad \text{and} \quad f^- = \frac{1}{2}(|f| - f).$$

Integral over a Set

- If E is a measurable set and if $\{f_n\}$ is a mean fundamental sequence of integrable simple functions converging in measure to the integrable function f, then it is easy to see that the sequence $\{\chi_E f_n\}$ is mean fundamental and converges in measure to $\chi_E f$.
- We define the **integral of f over E** by

$$\int_E f d\mu = \int \chi_E f d\mu.$$

- The theorems of the preceding subsections were stated for general integrable functions but were proved for integrable simple functions only.

  Next, we complete their proofs.

Linearity

Theorem

If f and g are integrable functions and α and β are real numbers, then

$$\int(\alpha f+\beta g)d\mu=\alpha\int fd\mu+\beta\int gd\mu.$$

- Let $\{f_n\}$ and $\{g_n\}$ be mean fundamental sequences of integrable simple functions, such that $f_n\to f$ and $g_n\to g$ in measure.

  Then, we have

$$\begin{array}{rcl}\int(\alpha f+\beta g)d\mu & = & \lim_n\int(\alpha f_n+\beta g_n)d\mu \\ & = & \alpha\lim_n\int f_nd\mu+\beta\lim_n\int g_nd\mu \\ & = & \alpha\int fd\mu+\beta\int gd\mu.\end{array}$$

Positivity

Theorem

If an integrable function f is non negative a.e., then $\int f d\mu \geq 0$.

- Let $\{f_n\}$ be a mean fundamental sequence of integrable simple functions, such that $f_n \to f$ in measure.

  By switching to $\{|f_n|\}$, if necessary, we may assume, without loss of generality, that $f_n \geq 0$, for all n.

  Then $\int f_n d\mu \geq 0$, for all n, and therefore,

$$\int f d\mu = \lim_n \int f_n d\mu \geq 0.$$

Comparison

Theorem

If f and g are integrable functions such that $f \geq g$ a.e., then

$$\int f d\mu \geq \int g d\mu.$$

- We get

$$\begin{array}{rcl} f \geq g \text{ a.e} & \text{iff} & f - g \geq 0 \text{ a.e.} \\ & \text{implies} & \int (f - g) d\mu \geq 0 \\ & \text{iff} & \int f d\mu - \int g d\mu \geq 0 \\ & \text{iff} & \int f d\mu \geq \int g d\mu. \end{array}$$

Absolute Values

Theorem

If f and g are integrable functions, then

$$\int |f+g|d\mu \leq \int |f|d\mu + \int |g|d\mu.$$

- We have $\int |f+g|d\mu \leq \int (|f|+|g|)d\mu = \int |f|d\mu + \int |g|d\mu$.

Theorem

If f is an integrable function, then

$$\left|\int f d\mu\right| \leq \int |f|d\mu.$$

- Again, $\int f d\mu \leq \int |f|d\mu$ and $\int (-f)d\mu \leq \int |f|d\mu$. Combining, we get $-\int |f|d\mu \leq \int f d\mu \leq \int |f|d\mu$, i.e., $|\int f d\mu| \leq \int |f|d\mu$.

Boundedness

Theorem

If f is an integrable function, α and β are real numbers, and E is a measurable set, such that, for x in E, $\alpha \leq f(x) \leq \beta$, then

$$\alpha\mu(E) \leq \int_E f d\mu \leq \beta\mu(E).$$

- The assumption is equivalent to the relation

$$\alpha\chi_E \leq \chi_E f \leq \beta\chi_E.$$

  So the result follows from the Comparison Theorem if $\mu(E) < \infty$.

  The case in which $\mu(E) = \infty$ is easily treated by direct application of the definition of integrability.

Indefinite Integral

- The **indefinite integral** of an integrable function f is the set function ν, defined, for every measurable set E, by

$$\nu(E) = \int_E f d\mu.$$

Theorem

If an integrable function f is non negative a.e., then its indefinite integral is monotone.

- If E and F are measurable sets, such that $E \subseteq F$, then $\chi_E f \leq \chi_F f$ a.e.. The desired result follows from the Comparison Theorem.

Absolute Continuity

Theorem

The indefinite integral of an integrable function f is absolutely continuous.

- Let $\{f_n\}$ be a mean fundamental sequence of integrable simple functions which converges in measure to f.

