Introduction to Model Theory

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 500

Subsection 1

Definable Subsets

Example 1: Algebraic Curves

- Regard the field $\mathbb{R}$ of reals as a structure.
- An **algebraic curve in the real plane** is a set of ordered pairs of elements of $\mathbb{R}$ given by an equation $p(x,y)=0$, where p is a polynomial with coefficients from $\mathbb{R}$.

  Example: The parabola $y=x^2$ is perhaps the most quoted example of an algebraic curve in the real plane.

  This equation can be written without naming any elements of $\mathbb{R}$ as parameters.

Example 2: Recursive Sets of Natural Numbers

- We use the structure $\mathbb{N} = (\omega, 0, 1, +, \cdot, <)$ of natural numbers.
- Any recursive subset X of ω can be defined, for example, by an algorithm for computing whether any given number is in X.
- Unlike the preceding example, the definition will usually be much too complicated to be written as an atomic formula.
- There is no need to use parameters in this case, since every element of $\mathbb{N}$ is named by a closed term of the signature of $\mathbb{N}$.

Example 3: Connected Components of Graphs

- Let G be a graph, and g an element of G.
- The **connected component** of g in G is the smallest set Y of vertices of G, such that:
  (1) $g \in Y$;
  (2) if $a \in Y$ and a is joined to b by an edge, then $b \in Y$.
- This description defines Y, using g as a parameter.
- There is generally no hope of expressing the definition as an atomic formula.
- Also, generally, we cannot define Y without mentioning any element as a parameter.

Atomic Formulas and Relations

- Given an L-structure A and an atomic formula $\phi(x_0,\ldots,x_{n-1})$ of L, we write $\phi(A^n)$ for the set of n-tuples $\{\overline{a} : A \models \phi(\overline{a})\}$.

$$\phi(A^n) = \{\overline{a} : A \models \phi(\overline{a})\}.$$

  Example: If R is a relation symbol of the signature L, then the relation R^A is of the form $\phi(A^n)$. Take $\phi(x_0,\ldots,x_{n-1}) = R(x_0,\ldots,x_{n-1})$.

- Allowing parameters, let $\psi(x_0,\ldots,x_{n-1},\overline{y})$ be an atomic formula of L and $\overline{b}$ a tuple from A. Then

$$\psi(A^n,\overline{b}) = \{\overline{a} : A \models \psi(\overline{a},\overline{b})\}.$$

  Example: If A consists of the real numbers and $\psi(x,y) = (x > y)$, then

$$\psi(A,0) = \{a \in A : a > 0\}$$

  is the set of all positive reals.

The Language $L_{\infty\omega}$

- Let L be a signature. The language $L_{\infty\omega}$ will be *infinitary*, which means that some of its formulas will be infinitely long.
- The symbols of $L_{\infty\omega}$ are those of L together with some logical symbols, variables and punctuation signs.
  - The **logical symbols** are

$$= \text{"equals"},\ \neg \text{ "not"},\ \bigwedge \text{ "and"},\ \bigvee \text{ "or"},\ \forall \text{ "for all"},\ \exists \text{ "there exists"}.$$

- The **terms**, the **atomic formulas** and the **literals** of $L_{\infty\omega}$ are the same as those of L.
- The class of **formulas** of $L_{\infty\omega}$ is defined to be the smallest class X, such that:
  1. All atomic formulas of L are in X;
  2. If ϕ is in X, then the expression $\neg\phi$ is in X, and if $\Phi \subseteq X$, then the expressions $\bigwedge\Phi$ and $\bigvee\Phi$ are both in X;
  3. If ϕ is in X and y is a variable, then $\forall y\phi$ and $\exists y\phi$ are both in X.

Subformulas and Free and Bound Variables

- The formulas which go into the making of a formula ϕ are called the **subformulas** of ϕ.
  - The formula ϕ is counted as a subformula of itself.
  - The **proper subformulas** are all its subformulas except itself.
- The quantifiers $\forall y$ ("for all y") and $\exists y$ ("there is y") bind variables just as in elementary logic.
- We distinguish between free and bound occurrences of variables.
- The **free variables** of a formula ϕ are those which have free occurrences in ϕ.

Notation for Formulas and Variables

- We sometimes introduce a formula ϕ as $\phi(\overline{x})$, for some sequence $\overline{x}$ of variables.
- This means that the variables in $\overline{x}$ are all distinct, and the free variables of ϕ all lie in $\overline{x}$.
- Then $\phi(\overline{s})$ means the formula that we get from ϕ by putting the terms s_i in place of the free occurrences of the corresponding variables x_i.
- This extends the notation applied previously to atomic formulas.

Satisfiability of $L_{\infty\omega}$-Formulas

- For any L-structure A and sequence $\overline{a}$ of elements of A, we extend the notation $A \models \phi[\overline{a}]$ or $A \models \phi(\overline{a})$ ("$\overline{a}$ **satisfies** ϕ **in** A") to all formulas $\phi(\overline{x})$ of $L_{\infty\omega}$ by induction on the construction of ϕ:
  1. If ϕ is atomic, then $A \models \phi[\overline{a}]$ holds or fails per previous conventions.
  2. $A \models \neg\phi[\overline{a}]$ iff it is not true that $A \models \phi[\overline{a}]$.
  3. $A \models \bigwedge\Phi[\overline{a}]$ iff, for every formula $\psi(\overline{x}) \in \Phi$, $A \models \psi[\overline{a}]$.
  4. $A \models \bigvee\Phi[\overline{a}]$ iff, for at least one formula $\psi(\overline{x}) \in \Phi$, $A \models \psi[\overline{a}]$.
  5. Suppose ϕ is $\forall y\psi$, where $\psi(y,\overline{x})$.
     Then $A \models \phi[\overline{a}]$ iff, for all elements b of A, $A \models \psi[b,\overline{a}]$.
  6. Suppose ϕ is $\exists y\psi$, where ψ is $\psi(y,\overline{x})$.
     Then $A \models \phi[\overline{a}]$ iff, for at least one element b of A, $A \models \psi[b,\overline{a}]$.
- If $\overline{x}$ is an n-tuple of variables, $\phi(\overline{x},\overline{y})$ is a formula of $L_{\infty\omega}$ and $\overline{b}$ is a sequence of elements of A whose length matches that of $\overline{y}$, we write $\phi(A^n,\overline{b})$ for the set $\{\overline{a} : A \models \phi(\overline{a},\overline{b})\}$.
- $\phi(A^n,\overline{b})$ is the **relation defined in** A by the formula $\phi(\overline{x},\overline{b})$.

Example 3: Connected Components of Graphs (cont'd)

- The vertex x_0 is in the same component as g if:
  - Either x_0 is g,
  - or x_0 is joined by an edge to g (in symbols $R(x_0,g)$),
  - or there is x_1 such that $R(x_0,x_1)$ and $R(x_1,g)$,
  - or there are x_1 and x_2, such that $R(x_0,x_1)$, $R(x_1,x_2)$ and $R(x_2,g)$,
  - or $\cdots$.
- In other words, the connected component of g is defined by the formula

$$\bigvee(\{x_0=g\}\cup\{\exists x_1\cdots\exists x_n\bigwedge(\{R(x_i,x_{i+1}):i<n\}\cup\{R(x_n,g)\}):n<\omega\}),$$

  with parameter g.
- This formula may not be easy to read, but it is very precise.

Complexity of Formulas in $L_{\infty\omega}$

- We define the **complexity** of a formula ϕ, $\text{comp}(\phi)$, so that it is greater than the complexity of any proper subformula of ϕ.
- Using ordinals, one possible definition is

$$\text{comp}(\phi) = \sup\{\text{comp}(\psi) + 1 : \psi \text{ is a proper subformula of } \phi\}.$$

- The notion of complexity helps us prove theorems about relations definable in $L_{\infty\omega}$, by using induction on the complexity of the formulas defining them.

Language Classifications

- The subscripts $_{\infty\omega}$ suggest language classifications.
  - The second subscript, ω means that we can put only finitely many quantifiers together in a row.
    - $L_{\infty 0}$ is the language consisting of those formulas of $L_{\infty\omega}$ in which no quantifiers occur; we call such formulas **quantifier-free**.
      Every atomic formula is quantifier-free.
    - Occasionally we shall want to go beyond the confines of $L_{\infty\omega}$ by applying a quantifier $\forall$ or $\exists$ to infinitely many variables at once: $\forall(x_i : i \in I)$ or $\exists(x_i : i \in I)$.
      The language we get by adding these quantifiers to $L_{\infty\omega}$ is written $L_{\infty\infty}$.
  - The first subscript in $L_{\infty\omega}$ means that we can join together arbitrarily many formulas by $\bigwedge$ or $\bigvee$.
    - The **first-order language** of L, in symbols $L_{\omega\omega}$, consists of those formulas in which $\bigwedge$ and $\bigvee$ are only used to join together finitely many formulas at a time, so that the whole formula is finite.

Fragments of $L_{\infty,\omega}$

- We can pick out many smaller languages inside $L_{\infty\omega}$, by choosing subclasses of the class of formulas of $L_{\infty\omega}$.
- We say that a set X of formulas of $L_{\infty\omega}$ is **first-order-closed** if:
  (1) X satisfies:
    (a) All atomic formulas of L are in X;
    (b) If ϕ is in X, then the expression $\neg\phi$ is in X, and if $\Phi \subseteq X$ is finite, then the expressions $\bigwedge\Phi$ and $\bigvee\Phi$ are both in X;
    (c) If ϕ is in X and y is a variable then $\forall y\phi$ and $\exists y\phi$ are both in X.
  (2) Every subformula of a formula in X is also in X.
- All the languages $L_{\kappa\omega}$ are first-order-closed.
- First-order-closed sublanguages of $L_{\infty\omega}$ are sometimes known as **fragments** of $L_{\infty\omega}$.

Usage of the Symbol L

- We use L as a symbol to stand for languages as well as signatures.
- Since a language determines its signature, there is no ambiguity if we talk about L-structures for a language L.
- If L is a first-order language, it is clear what is meant by $L_{\infty\omega}$, $L_{\kappa\omega}$ etc. They are infinitary languages extending L.
- If a set X of parameters are added to L, forming a new language $L(X)$, we shall refer to the formulas of $L(X)$ as **formulas of L with parameters from X**.

Definability

- Let L be a first-order language and A an L-structure.
- If $\phi(\overline{x})$ is a first-order formula, then a set or relation of the form $\phi(A^n)$ is said to be **first-order definable without parameters**, or more briefly $\emptyset$-**definable** (pronounced "zero-definable".)
- A set or relation of the form $\psi(A^n,\overline{b})$, where $\psi(\overline{x},\overline{y})$ is a first-order formula and $\overline{b}$ is a tuple from some set X of elements of A, is said to be X-**definable** and **first-order definable with parameters**.

Standard Abbreviations

- We use the abbreviations:
  - $x \neq y$ for $\neg(x = y)$;
  - $(\phi_1 \wedge \cdots \wedge \phi_n)$ for $\bigwedge\{\phi_1, \ldots, \phi_n\}$ (finite conjunction);
  - $(\phi_1 \vee \cdots \vee \phi_n)$ for $\bigvee\{\phi_1, \ldots, \phi_n\}$ (finite disjunction);
  - $\bigwedge_{i \in I} \phi_i$ for $\bigwedge\{\phi_i : i \in I\}$;
  - $\bigvee_{i \in I} \phi_i$ for $\bigvee\{\phi_i : i \in I\}$;
  - $(\phi \to \psi)$ for $(\neg\phi) \vee \psi$ ("if ϕ then ψ");
  - $(\phi \leftrightarrow \psi)$ for $(\phi \to \psi) \wedge (\psi \to \phi)$ ("ϕ iff ψ");
  - $\forall x_1 \ldots x_n$ or $\forall \overline{x}$ for $\forall x_1 \cdots \forall x_n$;
  - $\exists x_1 \ldots x_n$ or $\exists \overline{x}$ for $\exists x_1 \cdots \exists x_n$;
  - $\bot$ for $\bigvee \emptyset$ (empty disjunction, false everywhere).
- Brackets around $(\phi \wedge \psi)$ or $(\phi \vee \psi)$ can be omitted when either $\to$ or $\leftrightarrow$ stands immediately outside these brackets.

Example on Abbreviations and Logics

Example: $\phi \wedge \psi \to \chi$ always means $(\phi \wedge \psi) \to \chi$, not $\phi \wedge (\psi \to \chi)$.

