Introduction to Model Theory

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Subsection 1

Structures

Structures

Definition

A **structure** A is an object with the following four ingredients:

1. A set called the **domain** of A, written $\text{dom}(A)$ or $\text{dom}A$ (also called the **universe** or **carrier** of A). The elements of $\text{dom}(A)$ are called the **elements** of the structure A. The **cardinality** of A, in symbols $|A|$, is defined to be the cardinality $|\text{dom}A|$ of $\text{dom}(A)$.
2. A set of elements of A, called **constant elements**, each of which is named by one or more **constants**. If c is a constant, we write c^A for the constant element named by c.
3. For each positive integer n, a set of n-ary relations on $\text{dom}(A)$ (i.e., subsets of $(\text{dom}A)^n$), each of which is named by one or more n-ary **relation symbols**. If R is a relation symbol, we write R^A for the relation named by R.

Structures (Cont'd)

Definition (Cont'd)

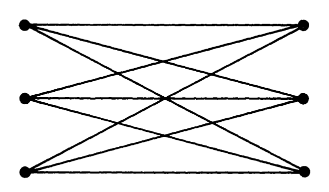
4. For each positive integer n, a set of n-ary operations on $\text{dom}(A)$ (i.e., maps from $(\text{dom}A)^n$ to $\text{dom}(A)$), each of which is named by one or more n-ary **function symbols**. If F is a function symbol, we write F^A for the function named by F.

- Any of the sets in 1-4 may be empty, unless stated otherwise.
- We shall use capital letters $A, B, C, \ldots$ for structures.
- Sequences of elements of a structure are written $\overline{a}, \overline{b}$ etc.
- A **tuple in** A (or **from** A) is a finite sequence of elements of A.

  It is an n-**tuple** if it has length n.

Example: Graphs

- A **graph** consists of a set V (the set of **vertices**) and a set E (the set of **edges**), where each edge is a set of two distinct vertices.
- An edge $\{v, w\}$ is said to **join** the two vertices v and w.
- We can picture a finite graph by putting dots for the vertices and joining two vertices v, w by a line when $\{v, w\}$ is an edge.



- One natural way to make a graph G into a structure is as follows:
  - The elements of G are the vertices.
  - There is one binary relation R^G. The ordered pair (v, w) lies in R^G if and only if there is an edge joining v to w.

Examples: Linear Orderings and Groups

- **Linear orderings**: Suppose $\leq$ linearly orders a set X. Then we can make $(X, \leq)$ into a structure A as follows:
  - The domain of A is the set X.
  - There is one binary relation symbol R, and its interpretation R^A is the ordering $\leq$.

  In practice we usually write the relation symbol as $\leq$ rather than R.
- **Groups**: We can think of a group as a structure G with:
  - One constant 1 naming the identity 1^G;
  - One binary function symbol $\cdot$ naming the group product operation $\cdot^G$;
  - One unary function symbol $^{-1}$ naming the inverse operation $^{(-1)^G}$.

  Another group H will have the same symbols $1, \cdot, ^{-1}$.

  Then 1^H is the identity element of H, $\cdot^H$ is the product operation of H, and so on.

Example: Vector Spaces

- One way to make a vector space into a structure is as follows:
  Suppose V is a vector space over a field of scalars K.
  - Take the domain of V to be the set of vectors of V.
  - There is one constant element 0^V, the origin of the vector space.
  - There is one binary operation, $+^V$, which is addition of vectors.
  - There is a unary operation $-^V$ for additive inverse.
  - For every scalar k, there is a unary operation k^V to represent multiplying a vector by k.

  Thus, each scalar serves as a unary function symbol.

  Note, the symbol "$-$" is redundant, because $-^V$ is the same operation as $(-1)^V$.
- When we speak of **vector spaces**, we shall assume that they are structures of this form (unless the contrary is explicitly stated).
- The same goes for **modules**, replacing the field K by a ring.

Signatures

- The **signature** of a structure A is specified by giving:
  - The set of constants of A;
  - For each separate $n > 0$,
    - The set of n-ary relation symbols;
    - The set of n-ary function symbols of A.
- We assume that the signature of a structure can be read off uniquely from the structure.
- The symbol L will be used to stand for signatures.
- If A has signature L, we say A is an L-**structure**.
- A signature L with no constants or function symbols is called a **relational signature**, and an L-structure is then said to be a **relational structure**.
- A signature with no relation symbols is sometimes called an **algebraic signature**.

Subsection 2

Homomorphisms and Substructures

Homomorphisms, Embeddings and Isomorphisms

- Let L be a signature and A, B be L-structures.

  A **homomorphism** f from A to B, written $f : A \to B$, is a function f from $\text{dom}(A)$ to $\text{dom}(B)$, with the following properties:

  1. For each constant c of L, $f(c^A) = c^B$;
  2. For each $n > 0$, each n-ary relation symbol R of L and n-tuple $\overline{a}$ from A, if $\overline{a} \in R^A$, then $f(\overline{a}) \in R^B$;
  3. For each $n > 0$, each n-ary function symbol F of L and n-tuple $\overline{a}$ from A, $f(F^A(\overline{a})) = F^B(f(\overline{a}))$.