  We have, for every measurable set E,

$$\left|\int_E f d\mu\right| \leq \left|\int_E f_n d\mu\right| + \left|\int_E f_n d\mu - \int_E f d\mu\right|.$$

  - The f_n are simple functions. So, by uniform absolute continuity, the first term on the right becomes arbitrarily small if the measure of E is taken sufficiently small.
  - The second term on the right approaches 0 as $n \to \infty$, by the definition of $\int_E f d\mu$.

Countable Additivity of Indefinite Integral

Theorem

The indefinite integral of an integrable function is countably additive.

- Let $\{f_n\}$ be a mean fundamental sequence of integrable simple functions which converges in measure to f.

  If ν_n is the indefinite integral of f_n, then, we know that $\nu(E) = \lim_n \nu_n(E)$ exists for every measurable E and ν is finite valued and countably additive.

  So, for every disjoint sequence of measurable sets $\{E_i\}_{i=1}^{\infty}$,

$$\begin{aligned}\nu(\bigcup_{i=1}^{\infty} E_i) &= \lim_n \nu_n(\lim_k \bigcup_{i=1}^{k} E_i) = \lim_{n,k} \nu_n(\bigcup_{i=1}^{k} E_i) \\ &= \lim_{n,k} \sum_{i=1}^{k} \nu_n(E_i) = \lim_k \sum_{i=1}^{k}(\lim_n \nu_n(E_i)) \\ &= \sum_{i=1}^{\infty} \nu(E_i).\end{aligned}$$

Mean Fundamental Sequences

- A sequence $\{f_n\}$ of integrable functions is **fundamental in the mean**, or **mean fundamental**, if

$$\rho(f_n, f_m) \overset{n,m\to\infty}{\longrightarrow} 0.$$

Theorem

A mean fundamental sequence $\{f_n\}$ of integrable functions is fundamental in measure.

- For fixed $\epsilon > 0$, set $E_{nm} = \{x : |f_n(x) - f_m(x)| \geq \epsilon\}$. Then

$$\rho(f_n, f_m) = \int |f_n - f_m| d\mu \geq \int_{E_{nm}} |f_n - f_m| d\mu \geq \epsilon\mu(E_{nm}),$$

so that $\mu(E_{mn}) \to 0$ as $n, m \to \infty$.

The Limit of Indefinite Integrals

Theorem

If $\{f_n\}$ is a mean fundamental sequence of integrable functions, and if the indefinite integral of f_n is ν_n, $n = 1, 2, \ldots$, then $\nu(E) = \lim_n \nu_n(E)$ exists for every measurable set E, and the set function ν is finite valued and countably additive.

- Since $|\nu_n(E) - \nu_m(E)| \leq \int |f_n - f_m| d\mu \overset{n,m \to \infty}{\to} 0$, the existence, finiteness, and uniformity of the limit are clear. It follows, by finite additivity of limits, that ν is finitely additive. If $\{E_n\}$ is a disjoint sequence of measurable sets whose union is E, then, for positive n, k,
$$\begin{aligned}|\nu(E) - \textstyle\sum_{i=1}^k \nu(E_i)| \leq{}& |\nu(E) - \nu_n(E)| \\ &+ |\nu_n(E) - \textstyle\sum_{i=1}^k \nu_n(E_i)| + |\nu_n(\bigcup_{i=1}^k E_i) - \nu(\bigcup_{i=1}^k E_i)|.\end{aligned}$$
The first and third terms of the right may be made arbitrarily small by choosing n sufficiently large. For fixed n, the middle term may be made arbitrarily small by choosing k sufficiently large. This proves that $\nu(E) = \lim_k \sum_{i=1}^k \nu(E_i) = \sum_{i=1}^\infty \nu(E_i)$.

Convergence in the Mean

- We shall say that a sequence $\{f_n\}$ of integrable functions **converges in the mean**, or **mean converges**, to an integrable function f if

$$\rho(f_n, f) = \int |f_n - f| d\mu \overset{n\to\infty}{\longrightarrow} 0.$$

Theorem

If $\{f_n\}$ is a sequence of integrable functions which converges in the mean to f, then $\{f_n\}$ converges to f in measure.

- For any $\epsilon > 0$, set $E_n = \{x : |f_n(x) - f(x)| \geq \epsilon\}$.

