Introduction to Model Theory

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Subsection 1

Theorems of Skolem

Skolem's Theorem

- Thoralf Skolem proved that for every infinite structure B of countable signature there is a countable substructure of B which is elementarily equivalent to B.
- We describe the quickest way to prove Skolem's result.
- Let B be any infinite structure with countable signature.
  By a previous theorem, we can build a chain $(A_n : n < \omega)$ of countable substructures of B, such that:

  > For each first-order formula $\phi(y, \overline{x})$, each $n < \omega$ and each tuple $\overline{a}$ of elements of A_n, such that $B \models \exists y \phi(y, \overline{a})$, if there is b in B, such that $B \models \phi(b, \overline{a})$, then there is such an element b in A_{n+1}.

  Put $A = \bigcup_{n<\omega} A_n$. Clearly A is countable. Also A is an elementary substructure of B by the Tarski-Vaught Criterion. So $A \equiv B$.
- Skolem proceeded differently: He added functions to B in such a way that every substructure of B which is closed under these functions is automatically an elementary substructure. Then took a generated substructure. The added functions are called **Skolem functions**.

Skolemization

- Suppose T is a theory in a first-order language L.
- A **skolemization** of T is a theory $T^+ \supseteq T$ in a first-order language $L^+ \supseteq L$, such that:
  1. Every L-structure which is a model of T can be expanded to a model of T^+;
  2. For every formula $\phi(\overline{x}, y)$ of L^+, with $\overline{x}$ non-empty, there is a term t of L^+, such that T^+ entails the sentence
$$\forall \overline{x}(\exists y \phi(\overline{x}, y) \to \phi(\overline{x}, t(\overline{x}))).$$
- The terms t of Clause 2 and the functions which they define in models of T^+ are called **Skolem functions** for T^+

Skolem Theories and the Skolem Hull

- We say that T **has Skolem functions** or that T is a **Skolem theory** if T is a skolemization of itself.
- Thus, T is a Skolem theory if Clause 2 holds with $L = L^+$ and $T = T^+$.
- Note that if T^+ is a skolemization of T, then T^+ has Skolem functions.
- Note also that these notions depend on the language:
  - If $L \subseteq L'$ and T is a Skolem theory in L, T will generally not be a Skolem theory in L'.
  - If T has Skolem functions and T' is a theory with $T \subseteq T'$, both in the first-order language L, then it's immediate that T' has Skolem functions too.
- Suppose T is a theory which has Skolem functions, in a first-order language. Let A be an L-structure and X a set of elements of A.

  The **Skolem hull** of X is defined to be $\langle X \rangle_A$, the substructure of A generated by X.

Properties of Skolem Theories

Theorem

Let T be a theory in a first-order language L, which has Skolem functions.

(a) Modulo T, each formula $\phi(\overline{x})$ of L (with $\overline{x}$ not empty) is equivalent to a quantifier-free formula $\phi^*(\overline{x})$ of L.

(b) If A is an L-structure and a model of T, and X is a set of elements of A, such that the Skolem hull $\langle X \rangle_A$ is non-empty, then $\langle X \rangle_A$ is an elementary substructure of A.

(a) In Condition 2 the formula $\phi(\overline{x}, t(\overline{x}))$ logically implies $\exists y \phi(\overline{x}, y)$.

So the $\rightarrow$ can be replaced by $\leftrightarrow$.

Hence Part (a) follows from a previous lemma, applied to the set Φ of quantifier-free formulas.

Properties of Skolem Theories (Part (b))

(b) To prove Part (b), put $B = \langle X \rangle_A$.

Let $\phi(\overline{x}, y)$ be a formula of L.

Let $\overline{b}$ be a tuple of elements of B, such that $A \models \exists y \phi(\overline{b}, y)$.

By Condition 2, there is a term t, such that $A \models \phi(\overline{b}, t(\overline{b}))$.

But B is closed under the functions of L.

So the element $t^A(\overline{b})$ is in B.

By the Tarski-Vaught Criterion, B is an elementary substructure of A.

Skolemization Theorem

Theorem (Skolemization Theorem)

Let L be a first-order language. Then there are a first-order language $L^\Sigma \supseteq L$ and a set Σ of sentences of L^Σ such that:

(a) Every L-structure A can be expanded to a model A^Σ of Σ;

(b) Σ is a Skolem theory in L^Σ;

(c) $|L^\Sigma| = |L|$.

- For each formula $\chi(\overline{x}, y)$ of L (where $\overline{x}$ is not empty), introduce a new function symbol $F_{\chi,\overline{x}}$ of the same arity as $\overline{x}$.
  The language L' is L with these new function symbols added.
  The set $\Sigma(L)$ consists of all the sentences

$$\forall \overline{x}(\exists y \chi(\overline{x}, y) \to \chi(\overline{x}, F_{\chi,\overline{x}}(\overline{x}))).$$

Skolemization Theorem (Auxiliary Lemma)

Claim: Every L-structure A can be expanded to a model of $\Sigma(L)$.

If A is empty it is already a model of $\Sigma(L)$.

If A is not empty, we expand it to an L'-structure A' as follows.

Let $\chi(\overline{x}, y)$ be any formula of L with $\overline{x}$ non-empty.

Let $\overline{a}$ be a tuple of elements of A.

- If there is an element b such that $A \models \chi(\overline{a}, b)$, choose one such element b and put $F_{\chi,\overline{x}}^{A'}(\overline{a}) = b$ (here we generally need the axiom of choice).
- If there is no such element, let $F_{\chi,\overline{x}}^{A'}(\overline{a})$ be, say, the first element in $\overline{a}$.

Then A' is a model of all the sentences in $\Sigma(L)$.

Skolemization Theorem (Cont'd)

- The theory Σ is built by iterating the construction of $\Sigma(L)$ ω times.

