Introduction to Measure Theory

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1 General Set Functions

- Signed Measures
- Hahn and Jordan Decompositions
- Absolute Continuity
- The Radon-Nikodym Theorem
- Derivatives of Signed Measures

Subsection 1

Signed Measures

Linear Combinations of Measures

- We generalize the notion of measure to set functions that are not required to be non negative.
- Suppose that μ_1 and μ_2 are two measures on a σ-ring $\boldsymbol{S}$ of subsets of a set X. If we define, for every set E in $\boldsymbol{S}$, $\mu(E) = \mu_1(E) + \mu_2(E)$, then it is clear that μ is a measure.

  This result, on the possibility of adding two measures, extends immediately to any finite sum.
- Another way of manufacturing new measures is to multiply a given measure by an arbitrary non negative constant.
- Combining these two methods, we see that, if $\{\mu_1, \ldots, \mu_n\}$ is a finite set of measures and $\{\alpha_1, \ldots, \alpha_n\}$ is a finite set of non negative real numbers, then the set function μ, defined,for every set E in $\boldsymbol{S}$, by $\mu(E) = \sum_{i=1}^n \alpha_i \mu_i(E)$, is a measure.

Allowing Negative Coefficients

- The situation is different if we allow negative coefficients.
- If μ_1 and μ_2 are two measures on $\boldsymbol{S}$, and if we define μ by $\mu(E) = \mu_1(E) - \mu_2(E)$, then we face two new possibilities.
  - μ may be negative on some sets. This is an interesting phenomenon worth investigating.
  - $\mu_1(E) = \mu_2(E) = \infty$. In this case, to avoid the difficulty of indeterminate forms, we shall agree to subtract two measures only if at least one of them is finite.
- This convention is analogous to the one we adopted in presenting the most general definition of the symbol $\int f d\mu$:
  - It is defined for a measurable function f if and only if at least one of the two functions f^+ and f^- is integrable, i.e., if and only if at least one of the two set functions ν^+ and ν^- defined by $\nu^+(E) = \int_E f^+ d\mu$ and $\nu^-(E) = \int_E f^- d\mu$ is a finite measure.
  - If f is a measurable function, such that $\int f d\mu$ is defined, then the set function ν, defined by $\nu(E) = \int_E f d\mu$ is the difference of two measures.

Signed Measures

- We define a **signed measure** as an extended real valued function μ on the class of all measurable sets of a measurable space $(X, \boldsymbol{S})$, such that:
  - $\mu(\emptyset) = 0$;
  - μ is countably additive;
  - μ assumes at most one of the values $+\infty$ and $-\infty$.
- If $\{E_n\}$ is a disjoint sequence of measurable sets, then the series $\sum_{n=1}^{\infty} \mu(E_n)$ is either convergent or definitely divergent (to $+\infty$ or $-\infty$).

  In any case, the symbol $\sum_{n=1}^{\infty} \mu(E_n)$ makes sense.

Finiteness and σ-Finiteness

- The words [**totally**] **finite** and [**totally**] σ-**finite** will be used for signed measures just as for measures, except that $\mu(E)$ has to be replaced by $|\mu(E)|$, or, equivalently, $\mu(E) < \infty$ has to be replaced by $-\infty < \mu(E) < \infty$.
- E.g., a signed measure μ is **totally finite** if X is measurable and $|\mu(X)| < \infty$.

Finite Additivity and Subtractiveness

- A signed measure is finitely additive.

  This follows from the fact that it is countably additive.
- A signed measure is subtractive.

  If $F \subseteq E$ are measurable, then, by additivity,

$$\mu(E) = \mu(F) + \mu(E - F).$$

  Therefore, $\mu(E - F) = \mu(E) - \mu(F)$.

Measurable Subsets of a Set of Finite Signed Measure

Theorem

If E and F are measurable sets and μ is a signed measure, such that $E \subseteq F$ and $|\mu(F)| < \infty$, then $|\mu(E)| < \infty$.