  Then

$$\int |f_n - f| d\mu \geq \int_{E_n} |f_n - f| d\mu \geq \epsilon\mu(E_n).$$

  So $\mu(E_n) \overset{n\to\infty}{\longrightarrow} 0$.

Vanishing a.e. and Vanishing Integral

Theorem

If f is an a.e. non negative integrable function, then a necessary and sufficient condition that $\int f d\mu = 0$ is that $f = 0$ a.e..

- If $f = 0$ a.e., then the sequence each of whose terms is identically zero is a mean fundamental sequence of integrable simple functions which converges in measure to f. It follows that $\int f d\mu = 0$.

  To prove the converse, we observe that, if $\{f_n\}$ is a mean fundamental sequence of integrable simple functions which converges in measure to f, then we may assume that $f_n \geq 0$, since we may replace each f_n by its absolute value.

  The assumption $\int f d\mu = 0$ implies that $\lim_n \int f_n d\mu = 0$, i.e., that $\{f_n\}$ mean converges to 0. It follows by the preceding theorem, that $\{f_n\}$ converges to 0 in measure. By a preceding result, $f = 0$ a.e..

Integrals over Sets of Measure Zero

Theorem

If f is an integrable function and E is a set of measure zero, then $\int_E f d\mu = 0$.

- By definition, $\int_E f d\mu = \int \chi_E f d\mu$.

  But the characteristic function of a set of measure zero vanishes a.e..

  Hence, the result follows from the preceding theorem.

Vanishing Integrals of Positive a.e. Functions

Theorem

If f is an integrable function which is positive a.e. on a measurable set E and if $\int_E f d\mu = 0$, then $\mu(E) = 0$.

- We write:

$$\begin{array}{rcl} F_0 & = & \{x : f(x) > 0\}; \\ F_n & = & \{x : f(x) \geq \frac{1}{n}\}, \quad n = 1, 2, \ldots. \end{array}$$

  Since the assumption of positiveness implies that $E - F_0$ is a set of measure zero, it suffices to show that $E \cap F_0$ is one also.

  But we have:
  - $F_0 = \bigcup_{n=1}^{\infty} F_n$;
  - $0 \leq \frac{1}{n}\mu(E \cap F_n) \leq \int_{E \cap F_n} f d\mu = 0$.

  Therefore, $\mu(E \cap F_0) \leq \sum_{n=1}^{\infty} \mu(E \cap F_n) = 0$.

Integrals Vanishing on All Measurable Sets

Theorem

If f is an integrable function such that $\int_F f d\mu = 0$, for every measurable set F, then $f = 0$ a.e..

- Let $E = \{x : f(x) > 0\}$.

  By hypothesis, $\int_E f d\mu = 0$.

  Therefore, by the preceding theorem, E is a set of measure zero.

  Applying the same reasoning to $-f$ shows that $\{x : f(x) < 0\}$ is a set of measure zero.

  Hence, $f = 0$ a.e..

Supports are σ-Finite in Measure

Theorem

If f is an integrable function, then the set $N(f) = \{x : f(x) \neq 0\}$ has σ-finite measure.

- Let $\{f_n\}$ be a mean fundamental sequence of integrable simple functions which converges in measure to f.

  For $n = 1, 2, \ldots$, $N(f_n)$ is a measurable set of finite measure.

  Let $E = N(f) - \bigcup_{n=1}^{\infty} N(f_n)$ and F a measurable subset of E.

  We have $\int_F f d\mu = \lim_n \int_F f_n d\mu = 0$.

  By the preceding theorem, $f = 0$ a.e. on E. Thus, $\mu(E) = 0$.

  Now we have $N(f) \subseteq \bigcup_{n=1}^{\infty} N(f_n) \cup E$.

Extended Real-Value Functions and Integrals

- If f is an extended real valued, measurable function such that $f \geq 0$ a.e. and if f is not integrable, then we write $\int f d\mu = \infty$.
- We may define $\int f d\mu$, for the class of all extended real valued measurable functions f for which at least one of the two functions f^+ and f^- is integrable:

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu.$$

- Since at most one of the two numbers $\int f^+ d\mu$, $\int f^- d\mu$ is infinite, the value of $\int f d\mu$ is always $+\infty$, $-\infty$, or a finite real number.
- We make free use of this extended notion of integration, but we apply the adjective "integrable" only to functions that are integrable in the sense of our former definitions.