  We define, by induction on n,
  - A chain of languages $(L_n : n < \omega)$;
  - A chain of theories $(L_n : n < \omega)$.

  The construction proceeds as follows.
  - $L_0 = L$ and Σ_0 is the empty theory;
  - L_{n+1} is $(L_n)'$ and $\Sigma_{n+1} = \Sigma_n \cup \Sigma(L_n)$;

  Finally, set $L^\Sigma = \bigcup_{n<\omega} L_n$ and $\Sigma = \bigcup_{n<\omega} \Sigma_n$.

  Part (a) is true by making repeated expansions as in the claim.

  Part (b) holds also. Every formula χ of L^Σ lies in some Σ_n. So the required sentence is, by construction, in Σ_{n+1}.

  Part (c) is clear.

Skolemization Preserving the Cardinality of the Language

Corollary

Let T be a theory in a first-order language L. Then T has a skolemization T^+ in a first-order language L^+ with $|L^+| = |L|$.

- Define:
  - $L^+ = L^\Sigma$;
  - $T^+ = T \cup \Sigma$.
- Note that Σ is a skolemization of the empty theory in L.

Downward Löwenheim-Skolem Theorem

Corollary (Downward Löwenheim-Skolem Theorem)

Let L be a first-order language, A an L-structure, X a set of elements of A, and λ a cardinal such that $|L|+|X|\leq\lambda\leq|A|$. Then A has an elementary substructure B of cardinality λ with $X\subseteq\operatorname{dom}(B)$.

- Expand A to a model A^Σ of Σ in L^Σ.

  Let Y be a set of λ elements of B, with $X\subseteq Y$.

  Let B' be the Skolem hull $\langle Y\rangle_A$.

  Let B be the reduct $B'|_L$.

  By a previous theorem,

  $$|B|\leq|Y|+|L^\Sigma|=\lambda+|L|=\lambda=|Y|\leq|B|.$$

  Since Σ is a Skolem theory, by a previous theorem, $B'\preccurlyeq A^\Sigma$.

  Hence, $B\preccurlyeq A$.

Example: Simple Subgroups of Simple Groups

- Let G be an infinite simple group. We show that for every infinite cardinal $\lambda \leq |G|$, G has a subgroup of cardinality λ which is simple.

  The language of groups is countable.

  By the Downward Löwenheim-Skolem Theorem, G has an elementary sub-structure H of cardinality λ.
  - Clearly H is a subgroup of G.
  - To show that H is simple it suffices to prove that if a, b are two elements of H and $b \neq 1$, then a is in the normal subgroup of H generated by b.
    Since G is simple, this is certainly true with G in place of H.
    Suppose, for example, that

    $$G \vDash \exists y \exists z (a = y^{-1} b y \cdot z^{-1} b^{-1} z).$$

    Since $H \preccurlyeq G$, the same sentence is true in H.
    Hence, there are c, d in H such that $a = c^{-1} bc \cdot d^{-1} b^{-1} d$.

Subsection 2

Back-and-forth Equivalence

Isomorphism and Elementary Equivalence

- Comparing the two relations $\cong$ (isomorphism) and $\equiv$ (elementary equivalence) between structures, we see that:
  - In one sense isomorphism is a more intrinsic property of structures, because it is defined directly in terms of structural properties, whereas $\equiv$ involves a language.
  - In another sense elementary equivalence is more intrinsic, because the existence of an isomorphism can depend on some subtle questions about the surrounding universe of sets.
- In the early 1950s, Roland Fraïssé discovered a family of equivalence relations which hover somewhere between $\cong$ and $\equiv$.
  - His equivalences are purely structural - there are no languages involved.
  - Moreover, they are independent of the surrounding universe of sets.

  The trick is to look at isomorphisms, but only between a finite number of elements at a time.
- Fraïssé's equivalence relations sometimes provide:
  - A way of proving that two structures are elementarily equivalent;
  - Proofs of isomorphism.

Playing a Game on Structures

- Let L be a signature and let A and B be L-structures.
- We imagine two people (or players), called $\forall$ and $\exists$, who are comparing these structures.
  - $\forall$ wants to prove that A is different from B.
  - $\exists$ tries to show that A is the same as B.
- Their conversation has the form of a game.
- Player $\forall$ wins if he manages to find a difference between A and B before the game finishes.
- Otherwise player $\exists$ wins.

Rules of the Game on Structures

- An ordinal γ (usually ω or finite) is given, which is the length of the game, i.e., the game is played in γ steps.
  - At the ith step of a play, player $\forall$ picks one of the structures A, B and chooses an element of this structure.
  - Then player $\exists$ chooses an element of the other structure.
  - So an element a_i of A and an element b_i of B are chosen.
- Apart from the fact that player $\exists$ must choose from the other structure from player $\forall$ at each step, both players have complete freedom to choose as they please (including elements which were chosen at an earlier step).
- At the end of the play, sequences $\overline{a} = (a_i : i < \gamma)$ and $\overline{b} = (b_i : i < \gamma)$ have been chosen. The pair $(\overline{a}, \overline{b})$ is known as the **play**.
- The play $(\overline{a}, \overline{b})$ is a **win for player** $\exists$, and we say that player $\exists$ **wins the play**, if there is an isomorphism $f : \langle a \rangle_A \to \langle b \rangle_B$, with $f(\overline{a}) = \overline{b}$.
- A play is a **win for player** $\forall$ if it is not a win for player $\exists$.