- We have $\mu(F) = \mu(F - E) + \mu(E)$.
  - If exactly one of the summands is infinite, then so is $\mu(F)$;
  - If they are both infinite, then (since μ assumes at most one of the values $+\infty$ and $-\infty$), they are equal and again $\mu(F)$ is infinite.
  - Only one possibility remains, namely that both summands are finite.

Thus, every measurable subset of a set of finite signed measure has finite signed measure.

Boundedness of Measure and Absolute Convergence

Theorem

If μ is a signed measure and $\{E_n\}$ is a disjoint sequence of measurable sets such that $|\mu(\bigcup_{n=1}^{\infty} E_n)| < \infty$, then the series $\sum_{n=1}^{\infty} \mu(E_n)$ is absolutely convergent.

- Write $E_n^+ = \begin{cases} E_n, & \text{if } \mu(E_n) \geq 0 \\ 0, & \text{if } \mu(E_n) < 0 \end{cases}$ and $E_n^- = \begin{cases} E_n, & \text{if } \mu(E_n) \leq 0 \\ 0, & \text{if } \mu(E_n) > 0 \end{cases}$. Then, $\mu(\bigcup_{n=1}^{\infty} E_n^+) = \sum_{n=1}^{\infty} \mu(E_n^+)$ and $\mu(\bigcup_{n=1}^{\infty} E_n^-) = \sum_{n=1}^{\infty} \mu(E_n^-)$. Since the terms of both series are of constant sign, and since μ takes on at most one of the values $+\infty$ and $-\infty$, at least one of these series is convergent. Since the sum of the two series is the convergent series $\sum_{n=1}^{\infty} \mu(E_n)$, they both converge. Since the convergence of the series of positive terms and the series of negative terms is equivalent to absolute convergence, the proof is complete.

Signed Measures of Limits of Sequences

Theorem

If μ is a signed measure, if $\{E_n\}$ is a monotone sequence of measurable sets, and if, in case $\{E_n\}$ is a decreasing sequence, $|\mu(E_n)| < \infty$, for at least one value of n, then $\mu(\lim_n E_n) = \lim_n \mu(E_n)$.

- Suppose, first, that $\{E_n\}$ is increasing.

  Set $E_0 = \emptyset$ and $F_i = E_i - E_{i-1}$, $i = 1, 2, \ldots$.

  Then, we have

$$\begin{array}{rcl}\mu(\lim_n E_n) & = & \mu(\bigcup_{i=1}^{\infty} E_i) = \mu(\bigcup_{i=1}^{\infty} F_i) \\ & = & \sum_{i=1}^{\infty} \mu(F_i) = \lim_n \sum_{i=1}^{n} \mu(F_i) \\ & = & \lim_n \mu(E_n).\end{array}$$

Signed Measures of Limits of Sequences (Cont'd)

- Suppose, next, that $\{E_n\}$ is decreasing and $|\mu(E_m)| < \infty$.
  By the first theorem, $|\mu(E_n)| < \infty$, for all $n \geq m$.
  The sequence $\{E_m - E_n\}$ is increasing and we have

$$\begin{array}{rcl}\mu(E_m) - \mu(\lim_n E_n) & = & \mu(E_m - \lim_n E_n) \\ & = & \mu(\lim_n (E_m - E_n)) \\ & = & \lim_n \mu(E_m - E_n) \\ & = & \lim_n (\mu(E_m) - \mu(E_n)) \\ & = & \mu(E_m) - \lim_n \mu(E_n).\end{array}$$

  It follows that $\mu(\lim_n E_n) = \lim_n \mu(E_n)$.