With these conventions, the transitive component formula in the language of graphs can be written

$$x_0 = g \vee \bigvee_{n<\omega} \exists x_1 \ldots x_n \left(\left(\bigwedge_{i<n} R(x_i, x_{i+1}) \right) \wedge R(x_n, g) \right).$$

- A family of languages which differ from each other only in signature is called a **logic**.
- **First-order logic** consists of the languages $L_{\omega\omega}$ as L ranges over all signatures.

Equivalence of Formulas

- We say that two formulas $\phi(\overline{x})$ and $\psi(\overline{x})$ are **equivalent** in the L-structure A if $\phi(A^n) = \psi(A^n)$.
- Thus, two formulas are equivalent in A iff they define the same relation in A.
- $\phi(\overline{x})$ and $\psi(\overline{x})$ are equivalent in A iff $A \models \forall \overline{x}(\phi(\overline{x}) \leftrightarrow \psi(\overline{x}))$.
- Likewise, sets of formulas $\Phi(\overline{x})$ and $\Psi(\overline{x})$ are **equivalent in** A if $\bigwedge \Phi(A^n) = \bigwedge \Psi(A^n)$.
- These definitions depend on the listing of variables.

  Example: If $\phi(x,y)$ and $\psi(y,x)$ are both $x < y$, then we should not expect $\phi(x,y)$ to be equivalent to $\psi(x,y)$.

Variants and Cardinality of a Language

- The following pairs of formulas are equivalent in any structure.
  - The formula $\phi(x_0,\ldots,x_{n-1})$ and the formula $\phi(y_0,\ldots,y_{n-1})$;
  - The formula $\forall yR(x,y)$ and the formula $\forall zR(x,z)$.
- We say that one formula is a **variant** of another formula if the two formulas differ only in the choice of variables, i.e., if each can be got from the other by a consistent replacement of variables.
- Variance is an equivalence relation on the class of formulas.
- We shall always take the **cardinality** $|L|$ of a first-order language L to be the number of equivalence classes of formulas of L under the relation of being variants.
- This agrees with the definition of $|L|$ for a signature L (as the least infinite cardinal $\geq$ the number of symbols in L).

Definable Subsets and Fixing

Lemma

Let L be a signature, A an L-structure, X a set of elements of A and Y a relation on $\mathrm{dom}(A)$. Suppose Y is definable by some formula of signature L with parameters from X. Then, for every automorphism f of A, if f fixes X pointwise (i.e., $f(a) = a$, for all a in X), then f fixes Y setwise (i.e., for every tuple $\overline{a}$ of A, $\overline{a} \in Y \Leftrightarrow f(\overline{a}) \in Y$).

- The lemma applies to formulas in logics other than $L_{\infty\omega}$.
- For formulas of $L_{\infty\omega}$ this can be proved by induction on complexity. E.g., for the base, given an n-ary relation symbol R,

$$\begin{array}{rll} \overline{a} \in R^A(A^n, \overline{b}) & \text{iff} & (\overline{a}, \overline{b}) \in R^A \\ & \text{iff} & (f(\overline{a}), f(\overline{b})) \in R^A \\ & \text{iff} & (f(\overline{a}), \overline{b}) \in R^A \\ & \text{iff} & f(\overline{a}) \in R^A(A^n, \overline{b}). \end{array}$$

Definable Subsets and Fixing (Cont'd)

- We also give two of the inductive cases:

  For $\bigwedge\Phi$, we have

$$\begin{array}{lll}\overline{a}\in\bigwedge\Phi(A^n,\overline{b}) & \text{iff} & \overline{a}\in\phi(A^n,\overline{b}),\ \text{for all } \phi\in\Phi,\\ & \text{iff} & f(\overline{a})\in\phi(A^n,\overline{b}), \text{for all } \phi\in\Phi,\\ & \text{iff} & f(\overline{a})\in\bigwedge\Phi(A^n,\overline{b}).\end{array}$$

  For $\exists z\varphi(z,\overline{x},\overline{y})$, we have

$$\begin{array}{lll}\overline{a}\in\exists z\phi(z,\overline{x},\overline{y})(A^n,\overline{b}) & \text{iff} & (c,\overline{a})\in\phi(A^{n+1},\overline{b}),\ \text{for some } c\in A,\\ & \text{iff} & (f(c),f(\overline{a}))\in\phi(A^{n+1},\overline{b}),\ \text{for some } c\in A,\\ & \text{iff} & f(\overline{a})\in\exists z\phi(z,\overline{x},\overline{y})(A^n,\overline{b}).\end{array}$$

Definability in Empty Signature with Parameters

Theorem

Let L be the empty signature and A an L-structure so that A is simply a set. Let X be any subset of A, and let Y be a subset of $\text{dom}(A)$ which is definable in A by a formula of some logic of signature L, using parameters from X. Then Y is either a subset of X, or the complement in $\text{dom}(A)$ of a subset of X.

- Immediate from the lemma.
- In this theorem, all finite subsets of X and their complements in A can be defined by first-order formulas with parameters in X.

  The set $\{a_0, \ldots, a_{n-1}\}$ is defined by the formula $x = a_0 \vee \cdots \vee x = a_{n-1}$ (which is $\perp$ if the set is empty).

  If we negate this formula we get a definition of the complement.

Minimal Structures and Minimal Definable Sets

- We say that a structure A is **minimal** if A is infinite but the only subsets of $\text{dom}(A)$ which are first-order definable with parameters are either finite or cofinite (i.e., complements of finite sets).
- More generally, a set $X \subseteq \text{dom}(A)$ which is first-order definable with parameters is said to be **minimal** if X is infinite, and for every set Z which is first-order definable in A with parameters, either $X \cap Z$ or $X \setminus Z$ is finite.

The Recursive Hierarchy

- Take $\mathbb{N} = (\omega, 0, 1, +, \cdot, <)$ and let L be its signature.
- We use the **bounded quantifiers** $(\forall x < y)$ and $(\exists x < y)$ as follows:
  - $(\forall x < y)\phi$ is shorthand for $\forall x(x < y \to \phi)$;
  - $(\exists x < y)\phi$ is shorthand for $\exists x(x < y \wedge \phi)$.
- We define a hierarchy of first-order formulas of L, as follows:
  1. A first-order formula of L is said to be a Π_0^0 formula, or equivalently a Σ_0^0 formula, if all quantifiers in it are bounded.
  2. A formula is said to be a Π_{k+1}^0 formula if it is of form $\forall \overline{x}\psi$ for some Σ_k^0 formula ψ. (The tuple $\overline{x}$ may be empty.)
  3. A formula is said to be a Σ_{k+1}^0 formula if it is of form $\exists \overline{x}\psi$ for some Π_k^0 formula ψ. (The tuple $\overline{x}$ may be empty.)

  Example: An Σ_3^0 formula consists of three blocks of quantifiers, $\exists \overline{x} \forall \overline{y} \exists \overline{z}$ followed by a formula with only bounded quantifiers.
- Because the blocks are allowed to be empty, every Π_k^0 formula is also a Σ_{k+1}^0 formula and a Π_{k+1}^0 formula.

Hierarchy of Definable Relations

- Let $\overline{x}$ be $(x_0, \ldots, x_{n-1})$.
- A set R of n-tuples of natural numbers is called a:
  - Π_k^0 **relation** if it is of the form $\phi(\mathbb{N}^n)$, for some Π_k^0 formula $\phi(\overline{x})$;
  - Σ_k^0 **relation** if it is of the form $\phi(\mathbb{N}^n)$, for some Σ_k^0 formula $\phi(\overline{x})$;
  - Δ_k^0 **relation** if it is both a Π_k^0 relation and a Σ_k^0 relation.
- A relation is said to be **arithmetical** if it is Σ_k^0 for some k.

  I.e., arithmetical relations are exactly the first-order definable ones.
- Intuitively the hierarchy measures how many times we have to run through the entire set of natural numbers if we want to check whether a particular tuple belongs to the relation R.

Some Results on the Hierarchy of Definable Relations

- An important theorem of Kleene [1943] says that:
  - The Δ_1^0 relations are exactly the recursive ones;
  - The Σ_1^0 relations are exactly the recursively enumerable ones.
- Another theorem of Kleene [1943] says that for each $k < \omega$, there is a relation R which is Σ_{k+1}^0 but neither Σ_k^0 nor Π_k^0.
- This last result ensures that the hierarchy keeps growing.

Subsection 2

Definable Classes of Structures

Sentences, Theories and Models

- A **sentence** is a formula with no free variables.
- A **theory** is a set of sentences.
- If ϕ is a sentence of $L_{\infty\omega}$ and A is an L-structure, then there is defined a relation "$A \models \phi[]$", i.e., "the empty sequence satisfies ϕ in A".
- We omit [] and write simply "$A \models \phi$".
- We say that A is a **model** of ϕ, or that ϕ is **true in** A, when "$A \models \phi$" holds.
- Given a theory T in $L_{\infty\omega}$, we say that A is a **model of** T, in symbols $A \models T$, if A is a model of every sentence in T.

Axiomatized Classes of Structures

- Let T be a theory in $L_{\infty\omega}$ and $\mathbf{K}$ a class of L-structures.
- We say that T **axiomatizes** $\mathbf{K}$, or is a **set of axioms** for $\mathbf{K}$, if $\mathbf{K}$ is the class of all L-structures which are models of T.
- This determines $\mathbf{K}$ uniquely, and so we can write $\mathbf{K} = \text{Mod}(T)$ to mean that T axiomatizes $\mathbf{K}$.

  Remarks:
  - T is also a theory in $L^+_{\infty\omega}$, where L^+ is any signature containing L.
  - $\text{Mod}(T)$ in L^+ is a different class from $\text{Mod}(T)$ in L.
  - So the notion of "model of T" depends on the signature.
  - If no signature is mentioned, we choose the smallest L such that T is in $L_{\infty\omega}$.
- If T is a theory, we say that a theory U **axiomatizes** T (or **is equivalent to** T) if $\text{Mod}(U) = \text{Mod}(T)$.
- In particular if A is an L-structure and T is a first-order theory, we say that T **axiomatizes** A if the first-order sentences true in A are exactly those which are true in every model of T.

Theories and Definability

- Let L be a language and $\mathbf{K}$ a class of L-structures.
- We define the L-**theory** of $\mathbf{K}$, $\mathrm{Th}_L(\mathbf{K})$, to be the set (or class) of all sentences ϕ of L, such that $A \models \phi$, for every structure A in $\mathbf{K}$.
- We omit the subscript L when L is first-order.
- The **theory** of $\mathbf{K}$, $\mathrm{Th}(\mathbf{K})$, is the set of all first-order sentences which are true in every structure in $\mathbf{K}$.
- We say that $\mathbf{K}$ is L-**definable** if $\mathbf{K}$ is the class of all models of some sentence in L.
- We say that $\mathbf{K}$ is L-**axiomatizable**, or **generalized** L-**definable**, if $\mathbf{K}$ is the class of models of some theory in L.
  - $\mathbf{K}$ is **first-order definable** if $\mathbf{K}$ is the class of models of some first-order sentence, or equivalently, of some finite set of first-order sentences.
  - $\mathbf{K}$ is **generalized first-order definable** if and only if $\mathbf{K}$ is the class of all L-structures which are models of $\mathrm{Th}(\mathbf{K})$.
  - First-order definable and first-order axiomatizable classes are also known as EC and EC_Δ classes, respectively.

Using Abbreviations

- When writing theories, we may use standard mathematical abbreviations, so long as they can be seen as abbreviations of genuine terms or formulas:
  - $x+y+z$ for $(x+y)+z$;
  - $x-y$ for $x+(-y)$;
  - n for $1+\cdots+1$ (n times), n a positive integer;
  - nx for $\begin{cases} x+\cdots+x \ (n \text{ times}), & n \text{ positve integer} \\ 0, & n \text{ is } 0 \\ -(-n)x, & n \text{ negative integer} \end{cases}$
  - xy for $x \cdot y$;
  - x^n for $x \cdots x$ (n times), n a positive integer;
  - $x \leq y$ for $x < y \vee x = y$;
  - $x \geq y$ for $y \leq x$.

There Exist at Least, A Most and Exactly n Elements

- Let $\phi(x,\overline{z})$ be a formula.

  Then we define $\exists_{\geq n} x\phi$ ("At least n elements x satisfy ϕ") by induction on n.

$$\begin{array}{lll} \exists_{\geq 0} x\phi & \text{is} & \forall x x = x \\ \exists_{\geq 1} x\phi & \text{is} & \exists x\phi \\ \exists_{\geq n+1} x\phi & \text{is} & \exists x(\phi(x,\overline{z}) \wedge \exists_{\geq n} y(\phi(y,\overline{z}) \wedge y \neq x)), \quad n \geq 1. \end{array}$$

- Then we put $\exists_{\leq n} x\phi$ for $\neg\exists_{\geq n+1} x\phi$.
- Finally, $\exists_{=n} x\phi$ is $\exists_{\geq n} x\phi \wedge \exists_{\leq n} x\phi$.