Example: Rationals versus Integers

- Suppose $\gamma \geq 2$.
- Let A be the additive group $\mathbb{Q}$ of rational numbers.
- Let B be the additive group $\mathbb{Z}$ of integers.
- Player $\forall$ can win by playing as follows:
  - He chooses a_0 to be any non-zero element of $\mathbb{Q}$.
  - Then player $\exists$ must choose b_0 to be a non-zero integer. Otherwise, she loses the game at once.
  - Now there is some integer n which does not divide b_0 in $\mathbb{Z}$. Player $\forall$ chooses a_1 in $\mathbb{Q}$ so that $na_1 = a_0$.
  - Player $\exists$ cannot choose b_1 in $\mathbb{Z}$ so that $nb_1 = b_0$.
- It follows that, if $\gamma \geq 2$, player $\forall$ can always arrange to win the game on $\mathbb{Q}$ and $\mathbb{Z}$.

Characterizing Winning for ∃

- We write $A \equiv_0 B$ to mean that, for every atomic sentence ϕ of L,

$$A \models \phi \quad \text{iff} \quad B \models \phi.$$

- We may replace "atomic" by "quantifier-free" without change.

Proposition

Player ∃ wins the play $(\overline{a}, \overline{b})$ if and only if $(A, \overline{a}) \equiv_0 (B, \overline{b})$.

- This is equivalent to the definition of a win for ∃ by a previous theorem.

Ehrenfeucht-Fraïssé game of length γ on A and B

- The game just described is called the **Ehrenfeucht-Fraïssé game of length γ on A and B**, in symbols $\mathrm{EF}_\gamma(A, B)$.
- The more A is like B, the better chance player $\exists$ has of winning.
- If player $\exists$ knows an isomorphism $i : A \to B$, then she can be sure of winning every time.
- All she has to do is choose:
  - $i(a)$ whenever player $\forall$ has just chosen an element a of A;
  - $i^{-1}(b)$ whenever player $\forall$ has just chosen an element b from B.

Winning Strategies

- A **strategy** for a player in a game is a set of rules which tell the player exactly how to move, depending on what has happened earlier in the play.
- We say that the player **uses the strategy** σ in a play if each of his or her moves in the play obeys the rules of σ.
- We say that the strategy σ is a **winning strategy** if the player wins every play in which he or she uses σ.
- We write $A \sim_\gamma B$ to mean that player $\exists$ has a winning strategy in the game $\mathrm{EF}_\gamma(A, B)$.
- We stipulate that for any positive ordinal γ, if at least one of A, B is empty, then $A \sim_\gamma B$ if and only if both are empty.

Example

- Consider a play over structures A and B.
- Suppose $\exists$ knows an isomorphism $i : A \to B$.
- Let σ be the strategy consisting of the rules:
  - Choose $i(a)$ whenever player $\forall$ has just chosen an element a of A.
  - Choose $i^{-1}(b)$ whenever player $\forall$ has just chosen an element b of B.
- This is a winning strategy for player $\exists$.
- On the other hand, we showed that $\exists$ does not have a winning strategy for the 2-round game played on the additive groups $\mathbb{Q}$ and $\mathbb{Z}$.

  Therefore, $\mathbb{Q} \not\sim_2 \mathbb{Z}$.

Properties of Winning Strategies

Lemma

Let L be a signature and let A, B be L-structures.

(a) If $A \cong B$, then $A \sim_\gamma B$, for all ordinals γ.

(b) If $\beta < \gamma$, and $A \sim_\gamma B$, then $A \sim_\beta B$.

(c) If $A \sim_\gamma B$ and $B \sim_\gamma C$, then $A \sim_\gamma C$; in fact $\sim_\gamma$ is an equivalence relation on the class of L-structures.

(a) This has already been proven.

(b) This is straightforward.

(c) It is clear from the definition that $\sim_\gamma$ is reflexive and symmetric on the class of L-structures. We prove transitivity.
Suppose $A \sim_\gamma B$ and $B \sim_\gamma C$. Then player $\exists$ has winning strategies σ and τ for $\text{EF}_\gamma(A, B)$ and $\text{EF}_\gamma(B, C)$, respectively.
Suppose the two players sit down to a match of $\text{EF}_\gamma(A, C)$.
We have to find a winning strategy for player $\exists$.

Properties of Winning Strategies (Cont'd)

- To respond in $\text{EF}_\gamma(A, C)$, $\exists$ will be playing two "private" games $\text{EF}_\gamma(A, B)$ and $\text{EF}_\gamma(B, C)$ on the side.
  - Suppose $\forall$ chooses a_i from A in $\text{EF}_\gamma(A, C)$.
    - $\exists$ uses σ to pick b_i from B in $\text{EF}_\gamma(A, B)$;
    - Assuming that b_i was chosen by $\forall$ in $\text{EF}_\gamma(B, C)$, $\exists$ uses τ to pick c_i from C in $\text{EF}_\gamma(B, C)$.

    $\exists$ chooses c_i from C as a response to a_i in $\text{EF}_\gamma(A, C)$.
  - Suppose $\forall$ chooses c_i from C in $\text{EF}_\gamma(A, C)$.
    - $\exists$ uses τ to pick b_i from B in $\text{EF}_\gamma(B, C)$;
    - Assuming that b_i was chosen by $\forall$ in $\text{EF}_\gamma(A, B)$, $\exists$ uses σ to pick a_i from A in $\text{EF}_\gamma(A, B)$.

    $\exists$ chooses a_i from A as a response to c_i in $\text{EF}_\gamma(A, C)$.

Properties of Winning Strategies (Conclusion)

- At the end of the contest, the players have constructed sequences
  - $\overline{a}$ from A;
  - $\overline{b}$ from B;
  - $\overline{c}$ from C.

  The play of the public game $\mathrm{EF}_\gamma(A,C)$ is $(\overline{a},\overline{c})$.
  - In the private game $\mathrm{EF}_\gamma(A,B)$, player $\exists$ used her winning strategy σ. So the play $(\overline{a},\overline{b})$ is a win for $\exists$.
  - In the private game $\mathrm{EF}_\gamma(B,C)$, player $\exists$ used her winning strategy τ. So the play $(\overline{b},\overline{c})$ is a win for $\exists$.

