Introduction to Measure Theory

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1. Product Spaces
   - Cartesian Products
   - Sections
   - Product Measures
   - Fubini's Theorem

Subsection 1

Cartesian Products

Cartesian Products

- If X and Y are any two sets (not necessarily subsets of the same space), the **Cartesian product** $X \times Y$ is the set of all ordered pairs (x, y), where $x \in X$ and $y \in Y$.

  Example: The **Euclidean plane** is most often viewed as the Cartesian product of two coordinate axes.

- If $A \subseteq X$ and $B \subseteq Y$, we call the set $E = A \times B$ (a subset of $X \times Y$) a **rectangle** and refer to the component sets A and B as its **sides**.

  Note: This usage differs from the terminology in the Euclidean plane which speaks of rectangles only if the sides are intervals.

Empty Rectangles

Theorem

A rectangle is empty if and only if one of its sides is empty.

- Suppose $A \times B \neq \emptyset$, say $(x, y) \in A \times B$.
  Then $x \in A$ and $y \in B$. So $A \neq \emptyset$ and $B \neq \emptyset$.
  Suppose, on the other hand, neither A nor B is empty.
  Then there is a point (x, y), such that $(x, y) \in A \times B$.
  Thus, $A \times B \neq \emptyset$.

Comparing Rectangles Using Their Sides

Theorem

If $E_1 = A_1 \times B_1$, and $E_2 = A_2 \times B_2$ are non empty rectangles, then $E_1 \subseteq E_2$ if and only if $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$.

- The "if" is obvious.
  Conversely, since $E_1 \neq \emptyset$, there exists $(x, y) \in A_1 \times B_1$.
  Suppose that there exists a point $x_1 \in A_1$ such that $x_1 \notin A_2$.
  Then $(x_1, y) \in A_1 \times B_1$ and $(x_1, y) \notin A_2 \times B_2$.
  It follows that no such point x_1 can exist. So $A_1 \subseteq A_2$.
  The same proof with only notational changes shows that $B_1 \subseteq B_2$.

Theorem

If $A_1 \times B_1 = A_2 \times B_2$ is a non empty rectangle, then $A_1 = A_2$ and $B_1 = B_2$.

- By the theorem, $A_1 \subseteq A_2 \subseteq A_1$ and $B_1 \subseteq B_2 \subseteq B_1$.

Disjointness of Rectangles

Theorem

If $E = A \times B$, $E_1 = A_1 \times B_1$ and $E_2 = A_2 \times B_2$ are non empty rectangles, then a necessary and sufficient condition that E be the disjoint union of E_1 and E_2 is that either A is the disjoint union of A_1 and A_2 and $B = B_1 = B_2$, or else B is the disjoint union of B_1 and B_2 and $A = A_1 = A_2$.

- Necessity: Since $E_1 \subseteq E$ and $E_2 \subseteq E$, it follows from the preceding theorem that $A_1 \subseteq A$ and $A_2 \subseteq A$, and, therefore, that $A_1 \cup A_2 \subseteq A$. Similarly, $B_1 \cup B_2 \subseteq B$. Since $E_1 \cup E_2 \subseteq (A_1 \cup A_2) \times (B_1 \cup B_2)$, it follows that $A \subseteq A_1 \cup A_2$ and $B \subseteq B_1 \cup B_2$, and, therefore, $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$.

  A similar argument shows that $\emptyset = E_1 \cap E_2 \supseteq (A_1 \cap A_2) \times (B_1 \cap B_2)$.

  By the first theorem at least one of the two sets $A_1 \cap A_2$ and $B_1 \cap B_2$ is empty.

Proof (Cont'd)

- Suppose, for instance, that $A_1 \cap A_2 = \emptyset$. We are to show that in this case $B = B_1 = B_2$. Suppose that there exists a point y in $B - B_1$. Then, if x is any point in A_1, we have $(x, y) \in E$, but $(x, y) \notin E_1$, and $(x, y) \notin E_2$. Since this contradicts the assumption $E = E_1 \cup E_2$, it follows that $B - B_1 = \emptyset$. By a similar argument, $B - B_2 = \emptyset$.