  Example: The first-order sentence $\exists_{=n} x(x = x)$ expresses that there are exactly n elements.

Intended Models and Term Algebra

- When a theory T is written down in order to describe a particular structure A, we say that A is the **intended model** of T.

  Example (The term algebra): Let L be an algebraic signature, X a set of variables and A the term algebra of L with basis X.

  We describe A by the set of all sentences of the following forms.

  1. $c \neq d$, where c, d are distinct constants.
  2. $\forall \overline{x} F(\overline{x}) \neq c$, where F is a function symbol and c a constant.
  3. $\forall \overline{x}\overline{y} F(\overline{x}) \neq G(\overline{y})$, where F, G are distinct function symbols.
  4. $\forall x_0 \ldots, x_{n-1} y_0 \ldots y_{n-1} (F(x_0, \ldots, x_{n-1}) = F(y_0, \ldots, y_{n-1}) \to \bigwedge_{i<n} x_i = y_i)$.
  5. $\forall x_0 \ldots x_{n-1} t(x_0, \ldots, x_{n-1}) \neq x_i$, where $i < n$ and t is any term containing x_i but distinct from x_i.
  6. [Use this axiom only when L is finite.] Write $\mathrm{Var}(x)$ for the formula $\bigwedge\{x \neq c : c \text{ a constant of } L\} \wedge \bigwedge\{\forall \overline{y} x \neq F(\overline{y}) : F \text{ a function symbol of } L\}$.
     Then, if X has finite cardinality n, we add the axiom $\exists_{=n} x \mathrm{Var}(x)$.
     If X is infinite, we add the infinitely many axioms $\exists_{\geq n} x \mathrm{Var}(x)$ $(n < \omega)$.
- Each axiom says something which is obviously true of A.

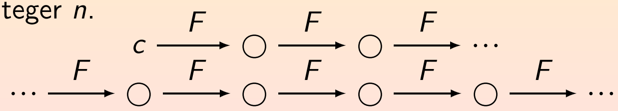
Intended Models and Term Algebra

- One can show Axioms 1-6 axiomatize A.
- They do not suffice to characterize A, even up to isomorphism.

  Example: Let L consist of one 1-ary function symbol F and one constant c, and let X be empty. Then 1-6 reduce to the following:

  $$\forall x F(x) \neq c$$
  $$\forall xy(F(x) = F(y) \to x = y)$$
  $$\forall x F(F(F(\cdots(F(x))\cdots))) \neq x \quad \text{any positive number of } F\text{'s}$$
  $$\forall x(x = c \vee \exists y x = F(y)).$$

  We can get a model B by taking the intended term model A and adding all the integers as new elements, putting $F^B(n) = n+1$, for each integer n.



  This model B is clearly not isomorphic to A.

The First-Order Peano Axioms

- This is a first-order theory with intended model the natural number structure $\mathbb{N}$. Gödel's Incompleteness Theorem [1931] says that the theory fails to axiomatize $\mathbb{N}$.
  1. $\forall x(x+1 \neq 0)$;
  2. $\forall xy(x+1 = y+1 \to x = y)$;
  3. $\forall \overline{z}(\phi(0,\overline{z}) \wedge \forall x(\phi(x,\overline{z}) \to \phi(x+1,\overline{z})) \to \forall x \phi(x,\overline{z}))$, for each first-order formula $\phi(x,\overline{z})$;
  4. $\forall x(x+0 = x)$; $\forall xy(x+(y+1) = (x+y)+1)$;
  5. $\forall x(x \cdot 0 = 0)$; $\forall xy(x \cdot (y+1) = x \cdot y + x)$;
  6. $\forall x \neg(x < 0)$; $\forall xy(x < (y+1) \leftrightarrow x < y \vee x = y)$.
- Clause 3 is an example of an **axiom schema**, i.e., a set of axioms consisting of all sentences of a certain pattern.
- This **first-order induction schema** expresses that:

  If X is a set which is first-order definable with parameters, and (1) $0 \in X$ and (2) if $n \in X$ then $n+1 \in X$, then every number is in X.
- Axioms 4-6 are the **recursive definitions** of $+, \cdot$ and $<$.

The First-Order Peano Arithmetic and Nonstandard Models

- Axioms 1-6 are known as **first-order Peano arithmetic**, or P for short.
- We will see later that P has other models besides the intended one.
- Models of P which are not isomorphic to the intended one are known as **nonstandard models**.
- They turn out to have important applications that nobody dreamed of beforehand.

Some Axiomatizable Classes I

- We provide a list of some classes which are definable or axiomatizable.
- The sentences given are referred to as the **theory of** the class.

1. **Groups (multiplicative)**:
   - $\forall xyz((xy)z = x(yz))$
   - $\forall x(x \cdot 1 = x)$;
   - $\forall x(x \cdot x^{-1} = 1)$.
2. **Groups of exponent** n **(**n **a fixed positive integer)**:
   - **Groups**;
   - $\forall x(x^n = 1)$.
3. **Abelian groups (additive)**:
   - $\forall xyz((x+y)+z = x+(y+z))$;
   - $\forall x(x+0 = x)$;
   - $\forall x(x-x = 0)$;
   - $\forall xy(x+y = y+x)$.
4. **Torsion-free abelian groups**:
   - **Abelian Groups**;
   - $\forall x(nx = 0 \rightarrow x = 0)$, for each positive integer n.

Some Axiomatizable Classes II

5. **Left R-modules**, where R is a ring:

   The module elements are the elements of the structures. Each ring element r is used as a 1-ary function symbol, i.e., $r(x)$ represents rx.
   - **Abelian groups**
   - $\forall xy(r(x+y)=r(x)+r(y))$, for all $r \in R$;
   - $\forall x((r+s)(x)=r(x)+s(x))$, for all $r,s \in R$;
   - $\forall x((rs)(x)=r(s(x)))$, for all $r,s \in R$;
   - $\forall x(1(x)=x)$.
6. **Rings**:
   - **Abelian groups**
   - $\forall xyz((xy)z=x(yz))$;
   - $\forall x(x1=x)$; $\forall x(1x=x)$;
   - $\forall xyz(x(y+z)=xy+xz)$; $\forall xyz((x+y)z=xz+yz)$.
7. **Von Neumann regular rings**:
   - **Rings**
   - $\forall x\exists y(xyx=x)$.

Some Axiomatizable Classes III

8. **Fields**:
   - **Rings**
   - $\forall xy(xy = yx)$;
   - $0 \neq 1$;
   - $\forall x(x \neq 0 \rightarrow \exists y(xy = 1))$.
9. **Fields of characteristic** p (p prime):
   - **Fields**
   - $p = 0$.
10. **Algebraically closed fields**:
    - **Fields**
    - $\forall x_1 \ldots x_n \exists y(y^n + x_1 y^{n-1} + \cdots + x_{n-1} y + x_n = 0)$, for each positive integer n.
11. **Real-closed fields**:
    - **Fields**
    - $\forall x_1 \ldots x_n (x_1^2 + \cdots + x_n^2 \neq -1)$, for each positive integer n;
    - $\forall x \exists y(x = y^2 \vee -x = y^2)$;
    - $\forall x_1 \ldots x_n \exists y(y^n + x_1 y^{n-1} + \cdots + x_{n-1} y + x_n = 0)$, for all odd n.

Some Axiomatizable Classes IV

12. **Lattices**:
    - $\forall x(x \wedge x = x)$; $\forall x(x \vee x = x)$;
    - $\forall xy(x \wedge y = y \wedge x)$; $\forall xy(x \vee y = y \vee x)$;
    - $\forall xy((x \wedge y) \vee y = y)$; $\forall xy((x \vee y) \wedge y = y)$;
    - $\forall xyz((x \wedge y) \wedge z = x \wedge (y \wedge z))$; $\forall xyz((x \vee y) \vee z = x \vee (y \vee z))$.

    In lattices we write $x \leq y$ as an abbreviation of $x \wedge y = x$.
13. **Boolean algebras**:
    - **Lattices**
    - $\forall xyz(x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z))$;
    - $\forall xyz(x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;
    - $\forall x(x \vee x^* = 1)$;
    - $\forall x(x \wedge x^* = 0)$;
    - $0 \neq 1$.
14. **Atomless boolean algebras**:
    - **Boolean algebras**
    - $\forall x \exists y(x \neq 0 \rightarrow 0 < y \wedge y < x)$, where $y < x$ is shorthand for $y \leq x \wedge y \neq x$.

Some Axiomatizable Classes V

15. **Linear orderings**:
    - $\forall x(x \nless x)$;
    - $\forall xy(x = y \vee x < y \vee y < x)$;
    - $\forall xyz(x < y \wedge y < z \rightarrow x < z)$.
16. **Dense linear orderings without endpoints**:
    - **Linear Orderngs**
    - $\forall xy(x < y \rightarrow \exists z(x < z \wedge z < y))$;
    - $\forall x \exists z(z < x)$;
    - $\forall x \exists z(x < z)$.

- Classes 1-16 are all generalized first-order definable.
- The following is a class with an infinitary definition.

17. **Locally finite groups**:
    - **Groups**
    - $\forall x_1 \dots x_n \bigvee_{m<\omega} (\exists y_1 \dots y_m \bigwedge_{t(\overline{x}) \text{ a term}} (t(\overline{x}) = y_1 \vee \cdots \vee t(\overline{x}) = y_m))$.

Subsection 3

Some Notions from Logic

Consequence, Validity and Consistency

- Let L be a signature, T a theory in $L_{\infty\omega}$ and ϕ a sentence of $L_{\infty\omega}$.
- We say that ϕ is a **consequence** of T, or that T **entails** ϕ, in symbols $T \vdash \phi$, if every model of T is a model of ϕ.

  (In particular, if T has no models then T entails ϕ.)
- We say that ϕ is **valid**, or is a **logical theorem**, in symbols $\vdash \phi$, if ϕ is true in every L-structure.
- We say that ϕ is **consistent** if ϕ is true in some L-structure.
- Likewise, we say that a theory T is **consistent** if it has a model.

Equivalence and Relative Equivalence

- We say that two theories S and T in $L_{\infty\omega}$ are **equivalent** if they have the same models, i.e., if $\mathrm{Mod}(S) = \mathrm{Mod}(T)$.
- When T is a theory in $L_{\infty\omega}$ and $\phi(\overline{x})$, $\psi(\overline{x})$ are formulas of $L_{\infty\omega}$, we say that ϕ is **equivalent to ψ modulo T** if for every model A of T and every sequence $\overline{a}$ from A, $A \models \phi(\overline{a}) \Leftrightarrow A \models \psi(\overline{a})$.
- Thus, $\phi(\overline{x})$ is equivalent to $\psi(\overline{x})$ modulo T if and only if $T \vdash \forall\overline{x}(\phi(\overline{x}) \leftrightarrow \phi(\overline{x}))$. (This sentence is not in $L_{\infty\omega}$ if ϕ and ψ have infinitely many free variables, but the sense is clear.)
- There is a metatheorem to the effect that:

  If ϕ is equivalent to ψ modulo T, and χ' comes from χ by putting ψ in place of ϕ somewhere inside χ, then χ' is equivalent to χ modulo T.
- Two sets of formulas $\Phi(\overline{x})$ and $\Psi(\overline{x})$ are **equivalent modulo T**, if $\bigwedge\Phi$ is equivalent to $\bigwedge\Psi$ modulo T.

Logical Equivalence

- A special case of relative equivalence is where T is empty.
- $\phi(\overline{x})$ and $\psi(\overline{x})$ are said to be **logically equivalent** if they are equivalent modulo the empty theory.
- This is the same as saying that they are equivalent in every L-structure.

  Example: $\neg\forall x\phi$ is logically equivalent to $\exists x\neg\phi$.

  $\exists x\bigvee_{i\in I}\psi_i$ is logically equivalent to $\bigvee_{i\in I}\exists x\psi_i$.