  If $\overline{a}$ is $(a_0, \ldots, a_{n-1})$, then $f(\overline{a})$ means $(f(a_0), \ldots, f(a_{n-1}))$.

- By an **embedding** of A into B we mean a homomorphism $f : A \to B$ which is injective and satisfies the following stronger version of 2:

  4. For each $n > 0$, each n-ary relation symbol R of L and each n-tuple $\overline{a}$ from A, $\overline{a} \in R^A$ if and only if $f(\overline{a}) \in R^B$.

- An **isomorphism** is a surjective embedding.

Endomorphisms, Automorphisms, Isomorphic Structures

- Homomorphisms $f: A \to A$ are called **endomorphisms** of A.
- Isomorphisms $f: A \to A$ are called **automorphisms** of A.

  Example: If G and H are groups and $f: G \to H$ is a homomorphism, then:
  - Property 1 says that $f(1^G) = 1^H$;
  - Property 3 says that, for all elements a, b of G, $f(a \cdot^G b) = f(a) \cdot^H f(b)$ and $f(a^{(-1)^G}) = f(a)^{(-1)^H}$.

  This is exactly the usual definition of homomorphism between groups.

  Since Property 4 is vacuous for groups, a homomorphism between groups is an embedding iff it is an injective homomorphism.
- We write 1_A for the identity map on $\text{dom}(A)$.

  It is a homomorphism from A to A, in fact an automorphism of A.
- We say that A is **isomorphic** to B, written $A \cong B$, if there is an isomorphism from A to B.

Elementary Properties of Homomorphisms

- The following facts are nearly all immediate from the definitions.

Theorem

Let L be a signature.

(a) If A, B, C are L-structures and $f : A \to B$ and $g : B \to C$ are homomorphisms, then the composition gf is a homomorphism from A to C. If moreover f and g are both embeddings, then so is gf.

(b) If A, B are L-structures and $f : A \to B$ is a homomorphism, $1_B f = f = f 1_A$.

(c) Let A, B, C be L-structures. Then 1_A is an isomorphism. If $f : A \to B$ is an isomorphism, the inverse $f^{-1} : \mathrm{dom}(B) \to \mathrm{dom}(A)$ exists and is an iso from B to A. If $f : A \to B$ and $g : B \to C$ are isomorphisms then so is gf.

(d) The relation $\cong$ is an equivalence relation on the class of L-structures.

(e) If A, B are L-structures, $f : A \to B$ is a homomorphism and there exist homomorphisms $g : B \to A$ and $h : B \to A$, such that $gf = 1_A$ and $fh = 1_B$, then f is an isomorphism and $g = h = f^{-1}$.

Proof of (e)

(e) We have

$$g = g1_B = gfh = 1_A h = h.$$

Since $gf = 1_A$, f is an embedding.

Since $fh = 1_B$, f is surjective.

Hence, f is an isomorphism, with $g = h = f^{-1}$.

Extensions and Substructures

- If A and B are L-structures with $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$ and the inclusion map $i : \mathrm{dom}(A) \to \mathrm{dom}(B)$ is an embedding, then we say that:
  - B is an **extension** of A;
  - A is a **substructure** of B.

  We write $A \subseteq B$.
- Note that if i is the inclusion map from $\mathrm{dom}(A)$ to $\mathrm{dom}(B)$, then:
  - Condition 1 says that $c^A = c^B$, for each constant c;
  - Condition 2 says that

    $$R^A = R^B \cap (\mathrm{dom}A)^n,$$

    for each n-ary relation symbol R;
  - Condition 3 says that

    $$F^A = F^B \restriction_{(\mathrm{dom}A)^n} \quad \text{(the restriction of } F^B \text{ to } (\mathrm{dom}A)^n\text{)},$$

    for each n-ary function symbol F.

Subsets Forming Domains of Substructures

Lemma

Let B be an L-structure and X a subset of $\mathrm{dom}(B)$. Then the following are equivalent:

(a) $X = \mathrm{dom}(A)$, for some $A \subseteq B$.

(b) For every constant c of L, $c^B \in X$; and for every $n > 0$, every n-ary function symbol F of L and every n-tuple $\overline{a}$ of elements of X, $F^B(\overline{a}) \in X$.

If (a) and (b) hold, then A is uniquely determined.

- Suppose (a) holds. Then, for every constant c of L, $c^B = c^A$. But $c^A \in \mathrm{dom}(A) = X$. So $c^B \in X$. Similarly, for each n-ary function symbol F of L and each n-tuple $\overline{a}$ of elements of X, $\overline{a}$ is an n-tuple in A. So $F^B(\overline{a}) = F^A(\overline{a}) \in \mathrm{dom}(A) = X$. This proves (b).