  Thus, we get $(A,\overline{a}) \equiv_0 (B,\overline{b}) \equiv_0 (C,\overline{c})$. Hence, $(A,\overline{a}) \equiv_0 (C,\overline{c})$.

  This shows that $(\overline{a},\overline{c})$ is a win for $\exists$ in $\mathrm{EF}_\gamma(A,C)$.

  Hence, the strategy described for $\exists$ is a winning strategy.

  Thus, $A \sim_\gamma B$.

Back-and-Forth Equivalence and Systems

- Two L-structures A and B are said to be **back-and-forth equivalent** if $A \sim_\omega B$, i.e. if player $\exists$ has a winning strategy for $\mathrm{EF}_\omega(A,B)$.
- A **back-and-forth system** from A to B is a set I of pairs $(\overline{a},\overline{b})$ of tuples, with $\overline{a}$ from A and $\overline{b}$ from B, such that:
  1. I is not empty;
  2. If $(\overline{a},\overline{b})$ is in I, then $\overline{a}$ and $\overline{b}$ have the same length and $(A,\overline{a}) \equiv_0 (B,\overline{b})$;
  3. For every pair $(\overline{a},\overline{b})$ in I and every element c of A, there is an element d of B, such that the pair $(\overline{a}c,\overline{b}d)$ is in I;
  4. For every pair $(\overline{a},\overline{b})$ in I and every element d of B, there is an element c of A, such that the pair $(\overline{a}c,\overline{b}d)$ is in I.
- By Condition 1 and a previous theorem, if $(\overline{a},\overline{b})$ is in I, then there is an isomorphism $f : \langle\overline{a}\rangle_A \to \langle\overline{b}\rangle_B$, such that $f(\overline{a}) = \overline{b}$.

  In fact, f is unique since $\overline{a}$ generates $\langle\overline{a}\rangle_A$.
- I^* denotes the set of all such functions f corresponding to pairs of tuples in I.

Properties of I^*

- Conditions 1-4 imply some similar conditions on the set $J = I^*$:
  1. J is not empty;
  2. Each $f \in J$ is an isomorphism from a finitely generated substructure of A to a finitely generated substructure of B;
  3. For every $f \in J$ and c in A there is $g \supseteq f$, such that $g \in J$ and $c \in \text{dom} g$;
  4. for every $f \in J$ and d in B, there is $g \supseteq f$, such that $g \in J$ and $d \in \text{im} g$.
- Conversely, if J is any set obeying these four conditions, then there is a back-and-forth system I, such that $J = I^*$.

  Take I to be the set of all pairs of tuples $(\overline{a}, \overline{b})$, such that:
  - $\overline{a}$ is from A;
  - $\overline{b}$ is from B;
  - J contains a map $f : \langle\overline{a}\rangle_A \to \langle\overline{b}\rangle_B$, such that $f(\overline{a}) = \overline{b}$.

Characterization of Back-and-Forth Equivalence

Lemma

Let L be a signature and let A, B be L-structures. Then A and B are back-and-forth equivalent if and only if there is a back-and-forth system from A to B.

- Suppose first that A is back-and-forth equivalent to B.
  Then player $\exists$ has a winning strategy σ for the game $\mathrm{EF}_\omega(A, B)$.
  Let I consist of the pairs of tuples which are of the form $(\overline{c}\,|_n, \overline{d}\,|_n)$, for some $n < \omega$ and some play $(\overline{c}, \overline{d})$ in which player $\exists$ uses σ.
  Claim: The set I is a back-and-forth system from A to B.
  First, putting $n = 0$ in the definition of I, we see that I contains the pair of 0-tuples $(\langle\rangle, \langle\rangle)$. This establishes Property 1.
  Property 2 holds because the strategy σ is winning.
  Properties 3 and 4 express that σ tells player $\exists$ what to do at each step of the game.

Characterization of Back-and-Forth Equivalence (Converse)

- Suppose that there exists a back-and-forth system I from A to B.
  Define the set I^* of maps corresponding to I, as above.
  Choose an arbitrary well-ordering of I^*.
  Consider the following strategy σ for player $\exists$ in $\text{EF}_\omega(A, B)$.
  Suppose the play so far is $(\overline{a}, \overline{b})$.
  - Suppose player $\forall$ has just chosen an element c from A.
    Find the first map f in I^*, such that
    $\overline{a}$ and c are in the domain of f and $f(\overline{a}) = \overline{b}$.
    Choose d to be $f(c)$.
  - Suppose player $\forall$ has just chosen an element d from B.
    Find the first map f in I^*, such that
    $\overline{b}$ and d are in the image of f and $f(\overline{a}) = \overline{b}$.
    Choose c such that $f(c) = d$.

  By Properties 1, 3, 4, there will always be a map f in I^* as required.
  So the strategy is well-defined.
  Suppose the resulting play is $(\overline{a}, \overline{b})$. Then, by Property 2 and a previous theorem, we have $(A, \overline{a}) \equiv_0 (B, \overline{b})$. So player $\exists$ wins.

Example: Algebraically Closed Fields

- Let A and B be algebraically closed fields of the same characteristic and infinite transcendence degree. We show that A is back-and-forth equivalent to B.
- Recall that a **finitely generated subfield** of A is the smallest subfield of A containing some given finite set of elements of A.
- Note, also, that it need not be finitely generated as a ring.
- Let J be the set of all isomorphisms $e: A' \to B'$, where A', B' are finitely generated subfields of A, B, respectively.

  A and B have the same characteristic. Hence, the prime subfields of A, B are isomorphic. So J is not empty. Thus, Condition 1 is satisfied.