Boolean Combinations and Disjunctive Normal Form

- A formula ϕ is said to be a **boolean combination** of formulas in a set Φ if ϕ is in the smallest set X such that:
  (1) $\Phi \subseteq X$;
  (2) X is closed under $\wedge, \vee$ and $\neg$.
- We say that ϕ is in **disjunctive normal form over** Φ if ϕ is a finite disjunction of finite conjunctions of formulas in Y, where Y is Φ together with the negations of all formulas in Φ.
- Every boolean combination $\phi(\overline{x})$ of formulas in a set Φ is logically equivalent to a formula $\psi(\overline{x})$ in disjunctive normal form over Φ.
- The same is true if we replace $\wedge$ and $\vee$ by $\bigwedge$ and $\bigvee$ respectively, dropping the word "finite".
- In this case we speak of **infinite boolean combinations** and **infinitary disjunctive normal form**.

Prenex Formulas

- A formula is **prenex** if it consists of a string of quantifiers (possibly empty) followed by a quantifier-free formula.
- Every first-order formula is logically equivalent to a prenex first-order formula.
- The proof of this result relies in a lemma asserting that the following pairs are logically equivalent, where z is a variable not appearing on the left:
  - $\neg\forall x\phi$ and $\exists x\neg\phi$;
  - $\neg\exists x\phi$ and $\forall x\neg\phi$;
  - $(\forall x\phi(x))\wedge\psi$ and $(\forall z)(\phi(z)\wedge\psi)$; $\phi\wedge(\forall x\psi(x))$ and $(\forall z)(\phi\wedge\psi(z))$;
  - $(\exists x\phi(x))\wedge\psi$ and $(\exists z)(\phi(z)\wedge\psi)$; $\phi\wedge(\exists x\psi(x))$ and $(\exists z)(\phi\wedge\psi(z))$;
  - $(\forall x\phi(x))\vee\psi$ and $(\forall z)(\phi(z)\vee\psi)$; $\phi\vee(\forall x\psi(x))$ and $(\forall z)(\phi\vee\psi(z))$;
  - $(\exists x\phi(x))\vee\psi$ and $(\exists z)(\phi(z)\vee\psi)$; $\phi\vee(\exists x\psi)$ and $(\exists z)(\phi\vee\psi(z))$.

Prenex Formulas

Lemma

Let T be a theory in a first-order language L, and Φ a set of formulas of L. Suppose:

(a) Every atomic formula of L is in Φ;
(b) Φ is closed under boolean combinations;
(c) For every formula $\psi(\overline{x}, y)$ in Φ, $\exists y \psi$ is equivalent modulo T to a formula $\phi(\overline{x})$ in Φ.

Then every formula $\chi(\overline{x})$ of L is equivalent modulo T to a formula $\phi(\overline{x})$ in Φ.

If (c) is weakened by requiring that $\overline{x}$ is non-empty, then the same conclusion holds provided $\overline{x}$ in $\chi(\overline{x})$ is also non-empty.

- By induction on the complexity of χ, using the fact that $\forall y \phi$ is equivalent to $\neg \exists x \neg \phi$.

Realizing and Omitting Types

- An n-**type** of a theory T is a set $\Phi(\overline{x})$ of formulas, with $\overline{x} = (x_0, \ldots, x_{n-1})$, such that for some model A of T and some n-tuple $\overline{a}$ of elements of A, $A \models \phi(\overline{a})$, for all ϕ in Φ.
- We say, then, that A **realizes** the n-type Φ, and that $\overline{a}$ **realizes** Φ in A.
- We say that A **omits** Φ if no tuple in A realizes Φ.
- A set Φ is a **type** if it is an n-type, for some $n < \omega$.
- If we work in a language L which is smaller than $L_{\infty\omega}$, then all formulas in a type will automatically be assumed to come from L.

L-Equivalence and Elementary Equivalence

- Let L be a language and A, B two L-structures.
- We say that A is **L-equivalent** to B, in symbols $A \equiv_L B$, if for every sentence ϕ of L,
$$A \models \phi \quad \text{iff} \quad B \models \phi.$$
- This means that A and B are indistinguishable by means of L.
- Two structures A and B are said to be **elementarily equivalent**, written $A \equiv B$, if they are first-order equivalent.
- We write $\equiv_{\infty\omega}$, $\equiv_{\kappa\omega}$ for equivalence in $L_{\infty\omega}$, $L_{\kappa\omega}$, respectively.

The Theory of a Structure

- If L is a language and A is an L-structure, the L-**theory** of A, $\mathrm{Th}_L(A)$, is the class of all sentences of L which are true in A.
- Thus, $A \equiv_L B$ if and only if $\mathrm{Th}_L(A) = \mathrm{Th}_L(B)$.
- The **complete theory** of A, $\mathrm{Th}(A)$ without a language L specified, always means the complete first-order theory of A.

Lemma

If L is a language, A is an L-structure and ϕ an L-sentence, then

$$\mathrm{Th}_L(A) \vdash \phi \quad \text{iff} \quad \phi \in \mathrm{Th}_L(A).$$

- The "if" is trivial.

  For the "only if" assume $\phi \notin \mathrm{Th}_L(A)$. Then $A \not\models \phi$. Thus, there exists a model, namely A, such that $A \models \mathrm{Th}_L(A)$ but $A \not\models \phi$. Therefore, by definition, $\mathrm{Th}_L(A) \nvdash \phi$.

Complete Theories

- Let L be a first-order language and T a theory in L.
- We say that T is **complete** if T has models and any two of its models are elementarily equivalent.

Proposition

A theory T in a first-order language L is complete if and only if, for every sentence ϕ of L, exactly one of ϕ and $\neg\phi$ is a consequence of T.

- Suppose T is complete. Then T has a model A.

  If $T \vdash \phi$ and $T \vdash \neg\phi$, then $A \models \phi$ and $A \models \neg\phi$, a contradiction. Thus, at most one of ϕ, $\neg\phi$ is a consequence of T.

  Suppose $T \nvdash \phi$ and $T \nvdash \neg\phi$. Then there exist models A and B of T, such that $A \models \phi$ and $B \models \neg\phi$. So $A \not\equiv B$. Thus, T is not complete.

Complete Theories (Cont'd)

- Assume, conversely, that, for every L-sentence ϕ, exactly one of ϕ and $\neg\phi$ is a consequence of T.

  T must have a model.

  Otherwise, every L-sentence would be vacuously a consequence of T, contrary to hypothesis.

  Let A and B be models of T, and ϕ an L-sentence, such that $A \models \phi$.

  Then $T \nvdash \neg\phi$. By hypothesis, $T \vdash \phi$. Since, $B \models T$, $B \models \phi$.

  By symmetry, for every L-sentence ϕ,

  $$A \models \phi \quad \text{iff} \quad B \models \phi.$$

  Thus, $A \equiv B$. This proves that T is complete.

Complete Theories and Theories of Structures

Proposition

A theory T in a first-order language L is complete if and only if it is equivalent to $\mathrm{Th}(A)$, for some L-structure A.

- Suppose $T = \mathrm{Th}(A)$, for some L-structure A.
  Recall that, for every L-sentence ϕ, $T \vdash \phi$ if and only if $\phi \in T$.
  Hence, exactly one of ϕ and $\neg\phi$ is a consequence of T.
  Therefore, T is complete.
  Suppose, conversely, that T is complete. Thus, T has a model A.
  Then $T \subseteq \mathrm{Th}(A)$. So $\mathrm{Mod}(\mathrm{Th}(A)) \subseteq \mathrm{Mod}(T)$.
  For the reverse inclusion, assume $B \notin \mathrm{Mod}(\mathrm{Th}(A))$.
  Towards a contradiction, suppose $B \in \mathrm{Mod}(T)$.
  Thus, there exists $\phi \in \mathrm{Th}(A)$, such that $B \not\models \phi$.
  Since $B \models T$, $T \nvdash \phi$. By completeness, $T \vdash \neg\phi$.
  But $A \models T$. Hence, $A \models \neg\phi$, contradicting $\phi \in \mathrm{Th}(A)$.

Categoricity

- We say that a theory T is **categorical** if T is consistent and all models of T are isomorphic.
- We will see that the only categorical first-order theories are the complete theories of finite structures, so the notion is not too useful.
- Let λ be a cardinal.
- We say that a class $\mathbf{K}$ of L-structures is λ-**categorical** if there is, up to isomorphism, exactly one structure in $\mathbf{K}$ which has cardinality λ.
- Likewise a theory T is λ-**categorical** if the class of all its models is λ-categorical.
- We say that a single structure A is λ-**categorical** if $\mathrm{Th}(A)$ is λ-categorical.

The Lemma on Constants

- We saw that if $(A,\overline{a})$ is an $L(\overline{c})$-structure, where A is an L-structure, then for every atomic formula $\phi(\overline{x})$ of L,

$$A \models \phi[\overline{a}] \quad \text{if and only if} \quad (A,\overline{a}) \models \phi(\overline{c}).$$

- This remains true for all formulas $\phi(\overline{x})$ of $L_{\infty\omega}$.
- We use the compromise notation $A \models \phi(\overline{a})$ to represent either.
- Thus, $\overline{a}$ are either elements of A satisfying $\phi(\overline{x})$, or added constants naming themselves in the true sentence $\phi(\overline{a})$.

Lemma (Lemma on Constants)

Let L be a signature, T a theory in $L_{\infty\omega}$ and $\phi(\overline{x})$ a formula in $L_{\infty\omega}$. Let $\overline{c}$ be a sequence of distinct constants which are not in L. Then

$$T \vdash \phi(\overline{c}) \quad \text{if and only if} \quad T \vdash \forall \overline{x} \phi.$$

The Lemma on Constants (Idea of Proof)

- Suppose, first, that $T \vdash \forall \overline{x} \phi(\overline{x})$.

  Let $(A, \overline{c})$ be an $L(\overline{c})$-model, where A is an L-model, such that $(A, \overline{c}) \models T$. Since $\overline{c}$ is not in L, $A \models T$. By hypothesis, $A \models \forall \overline{x} \phi(\overline{x})$.

  Hence, $A \models \phi(\overline{a}')$, for all $\overline{a}'$ in A. In particular, $A \models \phi(\overline{a})$, i.e., $(A, \overline{a}) \models \phi(\overline{c})$.

  This proves that $T \vdash \phi(\overline{c})$.

- Suppose, conversely, that $T \vdash \phi(\overline{c})$.

  Let A be an L-model, such that $A \models T$. Since T is an L-theory and $\overline{c}$ is not in L, we have, for any extension $(A, \overline{a})$ of A, $(A, \overline{a}) \models T$. Since, by hypothesis $T \vdash \phi(\overline{c})$, $(A, \overline{a}) \models \phi(\overline{c})$. Thus, $A \models \phi(\overline{a})$, for all $\overline{a}$ in A. By definition, $A \models \forall \overline{x} \phi(\overline{x})$.

  This proves that $T \vdash \forall \overline{x} \phi(\overline{x})$.

Hintikka Sets

- Consider an L-structure A which is generated by its constant elements.
- Let T be the class of all sentences of $L_{\infty\omega}$ which are true in A.
- Then T has the following properties:
  1. For every atomic sentence ϕ of L, if $\phi \in T$, then $\neg\phi \notin T$.
  2. For every closed term t of L, the sentence $t = t$ is in T.
  3. If $\phi(x)$ is an atomic formula of L, s and t are closed terms of L and $s = t \in T$, then $\phi(s) \in T$ if and only if $\phi(t) \in T$.
  4. If $\neg\neg\phi \in T$ then $\phi \in T$.
  5. If $\bigwedge\Phi \in T$, then $\Phi \subseteq T$; if $\neg\bigwedge\Phi \in T$, then there is $\psi \in \Phi$, such that $\neg\psi \in T$.
  6. If $\bigvee\Phi \in T$, then there is $\psi \in \Phi$, such that $\psi \in T$. In particular, $\bot \notin T$. If $\neg\bigvee\Phi \in T$, then $\neg\psi \in T$, for all $\psi \in \Phi$.
  7. Let ϕ be $\phi(x)$. If $\forall x\phi \in T$, then $\phi(t) \in T$, for every closed term t of L; if $\neg\forall x\phi \in T$, then $\neg\phi(t) \in T$, for some closed term t of L.
  8. Let ϕ be $\phi(x)$. If $\exists x\phi \in T$, then $\phi(t) \in T$, for some closed term t of L; if $\neg\exists x\phi \in T$, then for every closed term t of L, $\neg\phi(t) \in T$.
- A theory T with these properties is called a **Hintikka set** for L.

Models of Hintikka Sets

Theorem

Let L be a signature and T a Hintikka set for L. Then T has a model in which every element is of the form t^A for some closed term t of L. In fact the canonical model of the set of atomic sentences in T is a model of T.

- Write U for the set of atomic sentences in T, and let A be the canonical model of U.