Subsection 2

Hahn and Jordan Decompositions

Positive and Negative Measurable Sets

- Let μ be a signed measure on the class of all measurable sets of a measurable space $(X, \boldsymbol{S})$.
- We shall call a set E **positive** (with respect to μ) if, for every measurable set F,
  - $E \cap F$ is measurable;
  - $\mu(E \cap F) \geq 0$.
- Similarly, we shall call E **negative** if, for every measurable set F,
  - $E \cap F$ is measurable;
  - $\mu(E \cap F) \leq 0$.
- The empty set is both positive and negative in this sense.
- No assertion is made about the existence of any other, non trivial, positive sets or negative sets.

The Hahn Decomposition Theorem

Theorem (Hahn Decomposition Theorem)

If μ is a signed measure, then there exist two disjoint sets A and B whose union is X, such that A is positive and B is negative with respect to μ.

- The sets A and B are said to form a **Hahn decomposition** of X with respect to μ.
- Since μ assumes at most one of the values $+\infty$ and $-\infty$, we may assume that, say $-\infty < \mu(E) \leq +\infty$, for every measurable set E. Note that
  - the difference of two negative sets is negative;
  - the disjoint, countable union of negative sets is negative.

  So every countable union of negative sets is negative.

Proof of the Hahn Decomposition Theorem

- We write $\beta = \inf \mu(B)$, for all measurable negative sets B.

  Let $\{B_i\}$ be a sequence of measurable negative sets such that $\lim_i \mu(B_i) = \beta$. If $B = \bigcup_{i=1}^{\infty} B_i$, then B is a measurable negative set for which $\mu(B)$ is minimal.

  Claim: The set $A = X - B$ is a positive set.

  Suppose that, on the contrary, E_0 is a measurable subset of A, such that $\mu(E_0) < 0$. The set E_0 cannot be a negative set, for then $B \cup E_0$ would be a negative set with a smaller value of μ than $\mu(B)$, which is impossible. Let k_1 be the smallest positive integer with the property that E_0 contains a measurable set E_1, such that $\mu(E_1) \geq \frac{1}{k_1}$.

  Since $\mu(E_0) < 0$, $\mu(E_0)$ and $\mu(E_1)$ are both finite.

Proof of the Hahn Decomposition Theorem (Cont'd)

- Now observe that

$$\mu(E_0 - E_1) = \mu(E_0) - \mu(E_1) \leq \mu(E_0) - \frac{1}{k_1} < 0.$$

So the argument just applied to E_0 is applicable to $E_0 - E_1$ also.

Let k_2 be the smallest positive integer with the property that $E_0 - E_1$ contains a measurable subset E_2, with $\mu(E_2) \geq \frac{1}{k_2}$.

Then proceed ad infinitum.

μ is finite valued for measurable subsets of E_0. So $\lim_n \frac{1}{k_n} = 0$.

It follows that, for every measurable subset F of $F_0 = E_0 - \bigcup_{j=1}^\infty E_j$, we have $\mu(F) \leq 0$. i.e., that F_0 is a measurable negative set.

  - F_0 is disjoint from B.
  - $\mu(F_0) = \mu(E_0) - \sum_{j=1}^\infty \mu(E_j) \leq \mu(E_0) < 0$.

This contradicts the minimality of B.

We conclude that the hypothesis $\mu(E_0) < 0$ is untenable.

Upper, Lower and Total Variations of a Signed Measure

- It is not difficult to construct examples to show that a Hahn decomposition is not unique.
- But (as we show in the next slide) if $X = A_1 \cup B_1$ and $X = A_2 \cup B_2$ are two Hahn decompositions of X, then, for every measurable set E,

$$\mu(E \cap A_1) = \mu(E \cap A_2) \quad \text{and} \quad \mu(E \cap B_1) = \mu(E \cap B_2).$$

- Thus, the equations

$$\mu^+(E) = \mu(E \cap A) \quad \text{and} \quad \mu^-(E) = -\mu(E \cap B)$$

  unambiguously define two set functions μ^+ and μ^- on the class of all measurable sets.