Subsets Forming Domains of Substructures (Cont'd)

- Conversely, if (b) holds, then we can define A by putting:
  - $\text{dom}(A) = X$;
  - $c^A = c^B$, for each constant c of L;
  - $F^A = F^B|_{X^n}$, for each n-ary function symbol F of L;
  - $R^A = R^B \cap X^n$, for each n-ary relation symbol R of L.

  Then $A \subseteq B$. Moreover this is the only possible definition of A, given that $A \subseteq B$ and $\text{dom}(A) = X$.

Generated Substructures

- Let B be an L-structure and Y a set of elements of B.
- It follows from the lemma that there is a unique smallest substructure A of B whose domain includes Y, called the **substructure of** B **generated by** Y, or the **hull** of Y in B, in symbols $A = \langle Y \rangle_B$.
- We call Y a **set of generators** for A.
- A structure B is said to be **finitely generated** if B is of form $\langle Y \rangle_B$, for some finite set Y of elements.
- We write $\langle Y \rangle$ instead of $\langle Y \rangle_B$ if B is clear from context.
- If the generators in Y are listed as a sequence $\overline{a}$, then we write $\langle \overline{a} \rangle_B$ for $\langle Y \rangle_B$.

Cardinality of Languages

- Define the **cardinality of** L, in symbols $|L|$, to be the least infinite cardinal $\geq$ the number of symbols in L.
- We shall see that $|L|$ is equal to the number of first-order formulas of L, up to choice of variables.
- This is one reason why $|L|$ is taken to be infinite even when L contains only finitely many symbols.

  Warning: Occasionally it's important to know that a signature L contains only finitely many symbols.

  In this case we say that L is **finite**, in spite of the definition just given for $|L|$.

Cardinality of Generated Structures

Theorem

Let B be an L-structure and Y a set of elements of B. Then $|\langle Y\rangle_B| \leq |Y| + |L|$.

- We shall construct $\langle Y\rangle_B$ explicitly, thus proving its existence and uniqueness at the same time. We define a set $Y_m \subseteq \mathrm{dom}(B)$, for each $m < \omega$, by induction on m:

$$\begin{array}{rcl} Y_0 & = & Y \cup \{c^B : c \text{ a constant of } L\}, \\ Y_{m+1} & = & Y_m \cup \{F^B(\overline{a}) : \text{for some } n > 0,\ F \text{ is an } n\text{-ary function} \\ & & \text{symbol of } L \text{ and } \overline{a} \text{ is an } n\text{-tuple of elements of } Y_m\}. \end{array}$$

  Finally we put $X = \bigcup_{m<\omega} Y_m$.

  Clearly X satisfies condition (b) of the preceding lemma.

  So there is a unique substructure A of B with $X = \mathrm{dom}(A)$.

Cardinality of Generated Structures (Cont'd)

- Suppose A' is a substructure of B with $Y \subseteq \mathrm{dom}(A')$.

  Then by induction on m, we see that Y_m is included in $\mathrm{dom}(A')$ (by the implication (a)⇒(b) in the lemma). Hence, $X \subseteq \mathrm{dom}(A')$.

  So A is the unique smallest substructure of W whose domain includes Y, i.e., $A = \langle Y \rangle_B$.

  Now we estimate the cardinality of A. Put $\kappa = |Y| + |L|$. Clearly $|Y_0| \leq \kappa$. For each fixed n, if Z is a subset of $\mathrm{dom}(B)$ of cardinality κ, then the set

  $$\{F^B(\overline{a}) : F \text{ is an } n\text{-ary function symbol of } L \text{ and } \overline{a} \in Z^n\}$$

  has cardinality at most $\kappa \cdot \kappa^n = \kappa$, since κ is infinite. Hence, if $|Y_m| \leq \kappa$, then $|Y_{m+1}| \leq \kappa + \kappa = \kappa$. Thus, by induction on m, each $|Y_m| \leq \kappa$, and so $|X| \leq \omega \cdot \kappa = \kappa$. Since $|\langle Y \rangle_B| = |X|$ by definition, this proves the theorem.

Choice of Signature

- One and the same mathematical object can be interpreted as a structure in several different ways:
  - The same function or relation can be named by different symbols.
  - We also have some choice about which elements, functions or relations should be given names at all.

General Principle: Other things being equal, signatures should be chosen so that the notions of homomorphism and substructure agree with the usual notions for the relevant branch of mathematics.

Example: In the case of groups:

- If the only named operation is the product $\cdot$, then the substructures of a group will be its subsemigroups, closed under $\cdot$ but not necessarily containing inverses or identity.
- If we name $\cdot$ and the identity 1, then the substructures will be the submonoids.
- To ensure that "substructure equals subgroup", we also need to put in a symbol for $^{-1}$.