  Claim: For every sentence ϕ of $L_{\infty\omega}$, if $\phi \in T$, then $A \models \phi$, and if $\neg\phi \in T$, then $A \models \neg\phi$.

  The proof is by induction on the construction of ϕ, using the definition of $\models$.

  - By 2 and 3 above, U is $=$-closed in L. Hence if ϕ is atomic, the conclusion is immediate by 1 and the definition of A.

Models of Hintikka Sets (Cont'd)

- We continue with the induction:
  - Suppose ϕ is of the form $\neg\psi$ for some sentence ψ.
    If $\phi \in T$, then $\neg\psi \in T$. By induction, $A \models \neg\psi$. Hence, $A \models \phi$.
    Suppose, next, $\neg\phi \in T$. Then $\psi \in T$ by 4. Hence, $A \models \psi$ by induction. But then $A \models \neg\phi$.
  - Suppose next that ϕ is $\forall x\psi$.
    If $\phi \in T$, then by 7, $\psi(t) \in T$, for every closed term t of L. So $A \models \psi(t)$, by the induction hypothesis. Since every element of the canonical model is named by a closed term, $A \models \forall x\psi$.
    If $\neg\phi \in T$, then by 7 again, $\neg\psi(t) \in T$, for some closed term t. Hence, $A \models \neg\psi(t)$. Therefore $A \models \neg\forall x\psi$.
  - The remaining cases are similar.

  From the claim, it follows that A is a model of T.
- The theorem reduces the problem of finding a model to the problem of finding a particular kind of theory.

Identifying Hintikka Sets

Theorem

Let L be a first-order language (or, more generally, a first-order-closed language). Let T be a theory in L such that:

(a) Every finite subset of T has a model;
(b) For every sentence ϕ of L, either ϕ or $\neg\phi$ is in T;
(c) For every sentence $\exists x\psi(x)$ in T, there is a closed term t of L, such that $\psi(t)$ is in T;
(d) For every sentence $\bigvee\Phi$ in T with Φ infinite, there is $\psi\in\Phi$, such that $\psi\in T$, and, for every sentence $\neg\bigwedge\Phi$ in T with Φ infinite, there is $\psi\in\Phi$, such that $\neg\psi\in T$.

Then T is a Hintikka set for L. Note that clause (d) has no effect if L is first-order.

Claim: If U is a finite subset of T and ϕ is a sentence of L such that $U\vdash\phi$, then $\phi\in T$.

Identifying Hintikka Sets (Cont'd)

- Let U and ϕ be a counterexample. Then $\phi \notin T$. By (b), $\neg\phi \in T$.

  It follows by (a) that there is a model of $U \cup \{\neg\phi\}$, contradicting the assumption that $U \vdash \phi$.
- Condition 1 of the definition of Hintikka sets follows from (a).
- Conditions 2, 3 and 4 follow from the claim.
- Condition 5:
  - Its first part follows from the claim.
  - For the second part:
    - Suppose Φ is infinite and $\neg\bigwedge\Phi \in T$. Then, by (d), $\neg\psi \in T$, for some $\psi \in \Phi$.
    - Suppose $\neg(\phi_0 \wedge \cdots \wedge \phi_{n-1}) \in T$. Towards a contradiction, assume $\neg\phi_0, \ldots, \neg\phi_{n-1} \notin T$. By (b), $\phi_0, \ldots, \phi_{n-1} \in T$. Thus,

$$\{\neg(\phi_0 \wedge \cdots \wedge \phi_{n-1}), \phi_0, \ldots, \phi_{n-1}\} \subseteq T.$$

      By (a), this set has a model. This gives a contradiction.

Identifying Hintikka Sets (Conclusion)

- Condition 6:
  - For the first part:
    - Suppose Φ is infinite and $\bigvee \Phi \in T$. Then, by (d), $\psi \in T$, for some $\psi \in \Phi$.
    - Suppose $\phi_0 \vee \cdots \vee \phi_{n-1} \in T$. Towards a contradiction, assume $\phi_0, \ldots, \phi_{n-1} \notin T$. By (b), $\neg\phi_0, \ldots, \neg\phi_{n-1} \in T$. Thus,

$$\{\phi_0 \vee \cdots \vee \phi_{n-1}, \neg\phi_0, \ldots, \neg\phi_{n-1}\} \subseteq T.$$

      By (a), this set has a model. This gives a contradiction.
  - Its second part follows from the claim.
- Condition 7:
  - The first part follows from the claim.
  - If $\neg\forall x\phi(x) \in T$, by the claim, $\exists x\neg\phi(x) \in T$. Thus, by (c), $\neg\phi(t) \in T$, for some closed term t.
- Condition 8:
  - The first part is a consequence of (c).
  - The second part follows from the claim.

Subsection 4

Maps and the Formulas they Preserve

Preservation of Formulas by Homomorphisms

- Let $f : A \to B$ be a homomorphism of L-structures and $\phi(\overline{x})$ a formula of $L_{\infty\omega}$.
- We say that f **preserves** ϕ if, for every sequence $\overline{a}$ of elements of A,

$$A \models \phi(\overline{a}) \quad \Rightarrow \quad B \models \phi(f(\overline{a})).$$

  Example: In this terminology, we have seen that:
  - Homomorphisms preserve atomic formulas;
  - A homomorphism is an embedding if and only if it preserves literals.
- A formula ϕ is **absolute under** f if the displayed relation holds with $\Rightarrow$ replaced by $\Leftrightarrow$.

  Example: Thus atomic formulas are absolute under embeddings.
- The notion of preservation can be used in two ways.
  - To classify formulas in terms of the maps which preserve them.
  - To classify maps in terms of the formulas which they preserve.

Universal (A1) and Existential (E1) Formulas

- A formula ϕ is said to be an $\forall_1$ formula (pronounced "A1 formula"), or **universal**, if it is built up from quantifier-free formulas by means of $\bigwedge, \bigvee$ and universal quantification (at most).
- A formula ϕ is said to be an $\exists_1$ formula (pronounced "E1 formula"), or **existential**, if it is built up from quantifier-free formulas by means of $\bigwedge, \bigvee$ and existential quantification (at most).

The Quantifier Hierarchy

- Universal and existential formulas constitute the bottom end of a hierarchy:
    1. Formulas are said to be $\forall_0$, and $\exists_0$, if they are quantifier-free.
    2. A formula is an $\forall_{n+1}$ formula if it is in the smallest class of formulas which contains the $\exists_n$ formulas and is closed under $\bigwedge, \bigvee$ and adding universal quantifiers at the front.
    3. A formula is an $\exists_{n+1}$ formula if it is in the smallest class of formulas which contains the $\forall_n$ formulas and is closed under $\bigwedge, \bigvee$ and adding existential quantifiers at the front.
- $\forall_2$ formulas are sometimes known as $\forall\exists$ formulas.
- Every quantifier-free formula is $\forall_1$ and $\exists_1$.
- All $\forall_1$ or $\exists_1$ formulas are $\forall_2$.

Positive Boolean Combinations

- If a formula is formed from other formulas by means of just $\wedge$ and $\vee$, we say it is a **positive boolean combination** of these other formulas.
- If just $\bigwedge$ and $\bigvee$ are used, we talk of a **positive infinite boolean combination**.
- Note that, for any $n < \omega$, the class of $\forall_n$ formulas and the class of $\exists_n$ formulas of $L_{\infty\omega}$ are both closed under positive infinite boolean combinations.

Embeddings Preserve $\exists_1$ Formulas

Theorem

Let $\phi(\overline{x})$ be an $\exists_1$ formula of signature L and $f : A \to B$ an embedding of L-structures. Then f preserves ϕ.

- We first show that if $\phi(\overline{x})$ is a quantifier-free formula of L and $\overline{a}$ is a sequence of elements of A, $A \vDash \phi(\overline{a}) \Leftrightarrow B \vDash \phi(f\overline{a})$.
  This is proved by induction on the complexity of ϕ.
  - If ϕ is atomic, we have it by a precious result.
  - If ϕ is $\neg\psi$, $\bigwedge\Phi$ or $\bigvee\Phi$, then the result follows by induction hypothesis.

  We prove the theorem by showing that, for every $\exists_1$ formula $\phi(\overline{x})$ and every sequence $\overline{a}$ of elements of A, $A \vDash \phi(\overline{a}) \Rightarrow B \vDash \phi(f(\overline{a}))$.
  For quantifier-free ϕ this follows from the previous part.
  - $\bigwedge$ and $\bigvee$ are handled as before.
  - Let $\phi(\overline{x})$ be $\exists y \psi(y, \overline{x})$ and suppose $A \vDash \phi(\overline{a})$. Then $A \vDash \psi(c, \overline{a})$, for some c in A. By the induction hypothesis, $B \vDash \psi(f(c), f(\overline{a}))$. Hence, $B \vDash \phi(f(\overline{a}))$.

Substructures and $\forall_1$ Theories

- We say that a formula $\phi(\overline{x})$ is **preserved in substructures** if whenever A and B are L-structures, A is a substructure of B and $\overline{a}$ is a sequence of elements of A,

$$B \models \phi(\overline{a}) \quad \text{implies} \quad A \models \phi(\overline{a}).$$

- We say that a theory T is an $\forall_1$ **theory** if all the sentences in T are $\forall_1$ formulas.

Substructures Preserve $\forall_1$ Formulas

Corollary

(a) $\forall_1$ formulas are preserved in substructures.

(b) If T is an $\forall_1$ theory, then the class of models of T is closed under taking substructures.

(a) Suppose A is a substructure of B.
Then $i : A \to B$, with $i(a) = a$, for all $a \in A$, is an embedding.
Let $\phi(\overline{x})$ be a $\forall_1$ formula. Then $\neg\phi(\overline{x})$ is equivalent to an $\exists_1$ formula.
By the theorem, for all $\overline{a}$ in A,

$$A \models \neg\phi(\overline{a}) \quad \text{implies} \quad B \models \neg\phi(\overline{a}).$$

Now we get, for all $\overline{a}$ in A,

$$B \models \phi(\overline{a}) \quad \text{iff} \quad B \not\models \neg\phi(\overline{a}) \quad \text{implies} \quad A \not\models \neg\phi(\overline{a}) \quad \text{iff} \quad A \models \phi(\overline{a}).$$

(b) Suppose T is an $\forall_1$ theory. Let A be a model of T and B a substructure of A. By Part (a), B is also a model of T.

Example

- In a signature with just one binary function symbol, a substructure of a group need not be a group.
  - $(\mathbb{Z}, +)$ is a group;
  - $(2\mathbb{Z}, +)$ is a substructure that is not a group.
- By the corollary, in such a signature, groups cannot be axiomatized by an $\forall_1$ theory.

Positive and Positive Existential Formulas

- A formula of $L_{\infty\omega}$ is said to be **positive** if $\neg$ never occurs in it.
- Note $\rightarrow$ and $\leftrightarrow$ never occur in a positive formula, but $\bot$ may.
- We call a formula $\exists_1^+$ or **positive existential** if it is both positive and existential.

Theorem

Let $\phi(\overline{x})$ be a formula of signature L and $f : A \rightarrow B$ a homomorphism of L-structures.

(a) If ϕ is an $\exists_1^+$ formula then f preserves ϕ.

(b) If ϕ is positive and f is surjective, then f preserves ϕ.

(c) If f is an isomorphism then f preserves ϕ.

- There are many similar results for other types of homomorphism.

Unions of Chains

- Let L be a signature and $(A_i : i < \gamma)$ a sequence of L-structures.
- We call $(A_i : i < \gamma)$ a **chain** if, for all $i < j < \gamma$, $A_i \subseteq A_j$.
- If $(A_i : i < \gamma)$ is a chain, then we define an L-structure B as follows:
  - The domain of B is $\bigcup_{i<\gamma} \text{dom}(A_i)$.
  - For each constant c, c^{A_i} is independent of the choice of i.
    So we may define $c^B = c^{A_i}$, for any $i < \gamma$.
  - Likewise if F is an n-ary function symbol of L and $\overline{a}$ is an n-tuple of elements of B, then $\overline{a}$ is in $\text{dom}(A_i)$, for some $i < \gamma$.
    Without ambiguity, we can define $F^B(\overline{a})$ to be $F^{A_i}(\overline{a})$.
  - If R is an n-ary relation symbol of L, we define $\overline{a} \in R^B$ if $\overline{a} \in R^{A_i}$, for some (or all) A_i containing $\overline{a}$.
- By construction, $A_i \subseteq B$, for every $i < \gamma$.
- We call B the **union** of the chain $(A_i : i < \gamma)$, in symbols $B = \bigcup_{i<\gamma} A_i$.