Subsection 4

Sequences of Integrable Functions

Mean Fundamental Sequences of Simple Functions

Theorem

If $\{f_n\}$ is a mean fundamental sequence of integrable simple functions, which converges in measure to the integrable function f, then

$$\rho(f, f_n) = \int |f - f_n| d\mu \xrightarrow{n \to \infty} 0.$$

Hence, to every integrable function f and to every positive number ϵ, there corresponds an integrable simple function g, such that $\rho(f, g) < \epsilon$.

- For any fixed positive integer m, $\{|f_n - f_m|\}$ is a mean fundamental sequence of integrable simple functions which converges in measure to $|f - f_m|$. Therefore, $\int |f - f_m| d\mu = \lim_n \int |f_n - f_m| d\mu$. The fact that the sequence $\{f_n\}$ is mean fundamental implies the desired result.

Convergence of Mean Fundamental Sequences

Theorem

If $\{f_n\}$ is a mean fundamental sequence of integrable functions, then there exists an integrable function f, such that $\rho(f_n, f) \to 0$ (and, consequently, $\int f_n d\mu \to \int f d\mu$) as $n \to \infty$.

- By the preceding theorem, for each positive integer n, there is an integrable simple function g_n, such that $\rho(f_n, g_n) < \frac{1}{n}$. It follows that $\{g_n\}$ is a mean fundamental sequence of integrable simple functions. Let f be a measurable (and therefore integrable) function such that $\{g_n\}$ converges in measure to f. Then

$$\begin{array}{rcl} 0 & \leq & |\int f_n d\mu - \int f d\mu| \leq \int |f_n - f| d\mu \\ & = & \rho(f_n, f) \leq \rho(f_n, g_n) + \rho(g_n, f). \end{array}$$

  Now the desired result follows from the preceding theorem.

Continuity and Equicontinuity from Above at Zero

- A finite valued set function ν on a class $\boldsymbol{E}$ of sets is **continuous from above at 0** if, for every decreasing sequence $\{E_n\}$ of sets in $\boldsymbol{E}$, for which $\lim_n E_n = \emptyset$, we have $\lim_n \nu(E_n) = 0$.
- If $\{\nu_n\}$ is a sequence of such finite valued set functions on $\boldsymbol{E}$, we shall say that the terms of the sequence are **equicontinuous from above at 0** if, for every decreasing sequence $\{E_n\}$ of sets in $\boldsymbol{E}$, for which $\lim_n E_n = \emptyset$, and for every positive number ϵ, there exists a positive integer m_0, such that if $m \geq m_0$, then $|v_n(E_m)| < \epsilon$, $n = 1, 2, \ldots$.

Convergence in Mean in Measure and Equicontinuity

Theorem

A sequence $\{f_n\}$ of integrable functions converges in the mean to the integrable function f if and only if $\{f_n\}$ converges in measure to f and the indefinite integrals of $\{f_n\}$, $n = 1, 2, \ldots$, are uniformly absolutely continuous and equicontinuous from above at 0.

- We prove first the necessity of the conditions. Convergence in measure and uniform absolute continuity follow from preceding results. So, it suffices to show equicontinuity.

  The mean convergence of $\{f_n\}$ to f implies that, to every positive number ϵ, there corresponds a positive integer n_0, such that if $n \geq n_0$, then $\int |f_n - f| d\mu < \frac{\epsilon}{2}$. Since the indefinite integral of a non negative integrable function is a finite measure, it follows that such an indefinite integral is continuous from above at 0.

Proof of Necessity (Cont'd)

- To every positive number ϵ, there corresponds a positive integer n_0, such that if $n \geq n_0$, then $\int |f_n - f| d\mu < \frac{\epsilon}{2}$.

  By continuity from above at 0 of the indefinite integral of a non negative integrable function, if $\{E_m\}$ is a decreasing sequence of measurable sets with an empty intersection, then there exists a positive integer m_0, such that, for $m \geq m_0$, $\int_{E_m} |f| d\mu < \frac{\epsilon}{2}$ and $\int_{E_m} |f_n - f| d\mu < \frac{\epsilon}{2}$, $n = 1, \ldots, n_0$.