  J satisfies Condition 2 by construction.

  Suppose $f: A' \to B'$ is in J and c is an element of A.

  We want to find a matching element d in B.

  We need to distinguish two cases.

Example: Algebraically Closed Fields (Cont'd)

- First, suppose c is algebraic over A'. Then c is determined up to isomorphism over A' by its minimal polynomial $p(x)$ over A'.
  f carries $p(x)$ to a polynomial $fp(x)$ over B'.
  B contains a root d of $fp(x)$ since it is algebraically closed.
  Thus f extends to an isomorphism $g : A'(c) \to B'(d)$.
- Second, suppose c is transcendental over A'.
  B' is finitely generated and B has infinite transcendence degree.
  So there is an element d of B which is transcendental over B'.
  Thus again f extends to an isomorphism $g : A'(c) \to B'(d)$.

Either way, Condition 3 is satisfied.

By symmetry, Condition 4 is also satisfied.

So J defines a back-and-forth system from A to B.

By the lemma, A is back-and-forth equivalent to B.

Algebraically Closed Fields (Special Case)

- If $A \subseteq B$ in the example above, then we can say a little more.
- For every finitely generated subfield C of A, there is a system J as above, such that every map in J pointwise fixes C.
- In terms of back-and-forth systems, this says that if $\overline{e}$ is a tuple of elements which generate C, then there is a back-and-forth system I from A to B in which every pair has the form $(\overline{e}\,\overline{a}, \overline{e}\,\overline{b})$.

Winning Positions

- If two structures A and B are back-and-forth equivalent, they are in some sense hard to tell apart.
- A **position of length** n in a play of the back-and-forth game $\mathrm{EF}_\gamma(A,B)$ is a pair $(\overline{c},\overline{d})$ of n-tuples, where:
  - $\overline{c}$ lists in order the elements of A chosen in the first n moves;
  - $\overline{d}$ lists in order the elements of B chosen in the first n moves.
- A **position** is a position of some finite length.
- The position is **winning** for one of the players if the player has a wining strategy that enables him to win in $\mathrm{EF}_\gamma(A,B)$ whenever the first n moves are $(\overline{c},\overline{d})$.
- It is not hard to see that $(\overline{c},\overline{d})$ is a winning position for a player if and only if that player has a winning strategy for $\mathrm{EF}_\gamma((A,\overline{c}),(B,\overline{d}))$.
- In particular, the starting position is winning for $\exists$ if and only if A and B are back-and-forth equivalent.

Back-and-Forth Equivalence for Countable Structures

Theorem

Let L be any signature (not necessarily countable) and let A and B be L-structures.

(a) If $A \cong B$, then A is back-and-forth equivalent to B.

(b) Suppose A, B are at most countable. If A is back-and-forth equivalent to B, then $A \cong B$. In fact, if $\overline{c}, \overline{d}$ are tuples from A, B, respectively, such that $(\overline{c}, \overline{d})$ is a winning position for player $\exists$ in $\mathrm{EF}_\omega(A, B)$, then there is an isomorphism from A to B which takes $\overline{c}$ to $\overline{d}$.

(a) This is a special case of a preceding lemma.

(b) Suppose A and B are at most countable. The game $\mathrm{EF}_\omega(A, B)$ has infinite length. So player $\forall$ can list all the elements of A and of B among his choices. Let player $\exists$ play to win. Let $(\overline{a}, \overline{b})$ be the resulting play. By the Diagram Lemma $A = \langle a \rangle_A \overset{f}{\cong} \langle b \rangle_B = B$.
The last sentence is similar, but starting the play at $(\overline{c}, \overline{d})$.

Example: Dense Linear Orderings without Endpoints

- An old theorem of Cantor states that if A and B are countable dense linear orderings without endpoints, then $A \cong B$.
- This follows at once from Part (b) of the theorem, when we show that A is back-and-forth equivalent to B.
- The required back-and-forth system consists of all pairs of tuples $(\overline{a}, \overline{b})$ such that:
  - For some $n < \omega$, $\overline{a} = (a_0, \ldots, a_{n-1})$ is a tuple of elements of A;
  - $\overline{b} = (b_0, \ldots, b_{n-1})$ is a tuple of elements of B;
  - For all $i < j < n$,
  $$a_i \gtrless a_j \quad \text{iff} \quad b_i \gtrless b_j.$$

Example: Atomless Boolean Algebras

- Let A and B be countable atomless boolean algebras. Then $A \cong B$.
  Again we show this by Part (b) of the theorem. Let J be the set of all isomorphisms from finite subalgebras of A to finite subalgebras of B. Then Conditions 1 and 2 clearly hold.
  For Condition 3, suppose $f \in J$ and let $a_0, \ldots, a_{k-1}$ be the atoms of the boolean algebra A' which is the domain of f. Then the isomorphism type of any element c of A over A' is determined once we are told, for each $i < k$, whether $c \wedge a_i$ is 0, a_i or neither. Since B is atomless, there is an element d of B, such that, for each $i < k$,
  $$d \wedge f(a_i) \text{ is } 0 \text{ (resp. } f(a_i)) \quad \text{iff} \quad c \wedge a_i \text{ is } 0 \text{ (resp. } a_i).$$
  So f can be extended to an isomorphism whose domain includes c.
  Thus J satisfies Condition 3.
  By symmetry, it also satisfies Condition 4.
  We have proved that $A \cong B$.

The Case of Uncountable Structures

- The results of Examples 3 and 4 are as false as they possibly could be when we replace "countable" by an uncountable cardinal κ.
- We will see that there are:
  - 2^κ non-isomorphic dense linear orderings of cardinality κ;
  - 2^κ non-isomorphic atomless boolean algebras of cardinality κ.