Finite $\forall$ of $\exists_1$ Formula are Preserved in Unions

- We say that a formula $\phi(\overline{x})$ of L is **preserved in unions of chains** if whenever $(A_i : i < \gamma)$ is a chain of L-structures, $\overline{a}$ is a sequence of elements of A_0 and $A_i \models \phi(\overline{a})$, for all $i < \gamma$, then $\bigcup_{i<\gamma} A_i \models \phi(\overline{a})$.

Theorem

Let $\psi(\overline{y}, \overline{x})$ be an $\exists_1$ formula of signature L with $\overline{y}$ finite. Then $\forall \overline{y} \psi$ is preserved in unions of chains of L-structures.

- Let $(A_i : i < \gamma)$ be a chain of L-structures and $\overline{a}$ a sequence of elements of A_0, such that $A_i \models \forall \overline{y} \psi(\overline{y}, \overline{a})$, for all $i < \gamma$. Put $B = \bigcup_{i<\gamma} A_i$.

  To show that $B \models \forall \overline{y} \psi(\overline{y}, \overline{a})$, let $\overline{b}$ be any tuple of elements of B.

  Since $\overline{b}$ is finite, there is some $i < \gamma$, such that $\overline{b}$ lies in A_i.

  By assumption, $A_i \models \psi(\overline{b}, \overline{a})$. Since $A_i \subseteq B$ and $\psi(\overline{y}, \overline{x})$ is an $\exists_1$ formula, by a previous theorem, $B \models \psi(\overline{b}, \overline{a})$.

Example

- Any $\forall_2$ first-order formula can be brought to the form $\forall \overline{y} \psi$, with ψ existential.
- By the theorem, then, all $\forall_2$ first-order formulas are preserved in unions of chains.

  Example: Recall the axioms for dense linear orderings without endpoints:
  - $\forall x (x \not< x)$;
  - $\forall xy (x = y \vee x < y \vee y < x)$;
  - $\forall xyz (x < y \wedge y < z \rightarrow x < z)$;
  - $\forall xy (x < y \rightarrow \exists z (x < z \wedge z < y))$;
  - $\forall x \exists z (z < x)$;
  - $\forall x \exists z (x < z)$.

  These are all $\forall_2$ first-order.

  It follows that the union of a chain of dense linear orderings without endpoints is a dense linear ordering without endpoints.

Subsection 5

Classifying Maps by Formulas

Elementary Embeddings and Elementary Extensions

- Let L be a signature, $f: A \to B$ a homomorphism of L-structures and Φ a class of formulas of $L_{\infty\omega}$.
- We call f a **Φ-map** if f preserves all the formulas in Φ.
- An **elementary embedding** is a homomorphism (necessarily an embedding) which preserves all first-order formulas.
- We say that B is an **elementary extension** of A, or that A is an **elementary substructure** of B, in symbols $A \preccurlyeq B$, if:
  - $A \subseteq B$;
  - The inclusion map is an elementary embedding.
- If the inclusion map is an elementary embedding, it is called an **elementary inclusion**.
- We write $A \prec B$ when A is a proper elementary substructure of B.
- Note that $A \preccurlyeq B$ implies $A \equiv B$.
- However, $A \subseteq B$ and $A \equiv B$ *do not imply* $A \preccurlyeq B$.

Example

- We show that $A \subseteq B$ and $A \equiv B$ do not imply $A \preccurlyeq B$.
- Consider the language L with a single binary relation symbol $<$ and the two L-structures

$$\begin{array}{rcl} A & = & (\{0,2,4,\ldots\},<); \\ B & = & (\{0,1,2,\ldots\},<). \end{array}$$

- Clearly, $A \subseteq B$.
- Moreover $A \equiv B$.

  In fact, $f : A \to B$, with $a \mapsto \frac{a}{2}$ is an isomorphism.
- On the other hand, $A \not\preccurlyeq B$.

  For example, if $\phi(x,y) = \neg\exists z(x < z \wedge z < y)$, we have

$$A \vDash \phi(x,y)[0,2] \quad \text{but} \quad B \nvDash \phi(x,y)[0,2].$$

  Consequently, the inclusion is not an elementary embedding.

Tarski-Vaught Criterion for Elementary Substructures

Theorem (Tarski-Vaught Criterion for Elementary Substructures)

Let L be a first-order language and let A, B be L-structures with $A \subseteq B$. Then the following are equivalent:

(a) A is an elementary substructure of B.

(b) For every formula $\psi(\overline{x}, y)$ of L and all tuples $\overline{a}$ from A, if $B \models \exists y \psi(\overline{a}, y)$, then $B \models \psi(\overline{a}, d)$, for some element d of A.

- Let $f : A \to B$ be the inclusion map.

(a)⇒(b) Suppose $B \models \exists y \psi(\overline{a}, y)$. Since f is elementary, $A \models \exists y \psi(\overline{a}, y)$. Hence, there is d in A, such that $A \models \psi(\overline{a}, d)$. By applying f, $B \models \psi(\overline{a}, d)$.

(b)⇒(a) In the theorem on $\exists_1$ formulas and embeddings, Condition (b) is exactly what is needed to show that f is elementary.

- This theorem is not very useful for detecting elementary substructures.
- Its main use is for constructing elementary substructures.

Tarski-Vaught Theorem on Unions of Elementary Chains

- If Φ is a set of formulas, we say that a chain $(A_i : i < \gamma)$ of L-structures is a Φ-**chain** when each inclusion map $A_i \subseteq A_j$ is a Φ-map.
- In particular an **elementary chain** is a chain in which the inclusions are elementary.

Theorem (Tarski-Vaught Theorem on Unions of Elementary Chains)

Let $(A_i : i < \gamma)$ be an elementary chain of L-structures. Then $\bigcup_{i<\gamma} A_i$ is an elementary extension of each A_j, $j < \gamma$.

- Put $A = \bigcup_{i<\gamma} A_i$. Let $\phi(\overline{x})$ be a first-order formula of signature L.

  We show by induction on the complexity of ϕ that for every $j < \gamma$ and every tuple $\overline{a}$ of elements of A_j,

$$A_j \models \phi(\overline{a}) \quad \text{iff} \quad A \models \phi(\overline{a}).$$

Tarski-Vaught on Unions of Elementary Chains (Cont'd)

- When ϕ is atomic, this follows by a previous theorem.
- The cases $\neg\psi$, $(\psi \wedge \chi)$ and $(\psi \vee \chi)$ are straightforward.
  E.g.,
  $$\begin{aligned} A \models (\psi \vee \chi)(\overline{a}) \quad & \text{iff} \quad A \models \psi(\overline{a}) \text{ or } A \models \chi(\overline{a}) \\ & \text{iff} \quad A_j \models \psi(\overline{a}) \text{ or } A_j \models \chi(\overline{a}) \\ & \text{iff} \quad A_j \models (\psi \vee \chi)(\overline{a}). \end{aligned}$$
- Suppose then that ϕ is $\exists y \psi(\overline{x}, y)$.
  If $A \models \phi(\overline{a})$, then there is some b in A, such that $A \models \psi(\overline{a}, b)$.
  Choose $k < \gamma$ so that b is in $\text{dom}(A_k)$ and $k \geq j$.
  Then $A_k \models \psi(\overline{a}, b)$ by the induction hypothesis.
  Hence, $A_k \models \phi(\overline{a})$.
  Since the chain is elementary, $A_j \models \phi(\overline{a})$.
  The other direction is easier.
- The argument for $\forall y \psi(\overline{x}, y)$ is similar.

The Elementary Diagram Lemma

Lemma (Elementary Diagram Lemma)

Suppose L is a first-order language, A and B are L-structures, $\overline{c}$ is a tuple of distinct constants not in L, $(A,\overline{a})$ and $(B,\overline{b})$ are $L(\overline{c})$-structures, and $\overline{a}$ generates A. Then the following are equivalent:

(a) For every formula $\phi(\overline{x})$ of L, if $(A,\overline{a}) \models \phi(\overline{c})$, then $(B,\overline{b}) \models \phi(\overline{c})$.

(b) There is an elementary embedding $f : A \to B$ such that $f(\overline{a}) = \overline{b}$.

- Clearly, (b) implies (a). Suppose (a) holds. Every element of A is of the form $t^{(A,\overline{a})}$, for some closed term t of $L(\overline{c})$. Define $f : A \to B$ by $f(t^{(A,\overline{a})}) = t^{(B,\overline{b})}$.
  Let $\phi(\overline{z})$ be an L-formula and $\overline{a}'$ a tuple in A.
  By choosing a suitable sequence $\overline{x}$ of variables, we can write $\phi(\overline{z})$ as $\psi(\overline{x})$ so that $\phi(\overline{a}')$ is the same formula as $\psi(\overline{a})$.
  Then $A \models \phi(\overline{a}')$ implies $A \models \psi(\overline{a})$, which by (a) implies $B \models \psi(f(\overline{a}))$.
  Hence, $B \models \phi(f(\overline{a}'))$. So f is an elementary embedding.

The Elementary Diagram

- We define the **elementary diagram** of an L-structure A, in symbols eldiag(A), to be Th($A,\overline{a}$), where $\overline{a}$ is any sequence which generates A.
- By (a)⇒(b) of the Elementary Diagram Lemma, we have the following fact, which will be used constantly for constructing elementary extensions:

  If D is a model of the elementary diagram of the L-structure A, then there is an elementary embedding of A into the reduct $D|_L$.

Example: Pure Extensions

- This example is taken from abelian groups and modules.
- Let A and B be left R-modules, and A a submodule of B.
- We say that A is **pure** in B, or that B is a **pure extension** of A, if the following holds:

  For every finite set E of equations with parameters in A, if E has a solution in B, then E already has a solution in A.
- The statement that a certain finite set of equations with parameters $\overline{a}$ has a solution can be written

$$\exists \overline{x}(\psi_1(\overline{x},\overline{a}) \wedge \cdots \wedge \psi_k(\overline{x},\overline{a})),$$

  with $\psi_1,\ldots,\psi_k$ atomic.
- A first-order formula of this form is said to be **positive primitive**, or **p.p.** for short.
- So we can define a **pure embedding** to be one which preserves the negations of all positive primitive formulas.

Subsection 6

Translations

Paraphrase 1: Unnested Formulas

- We look at some paraphrases that do not alter the class of definable relations on a structure, but only affect formulas which can be used to define them.
- Let L be a signature. By an **unnested atomic formula** of signature L we mean an atomic formula of one of the following forms:
  1. $x = y$;
  2. $c = y$, for some constant c of L;
  3. $F(\overline{x}) = y$, for some function symbol F of L;
  4. $R\overline{x}$, for some relation symbol R of L.
- We call a formula **unnested** if all of its atomic subformulas are unnested.
- Unnested formulas are handy when we want to make definitions or proofs by induction on the complexity of formulas.

  For the atomic case we never need to consider any terms except variables, constants and terms $F(\overline{x})$, where F is a function symbol.

Atomic Formulas and Unnested Formulas

Theorem

Let L be a signature. Then every atomic formula $\phi(\overline{x})$ of L is logically equivalent to unnested first-order formulas $\phi^\forall(\overline{x})$ and $\phi^\exists(\overline{x})$ of signature L, such that $\phi^\forall$ is an $\forall_1$ formula and $\phi^\exists$ is an $\exists_1$ formula.

- The formula $F(G(x),z) = c$ is logically equivalent to

$$\forall uw(G(x) = u \wedge F(u,z) = w \rightarrow c = w)$$

and to

$$\exists uw(G(x) = u \wedge F(u,z) = w \wedge c = w).$$

- The formula $\phi^\exists$ is positive primitive.
- The formula $\phi^\forall$ is strict universal Horn (as will be defined later).

Existence of Unnested Equivalent Formulas

Corollary

Let L be a first-order language. Then every formula $\phi(\overline{x})$ of L is logically equivalent to an unnested formula $\psi(\overline{x})$ of L. More generally every formula of $L_{\infty\omega}$ is logically equivalent to an unnested formula of $L_{\infty\omega}$.

- Use the theorem to replace all atomic subformulas by unnested first-order formulas.
- If ϕ in the corollary is an $\exists_1$ formula, then by choosing wisely between $\theta^{\forall}$ and $\theta^{\exists}$ for each atomic subformula θ of ϕ, we can arrange that ψ in the corollary is an $\exists_1$ formula too.
- In fact we can always choose ψ to lie in the same place in the $\forall_n, \exists_n$ hierarchy as ϕ, unless ϕ is quantifier-free.