Natural Choice of Signature

- For some classes of objects there is a natural choice of signature.
  - For groups the natural choice is to name $\cdot, 1$ (or e) and $^{-1}$.
    We call this the **signature of groups**.
  - We shall always assume (unless otherwise stated) that rings have a 1.
    So the natural choice of signature for rings is to name $+, -, \cdot, 0$ and 1.
    We call this the **signature of rings**.
  - The **signature of partial orderings** has just the symbol $\leq$.
  - The **signature of lattices** has just $\wedge$ and $\vee$.

Reductions and Expansions

- Suppose L^- and L^+ are signatures, and L^- is a subset of L^+.
- If A is an L^+-structure, we can turn A into an L^--structure by simply forgetting the symbols of L^+ which are not in L^-.

  Remark: We don't remove any elements of A, though some constant elements in A may cease to be constant elements in the new structure.
- The resulting L^--structure is called the L^--**reduct** of A or the **reduct of** A **to** L^-, in symbols $A|_{L^-}$.
- If $f : A \to B$ is a homomorphism of L^+-structures, then the same map f is also a homomorphism $f : A|_{L^-} \to B|_{L^-}$ of L^--structures.
- When A is an L^+-structure and C is its L^--reduct, we say that A is an **expansion** of C to L^+.
- In general C may have many different expansions to L^+.

Notation on Reductions and Expansions

- There is a useful but imprecise notation for an expansion A of C.
- Suppose the symbols which are in L^+ but not in L^- are constants c, d and a function symbol F.
- Then c^A, d^A are respectively elements a, b of A, and F^A is some operation f on $\text{dom}(A)$.
- We write $A = (C, a, b, f)$ to express that A is an expansion of C got by adding symbols to name a, b and f.
- The notation is imprecise because it does not say what symbols are used to name a, b and f, respectively.
- However, often, the choice of symbols is unimportant or obvious from the context.

Subsection 3

Terms and Atomic Formulas

Variables and Terms

- Every language has a stock of **variables**, symbols written v, x, y, z, t, x_0, x_1 etc., one of whose purposes is to serve as temporary labels for elements of a structure.
- Any symbol not already used for something else can be used as a variable.
- The **terms** of the signature L are strings of symbols defined as follows:
  1. Every variable is a term of L;
  2. Every constant of L is a term of L;
  3. If $n > 0$, F is an n-ary function symbol of L and $t_1, \ldots, t_n$ are terms of L, then the expression $F(t_1, \ldots, t_n)$ is a term of L.
  4. Nothing else is a term of L.
- A term is said to be **closed** or **ground** if no variables occur in it.
- The **complexity** of a term is the number of symbols occurring in it, counting each occurrence separately.
- If t occurs as part of s, then s has higher complexity than t.

Substituting Terms for Variables in a Term

- If we introduce a term t as $t(\overline{x})$, this will always mean that $\overline{x}$ is a sequence $(x_0, x_1, \ldots)$, possibly infinite, of distinct variables, and every variable which occurs in t is among the variables in $\overline{x}$.
- Given a sequence of terms $\overline{s} = (s_0, s_1, \ldots)$, we write $t(\overline{s})$ for the term obtained from t by putting s_0 in place of x_0, s_1 in place of x_1, etc., throughout t.

  Example: Suppose $t(x, y)$ is the term $y + x$.

  - $t(0, 2y)$ is the term $2y + 0$;
  - $t(t(x, y), y)$ is the term $y + (y + x)$.

Interpretation of Terms in a Structure

- We make variables and terms stand for elements of a structure.
- Let $t(\overline{x})$ be a term of L, where $\overline{x} = (x_0, x_1, \ldots)$.
- Let A be an L-structure and $\overline{a} = (a_0, a_1, \ldots)$ a sequence of elements of A with $\overline{a}$ at least as long as $\overline{x}$.
- Then $t^A(\overline{a})$ (or $t^A[\overline{a}]$) is defined to be the element of A which is named by t when x_0 is interpreted as a name of a_0, x_1 as a name of a_1, and so on.
- More precisely, using induction on the complexity of t:
  5. If t is the variable x_i, then $t^A[\overline{a}]$ is a_i;
  6. If t is a constant c, then $t^A[\overline{a}]$ is the element c^A;
  7. If t of the form $F(s_1, \ldots, s_n)$, where each s_i is a term $s_i(\overline{x})$, then $t^A[\overline{a}]$ is the element $F^A(s_1^A[\overline{a}], \ldots, s_n^A[\overline{a}])$.
- If t is a closed term, then $\overline{a}$ plays no role and we write t^A for $t^A[\overline{a}]$.

Atomic Formulas

- The **atomic formulas** of L are the strings of symbols given by:
  8. If s and t are terms of L, then the string $s = t$ is an atomic formula of L.
  9. If $n > 0$, R is an n-ary relation symbol of L and $t_1, \ldots, t_n$ are terms of L, then the expression $R(t_1, \ldots, t_n)$ is an atomic formula of L.