  Sufficiency: If, for instance, A is the disjoint union of A_1 and A_2 and $B = B_1 = B_2$, then $A \supseteq A_1$, $A \supseteq A_2$, $B \supseteq B_1$ and $B \supseteq B_2$, so that $E \supseteq E_1 \cup E_2$. Also, if $(x, y) \in E$, then $(x, y) \in E_1$ or $(x, y) \in E_2$ according as $x \in A_1$ or $x \in A_2$, so that E is indeed the disjoint union of E_1 and E_2.

Finite Disjoint Unions of Rectangles

Theorem

If $\boldsymbol{S}$ and $\boldsymbol{T}$ are rings of subsets of X and Y respectively, then the class $\boldsymbol{R}$ of all finite, disjoint unions of rectangles of the form $A \times B$, where $A \in \boldsymbol{S}$ and $B \in \boldsymbol{T}$, is a ring.

- The intersection of two sets of the form $A \times B$ is a set of that form.

  If either of the two given sets, or their intersection, is empty, this result is trivial.

  Suppose $E_1 = A_1 \times B_1$, $E_2 = A_2 \times B_2$ and $(x, y) \in E_1 \cap E_2$. Then $x \in A_1 \cap A_2$ and $y \in B_1 \cap B_2$. So $E_1 \cap E_2 \subseteq (A_1 \cap A_2) \times (B_1 \cap B_2)$. On the other hand, by the second theorem, $(A_1 \cap A_2) \times (B_1 \cap B_2)$ is contained in E_1 and E_2 and, therefore, in $E_1 \cap E_2$. So $E_1 \cap E_2 = (A_1 \cap A_2) \times (B_1 \cap B_2)$. Since $\boldsymbol{S}$ and $\boldsymbol{T}$ are rings, $A_1 \cap A_2 \in \boldsymbol{S}$ and $B_1 \cap B_2 \in \boldsymbol{T}$. It follows that the class $\boldsymbol{R}$ is closed under the formation of finite intersections.

Finite Disjoint Unions of Rectangles (Cont'd)

- Note that

$$(A_1 \times B_1) - (A_2 \times B_2) = [(A_1 \cap A_2) \times (B_1 - B_2)] \cup [(A_1 - A_2) \times B_1].$$

So the difference of two sets of the given form is a disjoint union of two other sets of that form.

Also note that

$$\bigcup_{i=1}^{n} E_i - \bigcup_{j=1}^{m} F_j = \bigcup_{i=1}^{n} \bigcap_{j=1}^{m} (E_i - F_j).$$

It follows, using the result of the preceding paragraph, that the class $\boldsymbol{R}$ is closed under the formation of differences.

Since $\boldsymbol{R}$ is obviously closed under the formation of finite, disjoint unions, the proof is complete.

Cartesian Product of Measurable Spaces

- Suppose that, in addition to the two sets X and Y, we are also given two σ-rings $\boldsymbol{S}$ and $\boldsymbol{T}$ of subsets of X and Y, respectively.

  We shall denote by $\boldsymbol{S} \times \boldsymbol{T}$ the σ-ring of subsets of $X \times Y$ generated by the class of all sets of the form $A \times B$, where $A \in \boldsymbol{S}$ and $B \in \boldsymbol{T}$.

Theorem

If $(X, \boldsymbol{S})$ and $(Y, \boldsymbol{T})$ are measurable spaces, then $(X \times Y, \boldsymbol{S} \times \boldsymbol{T})$ is a measurable space.

- The measurable space $(X \times Y, \boldsymbol{S} \times \boldsymbol{T})$ is the **Cartesian product** of the two given measurable spaces.
- If $(x, y) \in X \times Y$, then there exist sets A and B such that $x \in A \in \boldsymbol{S}$ and $y \in B \in \boldsymbol{T}$. It follows that $(x, y) \in A \times B \in \boldsymbol{S} \times \boldsymbol{T}$.
- We have used (and will use) the fact that a measurable space is the union of its measurable sets.