Paraphrase 2: Definitional Expansions and Extensions

- Let L and L^+ be signatures with $L \subseteq L^+$.
- Let R be a relation symbol, c a constant and F a function symbol of L^+.
- An **explicit definition of R in terms of L** is a sentence of the form

  $$\forall \overline{x}(R\overline{x} \leftrightarrow \phi(\overline{x})),$$

  where ϕ is a formula of L.
- An **explicit definition of c in terms of L** is a sentence of the form

  $$\forall y(c = y \leftrightarrow \phi(y)),$$

  where ϕ is a formula of L.
- An **explicit definition of F in terms of L** is a sentence of the form

  $$\forall \overline{x}y(F(\overline{x}) = y \leftrightarrow \psi(\overline{x}, y)),$$

  where ψ is a formula of L.

Admissibility Conditions

- If c is a constant and F is a function symbol of L^+, **explicit definitions of c, F in terms of L** are sentences of the form
$$\forall y(c = y \leftrightarrow \phi(y)), \quad \forall \overline{x} y(F(\overline{x}) = y \leftrightarrow \psi(\overline{x}, y)),$$
where ϕ, ψ are formulas of L.
- These sentences imply respectively
$$\exists_{=1} y \phi(y), \quad \forall \overline{x} \exists_{=1} y \psi(\overline{x}, y).$$
We show $\forall \overline{x} y(F(\overline{x}) = y \leftrightarrow \psi(\overline{x}, y)) \vdash \forall \overline{x} \exists_{=1} y \psi(\overline{x}, y)$.
Suppose $(A, F^A) \models \forall \overline{x} y(F(\overline{x}) = y \leftrightarrow \psi(\overline{x}, y))$.
Thus, for all $\overline{a}, b$ in A, $(A, F^A) \models F(\overline{x}) = y \leftrightarrow \psi(\overline{x}, y)[\overline{a}, b]$.
Then, for all $\overline{a}, b$ in A, $F^A(\overline{a}) = b$ iff $A \models \psi(\overline{x}, y)[\overline{a}, b]$.
Since F^A is a function, for all $\overline{a}$ in A, $A \models \exists_{=1} y \psi(\overline{x}, y)[\overline{a}]$.
Thus, $A \models \forall \overline{x} \exists_{=1} y \psi(\overline{x}, y)$.
- We call these sentences the **admissibility conditions** of the explicit definitions of c and F, respectively, in terms of L.

Uniqueness of Definitional Expansions

Theorem (Uniqueness of Definitional Expansions)

Let L and L^+ be signatures with $L \subseteq L^+$ and A and B be L^+-structures.

- Let R be a relation symbol of L^+ and θ an explicit definition of R in terms of L. If A and B are models of θ, and $A|_L = B|_L$, then $R^A = R^B$.
- Let c be a constant of L^+ and θ an explicit definition of c in terms of L. If A and B are models of θ, and $A|_L = B|_L$, then $c^A = c^B$.
- Let F be a function symbol of L^+ and θ an explicit definition of F in terms of L. If A and B are models of θ, and $A|_L = B|_L$, then $F^A = F^B$.

- Suppose $\theta = \forall \overline{x}(R\overline{x} \leftrightarrow \phi(\overline{x}))$. Since $A|_L = B|_L$, $\mathrm{dom}(A) = \mathrm{dom}(B)$. Then, for all $\overline{a}$ in A,

$$\begin{array}{lll} R^A(\overline{a}) & \text{iff} & A \models \phi(\overline{a}) \quad (A \models \theta) \\ & \text{iff} & B \models \phi(\overline{a}) \quad (A|_L = B|_L) \\ & \text{iff} & R^B(\overline{a}). \quad (B \models \theta) \end{array}$$

Existence of Definitional Expansions

Theorem (Existence of Definitional Expansions)

Let L and L^+ be signatures with $L \subseteq L^+$. Suppose that for each symbol S of $L^+ \setminus L$, θ_S is an explicit definition of S in terms of L. Let U be the set of these definitions.

(a) If C is any L-structure which satisfies the admissibility conditions (if any) of the definitions θ_S, then we can expand C to form an L^+-structure C^+ which is a model of U.

(b) Every formula $\chi(\overline{x})$ of signature L^+ is equivalent modulo U to a formula $\chi^*(\overline{x})$ of signature L.

(c) If χ and all the sentences θ_S are first-order, then so is χ^*.

(a) Interpret the symbol S in C^+, using the definition θ_S.

(b) Use a previous theorem to replace every atomic formula in χ by an unnested formula. The explicit definitions translate each unnested atomic formula directly into a formula of signature L.

(c) Clear from the process of Part (b).

Definitional Expansions and Definitional Extensions

- A structure C^+ as in Part (a) of the theorem is called a **definitional expansion** of C.
- Let L and L^+ be signatures with $L \subseteq L^+$, and T an L-theory.
- A **definitional extension** of T to L^+ is a theory equivalent to

$$T \cup \{\theta_S : S \text{ a symbol in } L^+ \setminus L\},$$

  where, for each symbol S in $L^+ \setminus L$,
  1. θ_S is an explicit definition of S in terms of L;
  2. if S is a constant or function symbol and χ is the admissibility condition for θ_S, then $T \vdash \chi$.
- The preceding two theorems tell us that:
  - If T^+ is a definitional extension of T to L^+, then every model C of T has a unique expansion C^+ which is a model of T^+;
  - C^+ is a definitional expansion of C.

Explicit Definability

- Let T^+ be a theory in the language L^+, and L a language $\subseteq L^+$.
- We say that a symbol S of L^+ is **explicitly definable in T^+ in terms of L** if T^+ entails some explicit definition of S in terms of L.
- So, up to equivalence of theories, T^+ is a definitional extension of a theory T in L iff:
  (1) T and T^+ have the same consequences in L;
  (2) Every symbol of L^+ is explicitly definable in T^+ in terms of L.

Example

- Definitional extensions are useful for replacing complicated formulas by simple ones.
- Suppose Set Theory entails the admissibility condition
$$\forall x \forall y \exists_{=1} z (\forall t (t \in z \leftrightarrow (t \in x \vee t \in y))).$$
- Then, we may introduce a binary function symbol $\cup$, explicitly defined in terms of Set Theory by
$$\theta_{\cup} = \forall x \forall y (\forall t (t \in x \cup y \leftrightarrow (t \in x \vee t \in y))).$$
- So, instead of the formula
$$\exists z (\phi(z) \wedge \forall t (t \in z \leftrightarrow (t \in x \vee t \in y))),$$
we may write the simpler version
$$\phi(x \cup y).$$

Definitional Equivalence

- Suppose L_1 and L_2 are disjoint signatures.
- Let T_1 and T_2 be first-order theories of signature L_1, L_2, respectively.
- We say that T_1 and T_2 are **definitionally equivalent** if there is a first-order theory T in the signature $L_1 \cup L_2$ which is a definitional extension both of T_1 and of T_2.
- When theories T_1 and T_2 are definitionally equivalent as above, we can turn a model A_1 of T_1 into a model A_2 of T_2 by:
    - First expanding A_1 to a model of T;
    - Then restricting to the language L_2.
- We can get back to A_1 from A_2 by doing the same in the opposite direction.
- The structures A_1 and A_2 are then called **definitionally equivalent**.

Example: Term Algebras in Another Language

- We met the following sentences true in every term algebra of a fixed algebraic signature L.
  1. $c \neq d$, where c, d are distinct constants.
  2. $\forall \overline{x} F(\overline{x}) \neq c$, where F is a function symbol and c a constant.
  3. $\forall \overline{x}\overline{y} F(\overline{x}) \neq G(\overline{y})$, where F, G are distinct function symbols.
  4. $\forall x_0 \ldots, x_{n-1} y_0 \ldots y_{n-1} (F(x_0, \ldots, x_{n-1}) = F(y_0, \ldots, y_{n-1}) \to \bigwedge_{i<n} x_i = y_i)$.
  5. $\forall x_0 \ldots x_{n-1} t(x_0, \ldots, x_{n-1}) \neq x_i$, where $i < n$ and t is any term containing x_i but distinct from x_i.
  6. [Use this axiom only when L is finite.] Write $\mathrm{Var}(x)$ for the formula $\bigwedge\{x \neq c : c \text{ a constant of } L\} \wedge \bigwedge\{\forall \overline{y} x \neq F(\overline{y}) : F \text{ a function symbol of } L\}$.

    - If X has finite cardinality n, we add the axiom $\exists_{=n} x \mathrm{Var}(x)$.
    - If X is infinite, we add the infinitely many axioms $\exists_{\geq n} x \mathrm{Var}(x)$ $(n < \omega)$.
- Call the set of these sentences T_1 and let L_1 be their first-order language.

Example: Term Algebras in Another Language (Cont'd)

- Let L_2 be the first-order language whose signature consists of the following symbols:
  1. For each constant c of L_1, a unary relation symbol Is_c;
  2. For each function symbol F of L_1 a unary relation symbol Is_F;
  3. For each n-ary function symbol F of L_1 and each $i < n$, a unary function symbol F_i.

  Claim: T_1 is definitionally equivalent to the following theory T_2 in L_2.
  1. $\exists_{=1} y \mathrm{Is}_c(y)$, for each constant symbol c of L.
  2. $\forall x_0 \ldots x_n \exists_{=1} y(\mathrm{Is}_F(y) \wedge \bigwedge_{i<n} F_i(y) = x_i)$, for each function symbol F.
  3. $\forall x \neg(\mathrm{Is}_c(x) \wedge \mathrm{Is}_d(x))$, where c, d are distinct constant or function symbols.
  4. $\forall x(\neg \mathrm{Is}_F(x) \to F_i(x) = x)$, for each function symbol F_i.
  5. $\forall x(t(F_i(x)) = x \to \neg \mathrm{Is}_F(x))$, for each function symbol F_i and term $t(y)$ of L_2.

Example: Symbols of L_2 In Terms of L_1

- To prove definitional equivalence, we must write down:
  - Explicit definitions U_1 of the symbols of L_2 in terms of L_1;
  - Explicit definitions U_2 of the symbols of L_1 in terms of L_2;

  so that:
  - T_i implies the admissibility conditions for $U_i, i = 1, 2$;
  - $T_1 \cup U_1$ is equivalent to $T_2 \cup U_2$.
- Definitions of L_2 in terms of L_1.
  - $\forall y(\mathrm{Is}_c(y) \leftrightarrow y = c)$, c a constant of L_1.
  - $\forall y(\mathrm{Is}_F(y) \leftrightarrow \exists \overline{x} F\overline{x} = y)$, F a function symbol of L_1.
  - $\forall xy(F_i x = y \leftrightarrow (\exists y_0 \ldots y_{i-1} y_{i+1} \ldots y_{n-1} F(y_0, \ldots, y_{i-1}, y, y_{i+1}, \ldots, y_{n-1}) = x) \vee (x = y \wedge \neg \exists \overline{y} F\overline{y} = x))$, F an n-ary function symbol of L_1 and $i < n$.

Example: Symbols of L_1 In Terms of L_2

- Definitions of L_1 in terms of L_2.
  - $\forall y(y = c \leftrightarrow \mathrm{Is}_c(y))$, c a constant of L_1.
  - $\forall x_0 \ldots x_{n-1} y(F(x_0, \ldots, x_{n-1}) = y \leftrightarrow (\mathrm{Is}_F(y) \wedge \bigwedge_{i<n} F_i(y) = x_i))$, F a function symbol of L_1.
- T_1 and T_2 give opposite ways of looking at the term algebra.
  - T_1 generates the terms from their components.
  - T_2 recovers the components from the terms.
- Note that:
  - T_2 uses only unary function and relation symbols;
  - There is no bound on the arities of the symbols in T_1.

Paraphrase 3: Atomization

- We have a theory T in a language L, and a set Φ of formulas of L which are not sentences.
- The goal is to extend T to a theory T^+ in a larger language L^+ in such a way that every formula in Φ is equivalent modulo T^+ to an atomic formula.
- The set of new sentences $T^+ \setminus T$ will turn out to depend only on L and not on T.
- This process has been called **Morleyization**, even though it was introduced by Skolem.
- We call it **atomization**.

Atomization Theorem

Theorem (Atomization Theorem)

Let L be a first-order language. Then there are a first-order language $L^\Theta \supseteq L$ and a theory Θ in L^Θ such that:

(a) Every L-structure A can be expanded in just one way to an L^Θ-structure A^Θ which is a model of Θ;

(b) Every formula $\phi(\overline{x})$ of L^Θ is equivalent modulo Θ to a formula $\psi(\overline{x})$ of L, and also (when $\overline{x}$ is not empty) to an atomic formula $\chi(\overline{x})$ of L^Θ;

(c) Every homomorphism between non-empty models of Θ is an elementary embedding;

(d) $|L^\Theta| = |L|$.