  Remark: The symbol "=" is not assumed to be a relation symbol in the signature.
- An **atomic sentence** of L is an atomic formula in which there are no variables.
- Just as with terms, if we introduce an atomic formula ϕ as $\phi(\overline{x})$, then $\phi(\overline{s})$ means the atomic formula got from ϕ by putting terms from the sequence $\overline{s}$ in place of all occurrences of the corresponding variables from $\overline{x}$.

Truth and Models

- If the variables $\overline{x}$ in an atomic formula $\phi(\overline{x})$ are interpreted as names of elements $\overline{a}$ in a structure A, then ϕ makes a statement about A.
- Let $\phi(\overline{x})$ be an atomic formula of L, with $\overline{x} = (x_0, x_1, \ldots)$.
- Let A be an L-structure and $\overline{a}$ a sequence $(a_0, a_1, \ldots)$ of elements of A assumed to be at least as long as $\overline{x}$.
- We define the relation ϕ is **true of $\overline{a}$ in** A, or that $\overline{a}$ **satisfies ϕ in** A, in symbols $A \models \phi[\overline{a}]$, or equivalently $A \models \phi(\overline{a})$, as follows:
  10. If ϕ is the formula $s = t$ where $s(\overline{x}), t(\overline{x})$ are terms, then $A \models \phi[\overline{a}]$ iff $s^A[\overline{a}] = t^A[\overline{a}]$;
  11. If ϕ is the formula $R(s_1, \ldots, s_n)$, where $s_1(\overline{x}), \ldots, s_n(\overline{x})$ are terms, then $A \models \phi[\overline{a}]$ iff the ordered n-tuple $(s_1^A[\overline{a}], \ldots, s_n^A[\overline{a}])$ is in R^A.
- When ϕ is an atomic sentence, we can omit the sequence $\overline{a}$ and write simply $A \models \phi$ in place of $A \models \phi[\overline{a}]$.
- We say that A is a **model** of ϕ, or that ϕ is **true in** A, if $A \models \phi$.
- When T is a set of atomic sentences, we say that A is a **model** of T (in symbols, $A \models T$) if A is a model of every atomic sentence in T.

Homomorphisms and Satisfiability

Theorem

Let A and B be L-structures and f a map from $\mathrm{dom}(A)$ to $\mathrm{dom}(B)$.

(a) If f is a homomorphism then, for every term $t(\overline{x})$ of L and tuple $\overline{a}$ from A, $f(t^A[\overline{a}]) = t^B[f(\overline{a})]$.

(b) f is a homomorphism if and only if, for every atomic formula $\phi(\overline{x})$ of L and tuple $\overline{a}$ from A, $A \vDash \phi[\overline{a}] \Rightarrow B \vDash \phi[f(\overline{a})]$.

(c) f is an embedding if and only if, for every atomic formula $\phi(\overline{x})$ of L and tuple $\overline{a}$ from A, $A \vDash \phi[\overline{a}] \Leftrightarrow B \vDash \phi[f(\overline{a})]$.

(a) This is easily proved by induction on the complexity of t.

(b) Suppose first that f is a homomorphism. As a typical example, suppose $\phi(\overline{x})$ is $R(s,t)$, where $s(\overline{x})$ and $t(\overline{x})$ are terms. Assume $A \vDash \phi[\overline{a}]$. Then, we have $(s^A[\overline{a}], t^A[\overline{a}]) \in R^A$. Then, by Part (a), $(s^B[f(\overline{a})], t^B[f(\overline{a})]) = (f(s^A[\overline{a}]), f(t^A[\overline{a}])) \in R^B$. Hence, $B \vDash \phi[f(\overline{a})]$.

Homomorphisms and Satisfiability (Cont'd)

- Essentially the same proof works for every atomic formula ϕ.

  For the converse, again we take a typical example.

  Suppose the implication holds for all atomic ϕ and sequences $\overline{a}$.

  - Let c be a constant symbol and $c^A = a$. Then $A \models c = x_0[a]$. By hypothesis, $B \models c = x_0[f(a)]$. This gives $c^B = f(a) = f(c^A)$.
  - Let F be a binary function symbol and a_0, a_1 in A. Then, $A \models x_2 = F(x_0, x_1)[a_0, a_1, F^A(a_0, a_1)]$. By hypothesis, $B \models x_2 = F(x_0, x_1)[f(a_0), f(a_1), f(F^A(a_0, a_1))]$. This gives $f(F^A(a_0, a_1)) = F^B(f(a_0), f(a_1))$.
  - Let R be a binary relation symbol and a_0, a_1 in A, such that $(a_0, a_1) \in R^A$. We have $A \models R(x_0, x_1)[a_0, a_1]$. By hypothesis, $B \models R(x_0, x_1)[f(\overline{a})]$. Hence, $(f(a_0), f(a_1)) \in R^B$.

  Thus, f is a homomorphism.

(c) Similar to Part (b).