Back-and-Forth Equivalence for Uncountable Structures

Theorem

Let A, B be L-structures and $\overline{a}, \overline{b}$ n-tuples from A, B, respectively. If $(\overline{a}, \overline{b})$ is a winning position for player exists in $\text{EF}_\omega(A, B)$, then $(A, \overline{a}) \equiv_{\infty,\omega} (B, \overline{b})$. In particular, if A and B are back-and-forth equivalent, then they are $L_{\infty\omega}$-equivalent.

- We show that, if $\phi(\overline{x})$ is any formula of $L_{\infty\omega}$ and $(\overline{a}, \overline{b})$ is a winning position for $\exists$, then

$$A \models \phi(\overline{a}) \quad \text{iff} \quad B \models \phi(\overline{b}).$$

  By induction on the structure of ϕ.
  - If ϕ is atomic, the result follows by the Diagram Lemma and the definition of winning.
  - If ϕ is of the form $\neg\psi$, $\bigwedge\Phi$ or $\bigvee\Phi$, the the result follows easily by the induction hypothesis.

Back-and-Forth for Uncountable Structures (Cont'd)

- Let ϕ is $\exists y \psi(\overline{x}, y)$. Suppose $A \models \phi(\overline{a})$. Then, there is a c in A, such that $A \models \psi(\overline{a}, c)$. But $(\overline{a}, \overline{b})$ is winning for $\exists$. So she has a winning strategy from this position onward. This strategy gives $d \in B$, if player $\forall$ chooses c in his next move. So $(\overline{a}c, \overline{b}d)$ must still be a winning position for player $\exists$. By the induction hypothesis, $B \models \psi(\overline{b}, d)$. So $B \models \exists y \psi(\overline{x}, y)$.

  The other direction is handled similarly.
- If ϕ is $\forall y \psi$, we reduce it to the previous case by writing $\neg\exists\neg$ for $\forall y$.

- It is a little harder to prove, but the reverse is also true.

  If A and B are $L_{\infty\omega}$-equivalent, then they are back-and-forth equivalent.
- This shows that back-and-forth equivalence is not a good criterion for elementary equivalence because it proves too much.

Scott's Isomorphism Theorem

- We close by providing an important result without proof.

Theorem (Scott's Isomorphism Theorem)

Let L be a countable signature and B a countable L-structure.
Then, there is a sentence σ_B of $L_{\omega_1\omega}$ such that the models of σ_B are exactly the L-structures which are back-and-forth equivalent to B.
In particular, B is up to isomorphism the only countable model of σ_B.

- A sentence σ_B as in the theorem is called a **Scott sentence** of B.
- No satisfactory analog of the theorem is known for uncountable cardinalities.

Subsection 3

Games for Elementary Equivalence

Unnested Formulas

- Let L be a signature.
- Recall that by an **unnested atomic formula** of signature L we mean an atomic formula of one of the following forms:
  1. $x = y$;
  2. $c = y$, for some constant c of L;
  3. $F(\overline{x}) = y$, for some function symbol F of L;
  4. $R\overline{x}$, for some relation symbol R of L.
- Recall, also, that we call a formula **unnested** if all of its atomic subformulas are unnested.

Unnested Ehrenfeucht-Fraïssé Games

- We consider a game $\mathrm{EF}_k[A,B]$, played exactly like $\mathrm{EF}_k(A,B)$ but with a different criterion for winning.
  - The players between them make k pairs of choices;
  - At the end of the play, tuples $\overline{c}$ from A and $\overline{d}$ from B have been chosen.
  - Player $\exists$ wins the game $\mathrm{EF}_k[A,B]$ if, for every unnested atomic formula ϕ of L,
$$A \models \phi(\overline{c}) \quad \text{iff} \quad B \models \phi(\overline{d}).$$
- If the signature L contains no function symbols or constants, then every formula of L is unnested and $\mathrm{EF}_k(A,B)$ and $\mathrm{EF}_k[A,B]$ coincide.

  Example: This is the case, e.g., with linear orderings.
- The games $\mathrm{EF}_k[A,B]$ are called **unnested Ehrenfeucht-Fraïssé games**.
- We write $A \approx_k B$ to mean that player $\exists$ has a winning strategy for the game $\mathrm{EF}_k[A,B]$.
- $\approx_k$ is an equivalence relation on the class of L-structures.

Allowing Parameters

- We allow the structures to carry some parameters with them.
- Suppose $n < \omega$ and $\overline{a}$, $\overline{b}$ are n-tuples of elements of A, B, respectively.
- We write $(A, \overline{a}) \approx_k (B, \overline{b})$ to mean that player $\exists$ has a winning strategy for the game $\mathrm{EF}_k[(A, \overline{a}), (B, \overline{b})]$.
- The condition for player $\exists$ to win this game, when the play has chosen k-tuples $\overline{c}, \overline{d}$ from A, B, respectively, is that for every unnested atomic formula ϕ of L,
$$A \models \phi(\overline{a}, \overline{c}) \quad \text{iff} \quad B \models \phi(\overline{b}, \overline{d}).$$
- This is a restatement of the original condition with $(A, \overline{a})$ and $(B, \overline{b})$ in place of A and B.

Winning in Unnested Ehrenfeucht-Fraïssé Games

Lemma

Let A and B be structures of the same signature. Suppose $n, k < \omega$. Suppose $\overline{a}$, $\overline{b}$ are n-tuples of elements of A, B, respectively. Then the following are equivalent:

(a) $(A, \overline{a}) \approx_{k+1} (B, \overline{b})$.

(b) For every c in A, there is d in B, such that $(A, \overline{a}, c) \approx_k (B, \overline{b}, d)$;
For every d in B, there is c in A, such that $(A, \overline{a}, c) \approx_k (B, \overline{b}, d)$.