Measurable Sets in Cartesian Product

- We shall frequently use the concept of **measurable rectangle**.
  Two equally obvious and natural definitions of this phrase suggest themselves.
  - According to one, a rectangle in the Cartesian product of two measurable spaces $(X, \boldsymbol{S})$ and $(Y, \boldsymbol{T})$ is **measurable** if it belongs to $\boldsymbol{S} \times \boldsymbol{T}$.
  - According to the other, $A \times B$ is **measurable** if $A \in \boldsymbol{S}$ and $B \in \boldsymbol{T}$.
- It is an easy consequence of the results we shall obtain that for non empty rectangles the two concepts coincide.
- For the time being we adopt the second of our proposed definitions.
- Accordingly, the class of **measurable sets in the Cartesian product** of two measurable spaces is the σ-ring generated by the class of all measurable rectangles.

Subsection 2

Sections

Sections (Sets)

- Let $(X, \boldsymbol{S})$ and $(Y, \boldsymbol{T})$ be measurable spaces and let $(X \times Y, \boldsymbol{S} \times \boldsymbol{T})$ be their Cartesian product.
- If E is any subset of $X \times Y$ and x is any point of X, we shall call the set
$$E_x = \{y : (x, y) \in E\}$$
a **section** of E, or, more precisely, the **section determined by** x, or simply an X-**section**.
- A Y-**section determined by a point** y in Y is defined as the set
$$E^y = \{x : (x, y) \in E\}.$$
- We emphasize that a section of a set in a product space is not a set in that product space but a subset of one of the component spaces.

Sections (Functions)

- Let $(X, \boldsymbol{S})$ and $(Y, \boldsymbol{T})$ be measurable spaces and let $(X \times Y, \boldsymbol{S} \times \boldsymbol{T})$ be their Cartesian product.
- If f is any function defined on a subset E of the product space $X \times Y$ and x is any point of X, we shall call the function f_x, defined on the section E_x by
$$f_x(y) = f(x, y),$$
a **section** of f, or, more precisely, an X-**section** of f, or, still more precisely, the **section determined by** x.
- The concept of a Y-**section** of f, **determined by a point** y in Y is defined similarly by
$$f^y(x) = f(x, y).$$

Measurability of Sections of Measurable Sets

Theorem

Every section of a measurable set is a measurable set.

- Let $\boldsymbol{E}$ be the class of all those subsets of $X \times Y$ which have the property that each of their sections is measurable.
  - Every measurable rectangle $A \times B$ is in $\boldsymbol{E}$: Observe that every section of E is either empty or else equal to one of the sides, A or B, according as the section is a Y-section or an X-section.
  - $\boldsymbol{E}$ is a σ-ring:
    - Given $E, F \in \boldsymbol{E}$, $(E - F)_x = E_x - F_x$, and similarly for Y-sections. Thus, $E - F \in \boldsymbol{E}$.
    - Given $\{E^i\}_{i=1}^{\infty} \subseteq \boldsymbol{E}$, $(\bigcup_{i=1}^{\infty} E^i)_x = \bigcup_{i=1}^{\infty} E_x^i$, and similarly for y sections. Thus, $\bigcup_{i=1}^{\infty} E^i \in \boldsymbol{E}$.

So $\boldsymbol{E}$ is a σ-ring containing all measurable rectangles.

It follows that $\boldsymbol{S} \times \boldsymbol{T} \subseteq \boldsymbol{E}$.

Measurability of Sections of Measurable Functions

Theorem

Every section of a measurable function is a measurable function.

- Let f be a measurable function on $X \times Y$, x a point of X, and M a Borel set on the real line.

  The measurability of $N(f_x) \cap f_x^{-1}(M)$ follows from the preceding theorem and the following relations:

$$\begin{array}{rcl} f_x^{-1}(M) & = & \{y : f_x(y) \in M\} \\ & = & \{y : f(x,y) \in M\} \\ & = & \{y : (x,y) \in f^{-1}(M)\} \\ & = & (f^{-1}(M))_x. \end{array}$$

  (Observe that $N(f_x) = (N(f))_x$.)

  The proof of the measurability of an arbitrary Y-section of f is similar.