Negated Atomic Formulas and Literals

- By a **negated atomic formula** of L we mean a string $\neg\phi$, where ϕ is an atomic formula of L.
- We read the symbol $\neg$ as "not" and we define

$$\text{``}A \models \neg\phi[\overline{a}]\text{'' holds} \quad \text{iff} \quad \text{``}A \models \phi[\overline{a}]\text{'' does not hold,}$$

  where A is any L-structure, ϕ an atomic formula and $\overline{a}$ a sequence from A.
- A **literal** is an atomic or negated atomic formula.
- It is a **closed literal** if it contains no variables.

Corollary

Let A and B be L-structures and f a map from $\mathrm{dom}(A)$ to $\mathrm{dom}(B)$. Then f is an embedding if and only if, for every literal $\phi(\overline{x})$ of L and sequence $\overline{a}$ from A,

$$A \models \phi[\overline{a}] \ \Rightarrow \ B \models \phi[f(\overline{a})].$$

- Immediate from Part (c) of the theorem and the negation condition.

The Term or Absolutely Free Algebra With Basis X

- Let L be any signature and X a set of variables.
- We define the **term algebra of L with basis X** to be the following L-structure A:
  - The domain of A is the set of all terms of L whose variables are in X;
  - The constant, function and relation symbols are interpreted as follows:
    1. $c^A = c$, for each constant c of L;
    2. $F^A(\overline{t}) = F(\overline{t})$, for each n-ary function symbol F of L and n-tuple $\overline{t}$ of elements of $\text{dom}(A)$;
    3. R^A is empty for each relation symbol R of L.
- The term algebra of L with basis X is also known as the **absolutely free L-structure with basis X** (we will see why later).

Subsection 4

Parameters and Diagrams

Parameters

- We can avoid interpreting a variable as a name of the element b, by adding a new constant for b to the signature.
- Of course, the language changes every time another element is named.
- When constants are added to a signature, the new constants and the elements they name are called **parameters**.

  Example: Suppose that A is an L-structure, $\overline{a}$ is a sequence of elements of A, and we want to name the elements in $\overline{a}$.

  Choose a sequence $\overline{c}$ of distinct new constant symbols, of the same length as $\overline{a}$, and form the signature $L(\overline{c})$ by adding the constants $\overline{c}$ to L.

  Then $(A,\overline{a})$ is an $L(\overline{c})$-structure, and each element a_i is $c_i^{(A,\overline{a})}$.

  Likewise if B is another L-structure and $\overline{b}$ a sequence of elements of B of the same length as $\overline{c}$, then there is an $L(\overline{c})$-structure $(B,\overline{b})$ in which these same constants c_i name the elements of $\overline{b}$.

Homomorphisms and Parameters

Lemma

Let A, B be L-structures and suppose $(A, \overline{a})$, $(B, \overline{b})$ are $L(\overline{c})$-structures.

- A homomorphism $f : (A, \overline{a}) \to (B, \overline{b})$ is the same thing as a homomorphism $f : A \to B$, such that $f\overline{a} = \overline{b}$.
- An embedding $f : (A, \overline{a}) \to (B, \overline{b})$ is the same thing as an embedding $f : A \to B$, such that $f\overline{a} = \overline{b}$.

- In the situation above:
  - If $t(\overline{x})$ is a term of L, then $t^A[\overline{a}]$ and $t(\overline{c})^{(A, \overline{a})}$ are the same element.
  - If $\phi(\overline{x})$ is an atomic formula, then $A \models \phi[\overline{a}] \Leftrightarrow (A, \overline{a}) \models \phi(\overline{c})$.
- To avoid confusion between the two notations, we use the elements a_i as constants naming themselves.

  The expanded signature is $L(\overline{a})$, and we write $t^A(\overline{a})$ and $A \models \phi(\overline{a})$.
- However, special care is needed when:
  - $\overline{a}$ contains repetitions; or
  - Two separate $L(\overline{c})$- structures are under discussion.

Diagram and Positive Diagram

- Let $\overline{a}$ be a sequence of elements of A. We say that $\overline{a}$ **generates** A, in symbols $A = \langle \overline{a} \rangle_A$, if A is generated by the set of all elements in $\overline{a}$.
- Suppose that A is an L-structure, $(A, \overline{a})$ is an $L(\overline{c})$-structure and $\overline{a}$ generates A. Then every element of A is of the form $t^{(A,\overline{a})}$, for some closed term t of $L(\overline{c})$. So every element of A has a name in $L(\overline{c})$.
- The set of all closed literals of $L(\overline{c})$ which are true in $(A, \overline{a})$ is called the **(Robinson) diagram of** A, in symbols $\mathrm{diag}(A)$.
- The set of all atomic sentences of $L(\overline{c})$ which are true in $(A, \overline{a})$ is called the **positive diagram of** A, in symbols $\mathrm{diag}^+(A)$.
- $\mathrm{diag}(A)$ and $\mathrm{diag}^+(A)$ are not uniquely determined, because in general there are many ways of choosing $\overline{a}$ and $\overline{c}$ so that $\overline{a}$ generates A.
- There is always at least one possible choice of $\overline{a}$ and $\overline{c}$, namely listing all the elements of A without repetition.