  Hence, if $m \geq m_0$, then we have

  $$\int_{E_m} |f_n| d\mu \leq \int_{E_m} |f_n - f| d\mu + \int_{E_m} |f| d\mu < \epsilon,$$

  for every positive integer n. This is exactly the desired equicontinuity.

Proof of Sufficiency

- A countable union of measurable sets of σ-finite measure is a measurable set of σ-finite measure. So $E_0 = \bigcup_{n=1}^{\infty}\{x : f_n(x) \neq 0\}$ is such a set.

  Suppose $\{E_n\}$ is an increasing sequence of measurable sets of finite measure such that $\lim_n E_n = E_0$. If $F_n = E_0 - E_n$, $n = 1, 2, \ldots$, then $\{F_n\}$ is a decreasing sequence and $\lim_n F_n = 0$. By equicontinuity, for every $\delta > 0$, there exists an integer $k > 0$, such that $\int_{F_k} |f_n| d\mu < \frac{\delta}{2}$. Consequently,

$$\int_{F_k} |f_m - f_n| d\mu \leq \int_{F_k} |f_m| d\mu + \int_{F_k} |f_n| d\mu < \delta.$$

  For fixed $\epsilon > 0$, write $G_{mn} = \{x : |f_m(x) - f_n(x)| \geq \epsilon\}$. Then

$$\begin{array}{rcl} \int_{E_k} |f_m - f_n| d\mu & \leq & \int_{E_k - G_{mn}} |f_m - f_n| d\mu + \int_{E_k \cap G_{mn}} |f_m - f_n| d\mu \\ & \leq & \epsilon\mu(E_k) + \int_{E_k \cap G_{mn}} |f_m - f_n| d\mu. \end{array}$$

Proof of Sufficiency (Cont'd)

- We got
$$\int_{E_k} |f_m - f_n| d\mu \leq \epsilon\mu(E_k) + \int_{E_k \cap G_{mn}} |f_m - f_n| d\mu.$$
By convergence in measure and uniform absolute continuity, the second term on the right may be made arbitrarily small by choosing m and n sufficiently large. Hence, $\limsup_{m,n} \int_{E_k} |f_m - f_n| d\mu \leq \epsilon\mu(E_k)$. Since ϵ is arbitrary, it follows that $\lim_{m,n} \int_{E_k} |f_m - f_n| d\mu = 0$. But
$$\begin{aligned} \int |f_m - f_n| d\mu &= \int_{E_0} |f_m - f_n| d\mu \\ &= \int_{E_k} |f_m - f_n| d\mu + \int_{F_k} |f_m - f_n| d\mu. \end{aligned}$$
So $\limsup_{m,n} \int |f_m - f_n| d\mu < \delta$ and, since δ is arbitrary, $\lim_{m,n} \int |f_m - f_n| d\mu = 0$. I.e., $\{f_n\}$ is fundamental in the mean. By our second theorem, that there exists an integrable function g such that $\{f_n\}$ mean converges to g. Since mean convergence implies convergence in measure, we must have $f = g$ a.e..

Lebesgue's Bounded Convergence Theorem

Lebesgue's Bounded Convergence Theorem

If $\{f_n\}$ is a sequence of integrable functions which converges in measure to f (or else converges to f a.e.), and if g is an integrable function such that $|f_n(x)| \leq |g(x)|$ a.e., $n = 1, 2, \ldots$, then f is integrable and the sequence $\{f_n\}$ converges to f in the mean.

- Suppose $\{f_n\}$ convergences in measure. The given inequality ensures that the indefinite integrals of $\{f_n\}$ are uniformly absolutely continuous and equicontinuous from above at 0. The conclusion now follows from the preceding theorem.

  Suppose, next that $\{f_n\}$ converges a.e.. Assume, without loss of generality, that $|f_n(x)| \leq |g(x)|$ and $|f(x)| \leq |g(x)|$, for every x in X. Then, for every $\epsilon > 0$,

$$E_n := \bigcup_{i=n}^{\infty} \{x : |f_i(x) - f(x)| \geq \epsilon\} \subseteq \{x : |g(x)| \geq \frac{\epsilon}{2}\}.$$

Lebesgue's Bounded Convergence Theorem (Cont'd)

- We have, for all $\epsilon > 0$,

$$E_n = \bigcup_{i=n}^{\infty} \{x : |f_i(x) - f(x)| \geq \epsilon\} \subseteq \{x : |g(x)| \geq \frac{\epsilon}{2}\}.$$

  Therefore, $\mu(E_n) < \infty$, $n = 1, 2, \ldots$.