Subsection 3

Product Measures

Integrating Sections

Theorem

If $(X, \boldsymbol{S}, \mu)$ and $(Y, \boldsymbol{T}, \nu)$ are σ-finite measure spaces, and if E is any measurable subset of $X \times Y$, then the functions f and g, defined on X and Y, respectively, by $f(x) = \nu(E_x)$ and $g(y) = \mu(E^y)$ are nonnegative measurable functions such that $\int f d\mu = \int g d\nu$.

- Let $\boldsymbol{M}$ be the class of all those sets E for which the conclusion of the theorem is true. The proof involves many steps:
    - Show the result holds for finite measures.
        - Show that $\boldsymbol{M}$ includes the ring $\boldsymbol{R}$ of all finite disjoint unions of rectangles of the form $A \times B$, with $A \in \boldsymbol{S}$ and $B \in \boldsymbol{T}$;
        - Show that $\boldsymbol{M}$ is a monotone class.

      Since the class of measurable sets is the σ-ring generated by the ring $\boldsymbol{R}$, conclude that every measurable set is in $\boldsymbol{M}$.
    - Extend the result to σ-finite measures.

Integrating Sections ($R \subseteq M$)

- Suppose $A \times B$ is a measurable rectangle.
  Note that
  $$\begin{array}{rcl} f(x) & = & \nu((A \times B)_x) = \nu(B)\chi_A(x); \\ g(y) & = & \mu((A \times B)^y) = \mu(A)\chi_B(y). \end{array}$$
  Thus, f and g are measurable.
  Moreover, $\int f d\mu = \mu(A)\nu(B) = \int g d\nu$.
- Suppose, next, that $\bigcup_{i=1}^n (A^i \times B^i)$ is a finite disjoint union of measurable rectangles.
  Note that
  $$\begin{array}{rcl} f(x) & = & \nu((\bigcup_{i=1}^n (A^i \times B^i))_x) = \nu(\bigcup_{i=1}^n ((A^i \times B^i)_x)) \\ & = & \sum_{i=1}^n \nu((A^i \times B^i)_x) = \sum_{i=1}^n \nu(B^i)\chi_{A^i}(x); \\ g(y) & = & \sum_{i=1}^n \mu(A^i)\chi_{B^i}(y). \end{array}$$
  Thus, f and g are measurable.
  Moreover, $\int f d\mu = \sum_{i=1}^n \mu(A^i)\nu(B^i) = \int g d\nu$.

Integrating Sections (Monotonicity of $\boldsymbol{M}$)

- Suppose that $\{E^i\}$ is an increasing sequence of sets in $\boldsymbol{M}$.
  Then $\lim_n E^i = \bigcup_i E^i$. We must show $E = \bigcup_i E^i \in \boldsymbol{M}$.
  Let f_i and g_i be the functions associated with E^i and let f and g be the ones associates with E.
  - $\lim_n f_n = f$: We have:

    $$\begin{array}{rcl} f(x) & = & \nu((\bigcup_i E^i)_x) = \nu(\bigcup_i E^i_x) = \nu(\bigcup_i (E^{i+1}_x - E^i_x)) \\ & = & \sum_{i=1}^\infty (\nu(E^{i+1}_x) - \nu(E^i_x)) = \lim_n \nu(E^n_x) = \lim_n f_n(x). \end{array}$$

  - $|f_n(x)| \leq \nu(B)$: This is clear, since $|f_n(x)| = |\nu(E^n_x)| \leq \nu(B)$.

  By the Bounded Convergence Theorem, f is integrable.
  Analogously, we get that g is integrable.
- Finally, noting that $\{f_n\}$ and $\{g_n\}$ are increasing, nonnegative, with $\lim_n f_n = f$ and $\lim_n g_n = g$, by the Monotone Convergence Theorem,

  $$\begin{array}{rcl} \int f d\mu & = & \int (\lim_n f_n) d\mu = \lim_n \int f_n d\mu \\ & = & \lim_n \int g_n d\nu = \int (\lim_n g_n) d\nu = \int g d\nu. \end{array}$$

Integrating Sections (General Case: Sketch)

- We note that $\boldsymbol{M}$ is closed under the formation of countable, disjoint unions.