- First suppose (a) holds. Let c be an element of A.
  - $\exists$ views c as player $\forall$'s first choice in a play of $\text{EF}_{k+1}[(A, \overline{a}), (B, \overline{b})]$. She uses her winning strategy σ to choose d as her reply to c.
  - In playing $\text{EF}_k[(A, \overline{a}, c), (B, \overline{b}, d)]$, $\exists$ can win by regarding the steps as the last k steps in the play of $\text{EF}_{k+1}[(A, \overline{a}), (B, \overline{b})]$, for which she has a winning strategy.

This proves the first half of (b). The second follows by symmetry.

Winning in Unnested Ehrenfeucht-Fraïssé Games (Converse)

- Conversely, suppose (b) holds.

  Then ∃ can win the game $\mathrm{EF}_{k+1}[(A,\overline{a}),(B,\overline{b})]$ as follows:
  - If ∀ opens by choosing c in A, then ∃ chooses d as in (b). If ∀ opens by choosing d in B, then ∃ chooses c as in (b).
  - For the rest of the game ∃ follows her winning strategy for $\mathrm{EF}_k[(A,\overline{a},c),(B,\overline{b},d)]$.

The Quantifier Rank of a Formula

- We will prove a fundamental theorem about the equivalence relations $\approx_k$ between structures of the form $(A, \overline{a})$ with A an L-structure.
- It will say among other things that:
  - For each k, there are just finitely many equivalence classes of $\approx_k$;
  - Each equivalence class is definable by a formula of L;
  - A bound exists on the complexity of these defining formulas, in terms of the notion of *quantifier rank*.
- For any formula ϕ of the first-order language L, we define the **quantifier rank** $\text{qr}(\phi)$ **of** ϕ by induction on the construction of ϕ.
  - If ϕ is atomic then $\text{qr}(\phi) = 0$;
  - $\text{qr}(\neg\psi) = \text{qr}(\psi)$;
  - $\text{qr}(\bigwedge \Phi) = \text{qr}(\bigvee \Phi) = \max\{\text{qr}(\psi) : \psi \in \Phi\}$;
  - $\text{qr}(\forall x\psi) = \text{qr}(\exists x\psi) = \text{qr}(\psi) + 1$.
- Thus, $\text{qr}(\phi)$ measures the nesting of quantifiers in ϕ.

Fraïssé-Hintikka Theorem

Theorem (Fraïssé-Hintikka Theorem)

Let L be a first-order language with finite signature. Then we can effectively find, for each $k, n < \omega$, a finite set $\Theta_{n,k}$ of unnested formulas $\theta(x_0, \ldots, x_{n-1})$ of quantifier rank at most k, such that:

(a) For every L-structure A, all $k, n < \omega$ and each n-tuple $\overline{a} = (a_0, \ldots, a_{n-1})$ of elements of A, there is exactly one formula θ in $\Theta_{n,k}$, such that $A \models \theta(\overline{a})$.

(b) For all $k, n < \omega$ and every pair of L-structures A, B, if $\overline{a}$ and $\overline{b}$ are respectively n-tuples of elements of A and B, then $(A, \overline{a}) \approx_k (B, \overline{b})$ if and only if there is θ in $\Theta_{n,k}$, such that $A \models \theta(\overline{a})$ and $B \models \theta(\overline{b})$.

(c) For every $k < \omega$ and every unnested formula $\phi(\overline{x})$ of L with n free variables $\overline{x}$ and quantifier rank at most k, we can effectively find a disjunction $\theta_0 \vee \cdots \vee \theta_{m-1}$ of formulas $\theta_i(\overline{x})$ in $\Theta_{n,k}$ which is logically equivalent to ϕ.

Fraïssé-Hintikka Theorem (The Sets $\Theta_{n,k}$)

- We describe the sets $\Theta_{n,k}$.

  Write ϕ^1 for ϕ and ϕ^0 for $\neg\phi$.

  Let 2^m be the set of maps $s : m \to 2$, where $m = \{0,\ldots,m-1\}$ and $2 = \{0,1\}$.

  - Let $k = 0$ and $n < \omega$ fixed.
    There are finitely many unnested atomic formulas $\phi(x_0,\ldots,x_{n-1})$ of L.
    List them as $\phi_0,\ldots,\phi_{m-1}$.
    Define $\Theta_{n,0} = \left\{\phi_0^{s(0)} \wedge \cdots \wedge \phi_{m-1}^{s(m-1)} : s \in 2^m\right\}$.
    Thus, $\Theta_{n,0}$ lists all the possible unnested quantifier-free types of n-tuples of elements of an L-structure.
  - Suppose $\Theta_{n+1,k} = \{\chi_0(x_0,\ldots,x_n),\ldots,\chi_{j-1}(x_0,\ldots,x_n)\}$ has been defined.
  - Define $\Theta_{n,k+1} = \{\bigwedge_{i\in X} \exists x_n \chi_i(x_0,\ldots,x_n) \wedge \forall x_n \bigvee_{i\in X} \chi_i(x_0,\ldots,x_n) : X \subseteq j\}$.
    Thus, each formula in $\Theta_{n,k+1}$ lists the ways in which the n-tuple can be extended to an $(n+1)$-tuple, in terms of the formulas of quantifier rank k satisfied by the $(n+1)$-tuple.