  They are called, respectively, the **upper variation** and the **lower variation** of μ.
- The set function $|\mu|$, defined, for every measurable set E, by $|\mu|(E) = \mu^+(E) + \mu^-(E)$, is the **total variation** of μ.

Upper and Lower Variations of a Signed Measure

Proposition

If $X = A_1 \cup B_1$ and $X = A_2 \cup B_2$ are two Hahn decompositions of X, then, for every measurable set E,

$$\mu(E \cap A_1) = \mu(E \cap A_2) \quad \text{and} \quad \mu(E \cap B_1) = \mu(E \cap B_2).$$

- Observe that $E \cap (A_1 - A_2) \subseteq E \cap A_1$ and $E \cap (A_1 - A_2) \subseteq E \cap B_2$.
  Hence, $\mu(E \cap (A_1 - A_2)) \geq 0$ and $\mu(E \cap (A_1 - A_2)) \leq 0$.
  It follows that $\mu(E \cap (A_1 - A_2)) = 0$.
  By symmetry, $\mu(E \cap (A_2 - A_1)) = 0$.
  Therefore, $\mu(E \cap A_1) = \mu(E \cap (A_1 \cup A_2)) = \mu(E \cap A_2)$.

The Jordan Decomposition Theorem

Theorem (Jordan Decomposition Theorem)

The upper, lower, and total variations of a signed measure μ are measures and $\mu(E) = \mu^+(E) - \mu^-(E)$, for every measurable set E. If μ is [totally] finite or σ-finite, then so also are μ^+ and μ^-; at least one of the measures μ^+ and μ^- is always finite.

- The variations of μ are clearly non negative. If every measurable set is a countable union of measurable sets for which μ is finite, by the first theorem of the set, the same holds for μ^+ and μ^-. The equation $\mu = \mu^+ - \mu^-$ follows from the definitions of μ^+ and μ^-. The fact that μ takes on at most one of the values $+\infty$ and ∞ implies that at least one of the set functions μ^+ and μ^- is always finite. Since the countable additivity of μ^+ and μ^- is evident, the proof is complete.
- Thus, every signed measure is the difference of two measures (of which at least one is finite). The representation of μ as the difference of its upper and lower variations is the **Jordan decomposition** of μ.

Subsection 3

Absolute Continuity

Absolute Continuity With Respect to a Measure

- Let $(X, \boldsymbol{S})$ be a measurable space.

  Let μ and ν be signed measures on $\boldsymbol{S}$.

  We say that ν is **absolutely continuous with respect to** μ, in symbols $\nu \ll \mu$, if, for every measurable set E,

$$|\mu|(E) = 0 \quad \text{implies} \quad \nu(E) = 0.$$

- In a suggestively imprecise phrase, $\nu \ll \mu$ means that ν is small whenever μ is small.

Alternative Characterizations

Theorem

If μ and ν are signed measures, the following conditions are equivalent:

(a) $v \ll \mu$;

(b) $\nu^+ \ll \mu$ and $\nu^- \ll \mu$;

(c) $|\nu| \ll |\mu|$.

(a)⇒(b) Suppose (a) holds and let E be measurable, such that $|\mu|(E) = 0$. Consider a Hahn decomposition $X = A \cup B$ w.r.t. ν. Then, we have

$$|\mu|(E \cap A) \leq |\mu|(E) = 0 \quad \text{and} \quad |\mu|(E \cap B) \leq |\mu|(E) = 0.$$

Thus, by hypothesis, $\nu(E \cap A) = \nu(E \cap B) = 0$.

By definition, $\nu^+(E) = \nu^-(E) = 0$.

Alternative Characterizations (Cont'd)

(b)⇒(c) Suppose (b) holds and let E be measurable such that $|\mu|(E) = 0$.

By hypothesis, $\nu^+(E) = \nu^-(E) = 0$.

But then, we get $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$.