  Then, observe that the σ-finiteness of μ and ν implies that every set in $\boldsymbol{S} \times \boldsymbol{T}$ may be covered by a countable disjoint union of measurable rectangles, both sides of each of which have finite measure.

  We have showed that every measurable subset of every measurable rectangle with sides of finite measure belongs to $\boldsymbol{M}$.

  It now follows that every measurable set belongs to $\boldsymbol{M}$.

  This concludes the proof of the theorem.

Product Measures and Product Spaces

Theorem

If $(X, \boldsymbol{S}, \mu)$ and $(Y, \boldsymbol{T}, \nu)$ are σ-finite measure spaces, then the set function λ, defined, for every set E in $\boldsymbol{S} \times \boldsymbol{T}$, by

$$\lambda(E) = \int \nu(E_x) d\mu(x) = \int \mu(E^y) d\nu(y),$$

is a σ-finite measure with the property that, for every measurable rectangle $A \times B$, $\lambda(A \times B) = \mu(A) \cdot \nu(B)$.
The latter condition determines λ uniquely.

- The measure λ is called the **product** of the given measures μ and ν, in symbols $\lambda = \mu \times \nu$.
- The measure space $(X \times Y, \boldsymbol{S} \times \boldsymbol{T}, \mu \times \nu)$ is the **Cartesian product** of the given measure spaces.

Product Measures (Proof)

- λ is a measure:
  - $\lambda(\emptyset) = \int \nu(\emptyset_x) f\mu = \int 0 d\mu = 0.$
  - For disjoint measurable $\{E^i\}$, taking into account the Monotone Convergence Theorem:

$$\begin{array}{rcl}\lambda(\bigcup_i E^i) & = & \int \nu((\bigcup_i E^i)_x) d\mu = \int \nu(\bigcup_i E^i_x) d\mu \\ & = & \int \sum_i \nu(E^i_x) d\mu = \int \lim_n \sum_{i=1}^n \nu(E^i_x) d\mu \\ & = & \lim_n \int \sum_{i=1}^n \nu(E^i_x) d\mu \\ & = & \lim_n \sum_{i=1}^n \int \nu(E^i_x) d\mu \\ & = & \sum_n \lambda(E^i).\end{array}$$

The σ-finiteness of λ follows from the fact that every measurable subset of $X \times Y$ may be covered by countably many measurable rectangles of finite measure.

Uniqueness is given by the Extension Theorem of a σ-finite measure on a ring $\boldsymbol{R}$ to a measure on the σ-ring $\boldsymbol{S}(\boldsymbol{R})$ generated by $\boldsymbol{R}$.

Subsection 4

Fubini's Theorem

Double Integrals

- We assume that $(X, \boldsymbol{S}, \mu)$ and $(Y, \boldsymbol{T}, \nu)$ are σ-finite measure spaces and λ is the product measure $\mu \times \nu$ on $\boldsymbol{S} \times \boldsymbol{T}$.
- If a function h on $X \times Y$ is such that its integral is defined, then the integral is denoted by

$$\int h(x, y) d\lambda(x, y) \quad \text{or} \quad \int h(x, y) d(\mu \times \nu)(x, y)$$

and is called the **double integral** of h.

Iterated Integrals

- If h_x is such that $\int h_x(y)d\nu(y) = f(x)$ is defined, and if it happens that $\int fd\mu$ is also defined, it is customary to write

$$\int fd\mu = \iint h(x,y)d\nu(y)d\mu(x) = \int d\mu(x) \int h(x,y)d\nu(y).$$

- The symbols

$$\iint h(x,y)d\mu(x)d\nu(y) \quad \text{and} \quad \int d\nu(y) \int h(x,y)d\mu(x)$$

  are defined similarly, as the integral (if it exists) of the function g on Y, defined by $g(y) = \int h^y(x)d\mu(x)$.
- The integrals $\iint hd\mu d\nu$ and $\iint hd\nu d\mu$ are called the **iterated integrals** of h.