- For each formula $\phi(x_0,\ldots,x_{n-1})$ of L with $n > 0$, introduce a new n-ary relation symbol R_ϕ.
  - L^Θ is the first-order language got from L by adding all the symbols R_ϕ.
  - Θ is the set of all sentences of the form $\forall\overline{x}(R_\phi\overline{x} \leftrightarrow \phi(\overline{x}))$.

Atomization Theorem (cont'd)

(a) Θ is a definitional extension of the empty theory in L.

(b) This also implies the first part of (b).

The second part of (b) then follows by the sentences of Θ.

(c) By (b), every formula of L which is not a sentence is equivalent modulo Θ to an atomic formula.

If ϕ is a sentence of L then $\phi \wedge (x = x)$ is equivalent modulo Θ to an atomic formula $\chi(x)$.

Any homomorphism between non-empty models of Θ which preserves χ must also preserve ϕ. So (c) follows by a previous theorem.

(d) This is immediate.

- The same technique may be applied to a particular set Φ of formulas of L to study homomorphisms which preserve the formulas in Φ.

  If A and B are models of Θ, then every embedding (in fact every homomorphism) from A to B must preserve the formulas in Φ.

Atomization and Resulting Theories

Theorem

Let Θ be the theory constructed in the proof of the Atomization Theorem. Then for every theory T in L^Θ, $T \cup \Theta$ is equivalent to an $\forall_2$ theory.

- By (b) of the theorem, every formula of L^Θ with at least one free variable is equivalent modulo Θ to an atomic formula of L^Θ.

  So $T \cup \Theta$ is equivalent to a theory $T' \cup \Theta$, where every sentence of T' is $\forall_1$ at worst. We must show that Θ itself is equivalent to an $\forall_2$ theory.

  Let Θ' be the set of all sentences of the following forms:

  1. $\forall \overline{x}(\phi(\overline{x}) \leftrightarrow R_\phi(\overline{x}))$, where ϕ is an atomic formula of L;
  2. $\forall \overline{x}(R_\phi(\overline{x}) \wedge R_\psi(\overline{x}) \leftrightarrow R_{\phi \wedge \psi}(\overline{x}))$; and likewise for $\vee$;
  3. $\forall \overline{x}(\neg R_\phi(\overline{x}) \leftrightarrow R_{\neg \phi}(\overline{x}))$;
  4. $\forall \overline{x}(\forall y R_{\phi(\overline{x}, y)}(\overline{x}, y) \leftrightarrow R_{\forall y \phi(\overline{x}, y)}(\overline{x}))$ and likewise for $\exists$.

  After a slight rearrangement of the sentences 4, Θ' is an $\forall_2$ theory.

Atomization and Resulting Theories (Cont'd)

- Claim: Θ is equivalent to Θ'.
  Clearly Θ implies all the sentences in Θ'.
  Conversely assume that Θ' holds.
  Then $\forall\overline{x}(R_\phi\overline{x} \leftrightarrow \phi(\overline{x}))$ follows by induction on the complexity of ϕ.
  - The base case of ϕ atomic is covered by 1.
  - The steps for conjunction and disjunction are covered by 2.
  - The step for negation is covered by 3.
  - The steps for $\forall$ and $\exists$ are covered by 4.
    Suppose that $\Theta' \vdash \forall\overline{x}\forall y(R_{\phi(\overline{x},y)} \leftrightarrow \phi(\overline{x},y))$.
    We must show $\Theta' \vdash \forall\overline{x}(R_{\forall y\phi(\overline{x},y)} \leftrightarrow \forall y\phi(\overline{x},y))$.
    Let A be a model of Θ' and $\overline{a}, b$ in A.

$$\begin{array}{lll} A \models R_{\forall y\phi(\overline{x},y)}[\overline{a}] & \text{iff} & A \models \forall y R_{\phi(\overline{x},y)}(\overline{x},y)[\overline{a}] \quad (A \models \Theta') \\ & \text{iff} & A \models R_{\phi(\overline{x},y)}(\overline{x},y)[\overline{a},b], \text{ for all } b, \quad (\forall) \\ & \text{iff} & A \models \phi(\overline{x},y)[\overline{a},b], \text{ for all } b, \quad \text{(induction)} \\ & \text{iff} & A \models \forall y\phi(\overline{x},y)[\overline{a}]. \quad (\forall) \end{array}$$

    Therefore, $A \models \forall\overline{x}(R_{\forall y\phi(\overline{x},y)} \leftrightarrow \forall y\phi(\overline{x},y))$.

Model-Completeness

- A first-order theory is said to be **model-complete** if every embedding between its models is elementary.
- Atomization shows that we can turn any first-order theory into a model-complete theory in a harmless way.
- The real interest of the notion of model-completeness is that a number of theories in algebra have this property without any prior tinkering.

Subsection 7

Quantifier Elimination

Quantifier Elimination

- Take a first-order language L and a class $\mathbf{K}$ of L-structures.
- The class $\mathbf{K}$ might be, e.g., the class of all dense linear orderings, or it might be the singleton $\{\mathbb{R}\}$, where $\mathbb{R}$ is the field of real numbers.
- We say that a set Φ of formulas of L is an **elimination set** for $\mathbf{K}$ if:

  For every formula $\phi(\overline{x})$ of L, there is a formula $\phi^*(\overline{x})$ which is a boolean combination of formulas in Φ, and ϕ is equivalent to ϕ^* in every structure in $\mathbf{K}$.
- **Quantifier elimination**: Given $\mathbf{K}$, find an elimination set for $\mathbf{K}$.
  - Of course there always is at least one elimination set Φ for any class $\mathbf{K}$ of L-structures: We may take Φ to be the set of all formulas of L.
  - But with care and attention we can often find a much more revealing elimination set than this.

Example: Dense Linear Orderings

- A linear ordering is **dense** if for all elements $x < y$, there is z such that $x < z < y$.

Theorem

Let L be the first-order language whose signature consists of the binary relation symbol $<$, and let $\mathbf{K}$ be the class of all dense linear orderings. Let Φ consist of formulas of L which express each of the following:

- There is a first element.
- There is a last element.
- x is the first element.
- x is the last element.
- $x < y$.

Then Φ is an elimination set for $\mathbf{K}$.

Example: Dense Linear Orderings (Idea of Proof)

- The truth of a satisfiable formula $\varphi(x_1,\ldots,x_n)$ in a dense linear ordering A in $\mathbf{K}$ depends only on:
  - Whether the formula imposes the existence of a first and/or a last element and whether it stipulates that any of the x_i must be the first or the last element;
  - The relative positions imposed on $x_1,\ldots,x_n$.

  The particular ordering A is not important.
- So to write $\varphi(\overline{x})$ as a boolean combination of formulas in Φ, we have to take the conjunction of the following types of formulas:
  - For each i a formula stating whether x_i is a first or a last element, if that is stipulated by ϕ.
  - For each $i \neq j$, the disjunction of those of $x_i < x_j$, $x_i = x_j$ (which is equivalent to $\neg x_i < x_j \wedge \neg x_j < x_i$) and $x_j < x_i$ that hold for some $\overline{a}$ that realizes $\phi(\overline{x})$ in some structure A.

  The conjunction of those formulas of the two types outlined above is equivalent to ϕ in every structure in $\mathbf{K}$.

Example: Real Closed Fields

Theorem

Let L be the first-order language of rings, whose symbols are $+, -, \cdot, 0, 1$. Let $\mathbf{K}$ be the class of real-closed fields. Let Φ consist of the formulas

$$\exists y(y^2 = t(x)), \quad t \text{ a term of } L \text{ not containing the variable } y.$$

Then Φ is an elimination set for $\mathbf{K}$.

Note: $\exists y(y^2 = t(x))$ expresses $t(x) \geq 0$.

- We shall see an algebraic proof of this later.

Method vs. Property of Quantifier Elimination

- The name "quantifier elimination" refers to either of the following:
  - The process of reducing a formula to a boolean combination of formulas in Φ;
  - The process of discovering the appropriate set Φ in the first place.
- One should distinguish between:
  - The *method of quantifier elimination*;
  - The *property of quantifier elimination*, which is a property that some theories have.
- A theory T has **quantifier elimination** if the set of quantifier-free formulas forms an elimination set for the class of all models of T.

Usefulness of Quantifier Elimination

- The existence of an elimination set Φ for a class K of structures may prove useful in various contexts.
  (a) Classification of structures up to elementary equivalence;
  (b) Completeness proofs;
  (c) Decidability proofs;
  (d) Description of definable relations;
  (e) Description of elementary embeddings.

Quantifier Elimination and Axiomatization

- Suppose we have the following:
  - A first-order language L;
  - A class $\mathbf{K}$ of L-structures;
  - A theory T which is a candidate for an axiomatization of $\mathbf{K}$;
  - A set of formulas Φ which is a candidate for an elimination set.
- If $\mathbf{K}$ is defined as $\text{Mod}(T)$, then, of course, T does axiomatize $\mathbf{K}$.
- If $\mathbf{K}$ is given and T is a guess at an axiomatization, we may find during the course of the quantifier elimination that we have to adjust T.

Sufficient Conditions for an Elimination Set

Lemma

Given a set Φ of L-formulas, set $\Phi^- = \{\neg\phi : \phi \in \Phi\}$. Suppose that:

- Every atomic formula of L is in Φ;
- For every formula $\theta(\overline{x})$ of L which is of the form $\exists y \bigwedge_{i<n} \psi_i(\overline{x}, y)$, with each ψ_i in $\Phi \cup \Phi^-$, there is a formula $\theta^*(\overline{x})$ of L which:
  - (i) Is a boolean combination of formulas in Φ;
  - (ii) Is equivalent to θ in every structure in $\mathbf{K}$.

Then Φ is an elimination set for $\mathbf{K}$.

- Form the set Φ^B of all Boolean combinations of formulas in Φ.

  By a preceding lemma, it suffices to show the following:
  - (a) Every atomic L-formula is in Φ^B;
  - (b) Φ^B is closed under Boolean combinations;
  - (c) For every $\psi(\overline{x}, y)$ in Φ^B, $\theta(\overline{x}) := \exists y \psi(\overline{x}, y)$ is equivalent in $\mathbf{K}$ to some $\theta^*(\overline{x})$ in Φ^B.

Sufficient Conditions for an Elimination Set (Proof)

(a) This holds because of (i).

(b) Note that a Boolean combination of Boolean combinations of formulas from Φ is also a Boolean combination of formulas from Φ. Therefore, Φ^B is closed under Boolean combinations.

(c) Suppose $\psi(\overline{x}, y)$ is in Φ^B.

Then, taking disjunctive normal forms, $\psi(\overline{x}, y)$ is equivalent to $\bigvee_{i=1}^{n} \bigwedge_{j=1}^{k_i} \psi_{ij}(\overline{x}, y)$, for some ψ_{ij} in $\Phi \cup \Phi^{-}$.

Thus, $\theta(\overline{x}) := \exists y \psi(\overline{x}, y)$ is equivalent to $\bigvee_{i=1}^{n} \exists y \bigwedge_{j=1}^{k_i} \psi_{ij}(\overline{x}, y)$.

By hypothesis, for all i, there exists $\theta_i^*(\overline{x})$ in Φ^B equivalent to $\exists y \bigwedge_{j=1}^{k_i} \psi_{ij}(\overline{x}, y)$ in $\mathbf{K}$.

Therefore, $\theta(\overline{x})$ is equivalent in $\mathbf{K}$ to $\bigvee_{i=1}^{n} \theta_i^*(\overline{x})$ in Φ^B.

This proves Condition (c).

Exploiting the Lemma

- To find an elimination set, we must discover a way of getting rid of the quantifier $\exists y$ in $\exists y \bigwedge_{i<n} \psi_i(\overline{x}, y)$.

  Hence the name "quantifier elimination".
- We start with an arbitrary finite subset $\Theta(y, \overline{x})$ of $\Phi \cup \Phi^-$.

  We aim to find a boolean combination $\psi(\overline{x})$ of formulas in Φ so that $\exists y \bigwedge \Theta(y, \overline{x})$ is equivalent to ψ modulo T.
  - Typically the move from Θ to ψ takes several steps, depending on what kinds of formulas appear in Θ.
  - If we run into a dead end, we can add sentences to T and formulas to Φ until the process moves again.