The Diagram Lemma

Lemma (Diagram Lemma)

Let A and B be L-structures, $\overline{c}$ a sequence of constants, and $(A,\overline{a})$ and $(B,\overline{b})$ $L(\overline{c})$-structures. Then (a) and (b) are equivalent:

(a) For every atomic sentence ϕ of $L(\overline{c})$, if $(A,\overline{a}) \models \phi$, then $(B,\overline{b}) \models \phi$.

(b) There is a homomorphism $f : \langle\overline{a}\rangle_A \models B$, such that $f(\overline{a}) = \overline{b}$.

The homomorphism f is unique if it exists; it is an embedding if and only if

(c) for every atomic sentence ϕ of $L(\overline{c})$, $(A,\overline{a}) \models \phi \Leftrightarrow (B,\overline{b}) \models \phi$.

- Assume (a). The inclusion map embeds $\langle\overline{a}\rangle_A$ in A. By a preceding theorem, in (a) we can replace A by $\langle\overline{a}\rangle_A$. So, we can assume that $A = \langle\overline{a}\rangle_A$. By a previous lemma, it suffices to find a homomorphism $f : (A,\overline{a}) \to (B,\overline{b})$. We define f as follows: Since $\overline{a}$ generates A, each element of A is of the form $t^{(A,\overline{a})}$, for some closed term t of $L(\overline{c})$. We set $f(t^{(A,\overline{a})}) = t^{(B,\overline{b})}$.

The Diagram Lemma (Cont'd)

- The definition $f(t^{(A,\overline{a})}) = t^{(B,\overline{b})}$ is sound.

  Suppose, for two closed terms s and t, $s^{(A,\overline{a})} = t^{(A,\overline{a})}$. Then $(A,\overline{a}) \models s = t$. By hypothesis, $(B,\overline{b}) \models s = t$. Hence, $s^{(B,\overline{b})} = t^{(B,\overline{b})}$.

  f is a homomorphism by hypothesis and a previous theorem.

  Any homomorphism f from $(A,\overline{a})$ to $(B,\overline{b})$ must satisfy $f(t^{(A,\overline{a})}) = t^{(B,\overline{b})}$. So f is unique in (b).

  The converse follows at once from a previous theorem.

  The argument for embeddings and (c) is similar.

- There is a connection with Robinson diagrams.

  Suppose $\overline{a}$ generates A.

  - The implication (a)⇒(b) says that A can be mapped homomorphically to a reduct of B whenever $B \models \mathrm{diag}^+(A)$.
  - Similarly the last part of the lemma says that if $B \models \mathrm{diag}(A)$, then A can be embedded in a reduct of B.

Subsection 5

Canonical Models

=-Closed Sets of Atomic Sentences

- Let L be a signature, A an L-structure and T the set of all atomic sentences of L which are true in A.
- Then T has the following two properties:
  1. For every closed term t of L, the atomic sentence $t = t$ is in T.
  2. If $\phi(x)$ is an atomic formula of L and the equation $s = t$ is in T, then $\phi(s) \in T$ if and only if $\phi(t) \in T$.
- Any set T of atomic sentences which satisfies 1 and 2 will be said to be **=-closed (in L)**.

Canonical Model of =-Closed Sets of Sentences

Lemma

Let T be an =-closed set of atomic sentences of L. Then there is an L-structure A, such that:

(a) T is the set of all atomic sentences of L which are true in A;
(b) Every element of A is of the form t^A, for some closed term t of L.

- Let X be the set of all closed terms of L.
  We define a relation $\sim$ on X by $s \sim t$ iff $s = t \in T$.
  Claim: $\sim$ is an equivalence relation.
  (i) By the hypothesis, $\sim$ is reflexive.
  (ii) Suppose $s \sim t$. Then $s = t \in T$. Let $\phi(x)$ be the formula $x = s$. Then $\phi(s)$ is $s = s$ which is in T by hypothesis. So, again by hypothesis, T also contains $\phi(t)$. But $\phi(t)$ is $t = s$. Hence $t \sim s$.
  (iii) Suppose $s \sim t$ and $t \sim r$. Let $\phi(x)$ be $s = x$. By assumption, both $\phi(t)$ and $t = r$ are in T. By hypothesis, T also contains $\phi(r)$. But $\phi(r)$ is $s = r$. Hence, $s \sim r$.

Canonical Model of =-Closed Sets of Sentences (Cont'd)

- For each closed term t, let:
  - $t^\sim$ be the equivalence class of t under $\sim$;
  - Y be the set of all equivalence classes $t^\sim$ with $t \in X$.

  We define an L-structure A with $\mathrm{dom}(A) = Y$.
  - For each constant c of L, we put $c^A = c^\sim$.
  - If $0 < n < \omega$ and F is an n-ary function symbol of L, we define F^A by $F^A(s_0^\sim, \ldots, s_{n-1}^\sim) = (F(s_0, \ldots, s_{n-1}))^\sim$.
  - If $0 < n < \omega$ and R is an n-ary relation symbol of L, we define R^A by $(s_0^\sim, \ldots, s_{n-1}^\sim) \in R^A$ iff $R(s_0, \ldots, s_{n-1}) \in T$.