Double and Iterated Integrals over a Set

- To indicate the double integral of h over a measurable subset E of $X \times Y$, i.e., the integral of $\chi_E h$, we write
$$\int_E h d\lambda.$$
- To indicate the iterated integrals of h over a measurable subset E of $X \times Y$, i.e., the integrals of $\chi_E h$, we shall use the symbols
$$\iint_E h d\mu d\nu \quad \text{and} \quad \iint_E h d\nu d\mu.$$

"Almost Every Section"

- X-sections (of sets or functions) are determined by points in X.

  We say a proposition is true for **almost every X-section** if the set of those points x for which the proposition is not true is a set of measure zero in X.
- Y-sections (of sets or functions) are determined by points in Y.

  We say a proposition is true for **almost every Y-section** if the set of those points y for which the proposition is not true is a set of measure zero in Y.
- If a proposition is true simultaneously for a.e. X-section and a.e. Y-section, we say that it is true for **almost every section**.

Vanishing Almost Everywhere

Theorem

A necessary and sufficient condition that a measurable subset E of $X \times Y$ have measure zero is that almost every X-section (or almost every Y-section) have measure zero.

- By the definition of product measure,

$$\lambda(E) = \begin{cases} \int \nu(E_x) d\mu(x) \\ \int \mu(E^y) d\nu(y) \end{cases}.$$

If $\lambda(E) = 0$, then the integrals on the right are in particular finite. Thus, by a theorem on integrable functions, their non negative integrands must vanish a.e..

If, conversely, either of the integrands vanishes a.e., then $\lambda(E) = 0$.

Double and Iterated Integrals

Theorem

If h is a non negative, measurable function on $X \times Y$, then

$$\int hd(\mu \times \nu) = \iint hd\mu d\nu = \iint hd\nu d\mu.$$

- Suppose, first, $h = \chi_E(x, y)$ for a measurable set E.

$$\begin{array}{rcl} \int h(x,y)d\nu(y) & = & \int \chi_E(x,y)d\nu(y) = \nu(E_x); \\ \int h(x,y)d\mu(x) & = & \int \chi_E(x,y)d\mu(x) = \mu(E^y). \end{array}$$

Therefore,

$$\int h(x,y)d\lambda(x,y) = \left\{ \begin{array}{l} \int \nu(E_x)d\mu(x) \\ \int \mu(E^y)d\nu(y) \end{array} \right\} = \left\{ \begin{array}{l} \iint h(x,y)d\nu d\mu \\ \iint h(x,y)d\mu d\nu \end{array} \right. .$$

Double and Iterated Integrals (General Case)

- In the general case we may find an increasing sequence $\{h_n\}$ of non negative simple functions converging to h everywhere.

  Since a simple function is a finite linear combination of characteristic functions, the conclusion is valid for every h_n in place of h, i.e., $\int h_n d(\mu \times \nu) = \iint h_n d\mu d\nu = \iint h_n d\nu d\mu$.
  - By Monotone Convergence, $\lim_n \int h_n d\lambda = \int h d\lambda$.
  - Suppose $f_n(x) = \int h_n(x, y) d\nu(y)$. By the properties of $\{h_n\}$, $\{f_n\}$ is an increasing sequence of non negative measurable functions converging, for every x, to $f(x) = \int h(x, y) d\nu(y)$. Hence f is measurable (and nonnegative). By Monotone Convergence, $\lim_n \int f_n d\mu = \int f d\mu$.

  Thus, $\int h d\lambda = \iint h d\nu d\mu$.

  The truth of the other equality follows similarly.

Fubini's Theorem

Theorem (Fubini's Theorem)

If h is an integrable function on $X \times Y$, then almost every section of h is integrable. If the functions f and g are defined by

$$f(x) = \int h(x,y) d\nu(y) \quad \text{and} \quad g(y) = \int h(x,y) d\mu(x),$$

then f and g are integrable and $\int h d(\mu \times \nu) = \int f d\mu = \int g d\nu$.

- A real valued function is integrable if and only if its positive and negative parts are integrable.
  So it is sufficient to consider only nonnegative functions h.
  The asserted identity follows in this case from the preceding theorem.
  Since the nonnegative, measurable functions f and g have finite integrals, it follows that they are integrable.
  This implies that f and g are finite valued almost everywhere.
  Thus, the sections of h have the desired integrability properties.