(c)⇒(a) Suppose (c) holds and let E be measurable such that $|\mu|(E) = 0$.

By hypothesis, $|\nu|(E) = 0$.

Now we have

$$0 \leq |\nu(E)| = |\nu^+(E) - \nu^-(E)| \leq \nu^+(E) + \nu^-(E) = |\nu|(E) = 0.$$

Therefore, $\nu(E) = 0$.

Necessary Condition Under Finiteness

Theorem

If ν is a finite signed measure and if μ is a signed measure such that $\nu \ll \mu$, then, for every $\epsilon > 0$, there is a $\delta > 0$, such that, for every measurable set E,

$$|\mu|(E) < \delta \quad \text{implies} \quad |\nu|(E) < \epsilon.$$

- Suppose that it is possible, for some $\epsilon > 0$, to find a sequence $\{E_n\}$ of measurable sets, such that $|\mu|(E_n) < \frac{1}{2^n}$ and $|\nu|(E_n) \geq \epsilon$, $n = 1, 2, \ldots$. Let $E = \limsup_n E_n$. Then $|\mu|(E) \leq \sum_{i=n}^{\infty} |\mu|(E_i) < \frac{1}{2^{n-1}}$, $n = 1, 2, \ldots$. Therefore $|\mu|(E) = 0$.

  On the other hand, since ν is finite,

  $$|\nu|(E) = \lim_n |\nu|(E_n \cup E_{n+1} \cup \cdots) \geq \limsup_n |\nu|(E_n) \geq \epsilon.$$

  This contradicts the relation $\nu \ll \mu$.

Reflexivity, Transitivity and Equivalence

Proposition (Reflexivity)

If μ is a signed measure, $\mu \ll \mu$.

- Let E be measurable, such that $|\mu|(E) = 0$.
  Then $\mu^+(E) + \mu^-(E) = 0$, whence $\mu^+(E) = \mu^-(E) = 0$.
  Now we get $\mu(E) = \mu^+(E) - \mu^-(E) = 0$.

Proposition (Transitivity)

If μ_1, μ_2, μ_3 are signed measures, then $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_3$ imply $\mu_1 \ll \mu_3$.

- If E is measurable, such that $|\mu_3|(E) = 0$, then, by $\mu_2 \ll \mu_3$ and the characterization theorem, $|\mu_2|(E) = 0$. By $\mu_1 \ll \mu_2$, $\mu_1(E) = 0$.
- Two signed measures μ and μ for which both $\nu \ll \mu$ and $\mu \ll \nu$ hold are called **equivalent**, in symbols $\mu \equiv \nu$.

Singularity

- The antithesis of the relation of absolute continuity is **singularity**.
- Let $(X, \boldsymbol{S})$ be a measurable space and μ and ν signed measures on $\boldsymbol{S}$.
  We say that μ and ν are **mutually singular**, or more simply **singular**, in symbols $\mu \perp \nu$, if there exist two disjoint sets A and B whose union is X, such that, for every measurable set E,
  - $A \cap E$ and $B \cap E$ are measurable;
  - $|\mu|(A \cap E) = |\nu|(B \cap E) = 0$.
- Sometimes, despite the symmetry of the relation, we use an asymmetric expression, such as "ν is singular with respect to μ", instead of "μ and ν are singular".

The Modulo μ Notation

- In the discussion of absolute continuity and singularity we have to deal with several measures simultaneously.
- In such contexts, the following notation is useful.
- Let $(X, \boldsymbol{S})$ be a measurable space.

  Let $\pi(x)$ be a proposition concerning each point x of X.

  Let μ be a signed measure on $\boldsymbol{S}$.

  The symbol
  $$\pi(x)\ [\mu] \quad \text{or} \quad \pi\ [\mu]$$
  will mean that $\pi(x)$ is true for almost every x with respect to the measure $|\mu|$.