  $F^A(s_0^\sim, \ldots, s_{n-1}^\sim)$ is well-defined:

  Suppose $s_i \sim t_i$, for each $i < n$. By hypothesis, the sentence $F(s_0, \ldots, s_{n-1}) = F(s_0, \ldots, s_{n-1})$ is in T. By n applications of closedness, we find that the equation $F(s_0, \ldots, s_{n-1}) = F(t_0, \ldots, t_{n-1})$ is in T. Hence $F(s_0, \ldots, s_{n-1})^\sim = F(t_0, \ldots, t_{n-1})^\sim$.

  Well-definedness of R^A is justified similarly.

Canonical Model of =-Closed Sets of Sentences (Conclusion)

- Now it is easy to prove by induction on the complexity of t, that for every closed term t of L, $t^A = t^\sim$.

  From this we infer that:
  - If s and t are any closed terms of L, then
  $$A \models s = t \quad \text{iff} \quad s^A = t^A \quad \text{iff} \quad s^\sim = t^\sim \quad \text{iff} \quad s = t \in T.$$
  - If $t_0, \ldots, t_{n-1}$ are closed terms in L,
  $$A \models R(t_0, \ldots, t_{n-1}) \quad \text{iff} \quad (t_0^\sim, \ldots, t_{n-1}^\sim) \in R^A \quad \text{iff} \quad R(t_0, \ldots, t_{n-1}) \in T.$$

  Thus, T is the set of all atomic sentences of L which are true in A.

  Also, since $t^A = t^\sim$, every element of A is of the form t^A, for some closed term t of L.

Models of Sets of Atomic Sentences

- If T is any set of atomic sentences of L, there is a least set U of atomic sentences of L which contains T and is =-closed in L.
- We call U the =-**closure** of T in L.
- Any L-structure which is a model of U must also be a model of T since $T \subseteq U$.

Theorem

For any signature L, if T is a set of atomic sentences of L, then there is an L-structure A, such that:

(a) $A \models T$;

(b) Every element of A is of the form t^A, for some closed term t of L;

(c) If B is an L-structure and $B \models T$, then there is a unique homomorphism $f : A \to B$.

- Apply the lemma to the =-closure U of T to get the L-structure A.

Models of Sets of Atomic Sentences (Cont'd)

- We start with a set T of atomic sentences of L.

  We obtain the =-closure U of T.

  We apply the lemma to U to get the L-structure A.

  By the construction of A, Properties (a) and (b) are clear.

  By Property (b), $A = \langle \varnothing \rangle_A$.

  So Property (c) will follow from the Diagram Lemma if we can show that every atomic sentence true in A is true in all models B of T.

  Let B be a model of T.

  By the choice of A, every atomic sentence true in A is in U.

  The set of all atomic sentences true in B is an =-closed set containing T.

  So it must contain the =-closure of T, which is U.

Canonical Model and Herbrand Universe

- By Property (c) of the theorem, the model A of T is unique up to isomorphism.
- It is called the **canonical model** of T.
- Note that it will be the empty L-structure if and only if L has no constant symbols.
- If one does not include equations as atomic formulas, the canonical model is much easier to construct, because there is no need to factor out an equivalence relation.

  Thus, we get what has become known as the **Herbrand universe** of a set of atomic sentences.

Example: Adding Roots of Polynomials to a Field

- Let F be a field and $p(X)$ an irreducible polynomial over F in the indeterminate X.
- We can regard $F[X]$ as a structure in the signature of rings with constants added for X and all the elements of F.
- Let T be the set of all equations which are true in $F[X]$.

  Example: If a is $b \cdot (c+d)$ in $F[X]$, then T contains the equation "$a = b \cdot (c+d)$".

  Also rings satisfy the law $1 \cdot x = x$. So T contains the equation $1 \cdot t = t$, for every closed term t.
- T is a set of atomic sentences.
- The equation "$p(X) = 0$" is another atomic sentence.

Example: Adding Roots of Polynomials to a Field (Cont'd)

- Let C be the canonical model of the set $T \cup \{p(X) = 0\}$.
  - C is a model of T and every element is named by a closed term.
  - Thus, C is a homomorphic image of $F[X]$.
  - In particular C is a ring.
  - "$p(X) = 0$" holds in C.
  - So every element of the ideal I in $F[X]$ generated by $p(X)$ goes to 0 in C.
- Let θ be any root of p.
- Then the field extension $F[\theta]$ is also a model of $T \cup \{p(X) = 0\}$, with X read as a name of θ.
- By Part (c) of the theorem, $F[\theta]$ is a homomorphic image of C.
- On the other hand, $F[\theta] = F[X]/I$.
- So C is isomorphic to $F[\theta]$.