  Since $\{f_n\}$ converges a.e., $\mu(\bigcap_{n=1}^{\infty} E_n) = 0$.

  By one of our earlier results, $\lim_n \mu(E_n) = \mu(\lim_n E_n)$.

  Now we get

$$\limsup_n \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \leq \lim_n \mu(E_n) = \mu(\lim_n E_n) = 0.$$

  Thus, convergence a.e., together with being bounded by an integrable function, implies convergence in measure.

  So we can rely on the preceding case.

Subsection 5

Properties of Integrals

Measurability and Integrability

Theorem

If f is measurable, g is integrable, and $|f| \leq |g|$ a.e., then f is integrable.

- Consideration of the positive and negative parts of f shows that it is sufficient to prove the theorem for non negative functions f.
  - If f is a simple function, the result is clear.
  - In the general case, there is an increasing sequence $\{f_n\}$ of non negative simple functions such that $\lim_n f_n(x) = f(x)$, for all x in X. Since $0 \leq f_n \leq |g|$, each f_n is integrable. The desired result follows from the bounded convergence theorem.

Convergence a.e. and Integrability

Theorem

If $\{f_n\}$ is an increasing sequence of extended real valued non negative measurable functions and $\lim_n f_n(x) = f(x)$ a.e., then $\lim_n \int f_n d\mu = \int f d\mu$.

- If f is integrable, then the result follows from the bounded convergence theorem and the preceding theorem.
- The only novel feature of the present theorem is its application to the not necessarily integrable case: We must show that if $\int f d\mu = \infty$, then $\lim_n \int f_n d\mu = \infty$, i.e., that, if $\lim_n \int f_n d\mu < \infty$, then f is integrable. From the finiteness of the limit we may conclude that $\lim_{m,n} |\int f_m d\mu - \int f_n d\mu| = 0$. Since $f_m - f_n$ is of constant sign, for each fixed m and n, we have $|\int f_m d\mu - \int f_n d\mu| = \int |f_m - f_n| d\mu$, so that the sequence $\{f_n\}$ is mean convergent. Therefore, by a preceding result, it mean converges to an integrable function g. But mean convergence implies convergence in measure, and therefore a.e. convergence for some subsequence. So $f = g$ a.e..

Integrability from Integrability of Absolute Value

Theorem

A measurable function is integrable if and only if its absolute value is integrable.

- The new part of this theorem is the assertion that the integrability of $|f|$ implies that of f.

  This follows from the first theorem, with $|f|$ in place of g.

Integrability and Essential Boundedness

Theorem

If f is integrable and g is an essentially bounded measurable function, then fg is integrable.

- If $|g| \leq c$ a.e., then $|fg| \leq c|f|$ a.e..

  By hypothesis and the preceding theorem, fg is integrable.

Theorem

If f is an essentially bounded measurable function and E is a measurable set of finite measure, then f is integrable over E.

- The characteristic function of a measurable set of finite measure is an integrable function.

  The result follows from the preceding theorem with χ_E and f in place of f and g.

Fatou's Lemma

Theorem (Fatou's Lemma)

If $\{f_n\}$ is a sequence of non negative integrable functions for which $\liminf_n \int f_n d\mu < \infty$, then the function f, defined by $f(x) = \liminf_n f_n(x)$, is integrable and $\int f d\mu \leq \liminf_n \int f_n d\mu$.

- Let $g_n(x) = \inf\{f_i(x) : n \leq i < \infty\}$.
  - $g_n \leq f_n$.
  - $\{g_n\}$ is increasing.

  Since $\int g_n d\mu \leq \int f_n d\mu$,
  $$\lim_n \int g_n d\mu \leq \liminf_n \int f_n d\mu < \infty.$$
  But $\lim_n g_n(x) = \liminf_n f_n(x) = f(x)$.
  So, by the second theorem, f is integrable and
  $$\int f d\mu = \lim_n \int g_n d\mu \leq \liminf_n \int f_n d\mu.$$