  Example: If f and g are two functions on X, we write $f = g\ [\mu]$ for the statement that $\{x : f(x) \neq g(x)\}$ is a measurable set of measure zero with respect to $|\mu|$.
- The symbol $[\mu]$ may be read as "**modulo** μ".

Subsection 4

The Radon-Nikodym Theorem

Absolute Continuity and Signed Measures

Theorem

If μ and ν are totally finite measures such that $\nu \ll \mu$ and ν is not identically zero, then there exists a positive number ϵ and a measurable set A, such that $\mu(A) > 0$ and such that A is a positive set for the signed measure $\nu - \epsilon\mu$.

- Let $X = A_n \cup B_n$ be a Hahn decomposition with respect to the signed measure $\nu - \frac{1}{n}\mu$, $n = 1, 2, \ldots$. Write $A_0 = \bigcup_{n=1}^{\infty} A_n$ $B_0 = \bigcap_{n=1}^{\infty} B_n$. Since $B_0 \subseteq B_n$, we have $0 \leq \nu(B_0) \leq \frac{1}{n}\mu(B_0)$, $n = 1, 2, \ldots$. Consequently, $\nu(B_0) = 0$. It follows that $\nu(A_0) > 0$. Therefore, by absolute continuity, $\mu(A_0) > 0$. Hence, we must have $\mu(A_n) > 0$, for at least one value of n. If, for such a value of n, we write $A = A_n$ and $\epsilon = \frac{1}{n}$, the requirements of the theorem are all satisfied.

The Radon-Nikodym Theorem

Theorem (The Radon-Nikodym Theorem)

If $(X, \boldsymbol{S}, \mu)$ is a totally σ-finite measure space and if a σ-finite signed measure ν on $\boldsymbol{S}$ is absolutely continuous with respect to μ, then there exists a finite valued measurable function f on X, such that

$$\nu(E) = \int_E f d\mu, \text{ for every measurable set } E.$$

The function f is unique in the sense that, if also $\nu(E) = \int_E g d\mu$, for $E \in \boldsymbol{S}$, then $f = g$ $[\mu]$.

- We emphasize the fact that f is not asserted to be integrable.
  - It is, in fact, clear that a necessary and sufficient condition that f be integrable is that ν be finite.
  - The use of the symbol $\int f d\mu$ implicitly asserts that either the positive or the negative part of f is integrable, corresponding to the fact that either the upper or the lower variation of ν is finite.

Proof of the Radon-Nikodym Theorem (Reductions)

- Since X is a countable, disjoint union of measurable sets on which both μ and ν are finite, there is no loss of generality in assuming finiteness in the first place.

  For uniqueness, assume $\int_E f d\mu = \int_E g d\mu$, for every measurable E. Then $\int_E (f-g) d\mu = 0$, for every measurable E. By a result on integrable functions, $f - g = 0$ $[\mu]$, i.e., $f = g$ $[\mu]$.

  Recall, by the characterization of absolute continuity, that the assumption $\nu \ll \mu$ is equivalent to the simultaneous validity of the conditions $\nu^+ \ll \mu$ and $\nu^- \ll \mu$. It is sufficient to prove the existence of f in the case in which both μ and ν are finite measures. We would then have

$$\nu(E) = \nu^+(E) - \nu^-(E) = \int_E f_1 d\mu - \int_E f_2 d\mu = \int_E (f_1 - f_2) f d\mu.$$

Proof of the Radon-Nikodym Theorem

- Let $\mathcal{K}$ be the class of all non negative functions f, integrable with respect to μ, such that $\int_E f d\mu \leq \nu(E)$, for every measurable set E. Set $\alpha = \sup\{\int f d\mu : f \in \mathcal{K}\}$.
  Let $\{f_n\}$ be a sequence of functions in $\mathcal{K}$, such that $\lim_n \int f_n d\mu = \alpha$.
  Let E be measurable, n a positive integer, and $g_n = f_1 \cup \cdots \cup f_n$.
  Then E may be written as a finite, disjoint union of measurable sets, $E = E_1 \cup \cdots \cup E_n$, so that $g_n(x) = f_j(x)$, for x in E_j, $j = 1, \ldots, n$.
  Consequently we have

$$\int_E g_n d\mu = \sum_{j=1}^n \int_{E_j} f_j d\mu \leq \sum_{j=1}^n \nu(E_j) = \nu(E).$$

  Write $f_0(x) = \sup\{f_n(x) : n = 1, 2, \ldots\}$. Then $f_0(x) = \lim_n g_n(x)$.
  It follows by an integration theorem that $f_0 \in \mathcal{K}$ and $\int f_0 d\mu = \alpha$.
  Since f_0 is integrable, there exists a finite valued f, with $f_0 = f\ [\mu]$.

Proof of the Radon-Nikodym Theorem (Cont'd)

- Claim: If $\nu_0(E) = \nu(E) - \int_E f d\mu$, the measure ν_0 is identically zero.

  Suppose ν_0 is not identically zero.

  By the first theorem, there exists $\epsilon > 0$ and a measurable A, such that $\mu(A) > 0$ and, for every measurable E,

  $$\epsilon\mu(E \cap A) \leq \nu_0(E \cap A) = \nu(E \cap A) - \int_{E \cap A} f d\mu.$$

  If $g = f + \epsilon\chi_A$, then, for every measurable E,

  $$\int_E g d\mu = \int_E f d\mu + \epsilon\mu(E \cap A) \leq \int_{E-A} f d\mu + \nu(E \cap A) \leq \nu(E).$$

  Hence, $g \in \mathcal{K}$.

  But $\int g d\mu = \int f d\mu + \epsilon\mu(A) > \alpha$.

  This contradicts the maximality of $\int f d\mu$.

Subsection 5

Derivatives of Signed Measures

The Radon-Nikodym Derivative

- The functions which occur as integrands in the Radon-Nikodym theorem are called **Radon-Nikodym derivatives**.
- If μ is a totally σ-finite measure and if $\nu(E) = \int_E f d\mu$, for every measurable set E, we write

$$f = \frac{d\nu}{d\mu} \quad \text{or} \quad d\nu = f d\mu.$$

Properties of the Radon-Nikodym Derivative

- All the properties of Radon-Nikodym derivatives, suggested by the well known differential formalism, correspond to true theorems.
  - Some of these are trivial, e.g. $\frac{d(\nu_1+\nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}$.
  - Others are more or less deep properties of integration:
    - The chain rule for differentiation;
    - The substitution rule for the differentials occurring under an integral sign.
- A Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ is unique only a.e. with respect to μ.

  Therefore, differential formulas will be interpreted to hold "almost everywhere".

The Chain Rule

Theorem

If λ and μ are totally σ-finite measures, such that $\mu \ll \lambda$, and if ν is a totally σ-finite signed measure, such that $\nu \ll \mu$, then

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \; [\lambda].$$

- The validity of the desired equation for the upper and lower variations of ν implies its validity for ν. So we may assume that ν is a measure.

  Set $\frac{d\nu}{d\mu} = f$ and $\frac{d\mu}{d\lambda} = g$.

  Since ν is nonnegative, $f \geq 0$ $[\mu]$. Therefore, there is no loss of generality in assuming that f is everywhere non negative.

The Chain Rule (Cont'd)

- By measurability of f, there exists an increasing sequence $\{f_n\}$ of nonnegative simple functions converging at every point to f.

  By properties of integration, for every measurable set E,
  - $\lim_n \int_E f_n d\mu = \int_E f d\mu$;
  - $\lim_n \int_E f_n g d\lambda = \int_E f g d\lambda$.

  But, for every measurable set F,

  $$\int_E \chi_F d\mu = \mu(E \cap F) = \int_{E \cap F} g d\lambda = \int_E \chi_F g d\lambda.$$

  Since $\{f_n\}$ consists of simple functions, it follows that

  $$\int_E f_n d\mu = \int_E f_n g d\lambda, \quad n = 1, 2, \ldots.$$

  Hence, $\nu(E) = \int_E f d\mu = \int_E f g d\lambda$.

Integrals and Derivatives

Theorem

If λ and μ are totally σ-finite measures such that $\mu \ll \lambda$, and if f is a finite valued measurable function for which $\int f d\mu$ is defined, then

$$\int f d\mu = \int f \frac{d\mu}{d\lambda} d\lambda.$$

- Write $\nu(E) = \int_E f d\mu$, for every measurable set E.

  Applying the preceding theorem, we get

  $$\nu(E) = \int_E f \frac{d\mu}{d\lambda} d\lambda, \text{ for every measurable set } E.$$

  The desired result follows by putting $E = X$.

Lebesgue Decomposition Theorem

Theorem (Lebesgue Decomposition)

If $(X, \boldsymbol{S})$ is a measurable space and μ and ν are totally σ-finite signed measures on $\boldsymbol{S}$, then there exist two uniquely determined totally σ-finite signed measures ν_0 and ν_1 whose sum is ν, such that $\nu_0 \perp \mu$ and $\nu_1 \ll \mu$.

- As usual we may assume that μ and v are finite.

  Since ν_i, $i = 0, 1$, will be absolutely continuous or singular with respect to μ according as it is absolutely continuous or singular with respect to $|\mu|$, we may assume that μ is a measure.

  Since, finally, we may treat ν^+ and ν^- separately, we may also assume that ν is a measure.

  The proof of the theorem for totally finite measures is a useful trick, based on the elementary observation that ν is absolutely continuous with respect to $\mu + \nu$.

Proof of the Decomposition Theorem (Existence)

- Since $\nu \ll \mu + \nu$, there exists measurable f, such that

$$\nu(E) = \int_E f d\mu + \int_E f d\nu, \text{ for every measurable set } E.$$

Since $0 \le \nu(E) \le \mu(E) + \nu(E)$, we have $0 \le f \le 1$ $[\mu + \nu]$.
Therefore, $0 \le f \le 1$ $[\nu]$. Set

$$A = \{x : f(x) = 1\} \quad \text{and} \quad B = \{x : 0 \le f(x) < 1\}.$$

Then $\nu(A) = \int_A d\mu + \int_A d\nu = \mu(A) + \nu(A)$.
Since ν is finite, $\mu(A) = 0$.
Set, for every measurable set E,

$$\nu_0(E) = \nu(E \cap A) \quad \text{and} \quad \nu_1(E) = \nu(E \cap B).$$

Then $\nu_0 \perp \mu$. It remains to prove that $\nu_1 \ll \mu$.
If $\mu(E) = 0$, then $\int_{E \cap B} d\nu = \nu(E \cap B) = \int_{E \cap B} f d\nu$ and, therefore, $\int_{E \cap B} (1 - f) d\nu = 0$. Since $1 - f > 0$ $[\nu]$, $\nu_1(E) = \nu(E \cap B) = 0$.

Proof of the Decomposition Theorem (Uniqueness)

- Suppose
$$\nu = \nu_0 + \nu_1 \quad \text{and} \quad \nu = \overline{\nu}_0 + \overline{\nu}_1$$
are two Lebesgue decompositions of ν.
Then $\nu_0 - \overline{\nu}_0 = \overline{\nu}_1 - \nu_1$.
But:
  - $v_0 - \overline{\nu}_0$ is singular with respect to μ;
  - $\overline{\nu}_1 - \nu_1$ is absolutely continuous with respect to μ.

Therefore, $\nu_0 = \overline{\nu}_0$ and $\nu_1 = \overline{\nu}_1$.