Fraïssé-Hintikka Theorem (Proof of Property (b))

- We use induction on k, for all n simultaneously. Let A, B be L-structures and $\overline{a}, \overline{b}$ n-tuples of elements of A, B, respectively.
  - If $k = 0$, by the definition of $\approx_0$, $(A, \overline{a}) \approx_0 (B, \overline{b})$ iff, for every unnested atomic formula ϕ of L, $A \models \phi(\overline{a}) \Leftrightarrow B \models \phi(\overline{b})$. This holds iff $\overline{a}$ and $\overline{b}$ have the same unnested quantifier-free type in A and B, respectively, i.e., iff there is some $\theta \in \Theta_{n,0}$, such that $A \models \theta(\overline{a})$ and $B \models \theta(\overline{b})$. Clearly, this θ is unique.
  - Assume the result is proved for k. Let X be the set of i, such that A has an element c for which $A \models \chi_i(\overline{a}, c)$. For this choice of X, let $\theta'(x_0, \ldots, x_{n-1}) = \bigwedge_{i \in X} \exists x_n \chi_i(x_0, \ldots, x_n) \wedge \forall x_n \bigvee_{i \in X} \chi_i(x_0, \ldots, x_n) \in \Theta_{n,k+1}$. Certainly $A \models \theta'(\overline{a})$. Using Property (a) of the preceding lemma and the induction hypothesis, $(A, \overline{a}) \approx_{k+1} (B, \overline{b})$ means that:
    - For every $i \in X$, there is $d \in B$, such that $B \models \chi_i(\overline{b}, d)$;
    - For every $d \in B$, there is $i \in X$, such that $B \models \chi_i(\overline{b}, d)$.

    In short it means that $B \models \theta'(\overline{b})$.

Formulas in Game-Normal Form

- The formulas in the sets $\Theta_{n,k}$ are called **formulas in game normal form**, or more briefly **game normal formulas**.
- By Part (c) of the theorem, every first-order formula ϕ is logically equivalent to a disjunction of formulas in game normal form with at most the same free variables as ϕ.
  - If ϕ was unnested, the game normal formulas can be chosen to be of the same quantifier rank as ϕ.
  - However, the process of reducing a formula to unnested form (given previously) will generally raise the quantifier rank.

Equivalence and Unnested Equivalence of Structures

Corollary

Let L be a first-order language of finite signature. For any two L-structures A and B, the following are equivalent:

(a) $A \equiv B$.

(b) For every $k < \omega$, $A \approx_k B$.

- By the theorem, (b) says that A and B agree on all unnested sentences of finite quantifier rank. So (a) certainly implies (b).

  By a previous corollary, every first-order sentence is logically equivalent to an unnested sentence of finite quantifier rank. So (b) implies (a).

Graded Back-and-Forth Systems

- Suppose **K** is a class of L-structures.
- For each structure A in **K**, write tup(A) for the set of all pairs $(A, \overline{a})$, where $\overline{a}$ is a tuple of elements of A.
- Write tup(**K**) for the union of the sets tup(A) with A in **K**.
- By an (**unnested**) **graded back-and-forth system** for **K** we mean a family of equivalence relations $(E_k : k < \omega)$ on tup(**K**), that satisfy the following properties:
  1. If $\overline{a}, \overline{b}$ are in tup(A), tup(B), respectively, and $\overline{a}\ E_0\ \overline{b}$, then, for every unnested atomic formula $\phi(\overline{x})$ of L, $A \models \phi(\overline{a})$ iff $B \models \phi(\overline{b})$;
  2. if $\overline{a}, \overline{b}$ are in tup(A), tup(B), respectively, $\overline{a}\ E_{k+1}\ \overline{b}$ and c is any element of A, then exists an element d of B, such that $\overline{a}c\ E_k\ \overline{b}d$.

Graded Back-and-Forth Systems and Unnested Equivalence

Lemma

Suppose $(E_k : k < \omega)$ is a graded back-and-forth system for $\mathbf{K}$. Then

$$(A, \overline{a})\ E_k\ (B, \overline{b}) \quad \text{implies} \quad (A, \overline{a}) \approx_k (B, \overline{b}).$$

- By Condition 2, player $\exists$ can choose so that:
  - After the 0-th step in $\text{EF}_k[(A, \overline{a}), (B, \overline{b})]$, we have $\overline{a}c_0\ E_{k-1}\ \overline{b}d_0$;
  - After the 1-st step, we have $\overline{a}c_0c_1\ E_{k-2}\ \overline{b}d_0d_1$;

    $\vdots$
  - After k steps, $\overline{ac}\ E_0\ \overline{bd}$.

  But then player $\exists$ wins by Condition 1.

Graded Back-and-Forth Systems and Elimination Sets

Lemma

Suppose $(E_k : k < \omega)$ is a graded back-and-forth system for **K**. Suppose that, for each n and k, E_k has just finitely many equivalence classes on n-tuples, and each of these classes is definable by a formula $\chi_{k,n}(\overline{x})$. Then the set of all formulas $\chi_{k,n}, k, n < \omega$, forms an elimination set for **K**.

- We have to show that each formula $\phi(\overline{x})$ of the language L is logically equivalent to a boolean combination of formulas $\chi_{k,n}(\overline{x})$, $k, n < \omega$.
  By a previous corollary, we can suppose that ϕ is unnested.
  So by Part (c) of the Fraïssé-Hintikka Theorem ϕ is logically equivalent to a boolean combination of game normal formulas $\theta(\overline{x})$.
  By the lemma, each equivalence class under $\approx_k$ is a union of equivalence classes of E_k.
  By Part (b) of the Fraïssé-Hintikka Theorem, each game normal formula $\phi(\overline{x})$ is equivalent to a disjunction of formulas $\chi_{k,n}(\overline{x})$.

Application: The Ordered Group of Integers

- Consider the ordered group of integers over the language L whose symbols are $+,-,0,1$ and $<$. The ordered group of integers forms an L- structure which we shall write as $\mathbb{Z}$. Our aim is to find an elimination set for $\mathrm{Th}(\mathbb{Z})$.

  Suppose $\overline{x}$ is $(x_0,\ldots,x_{n-1})$ and m is a positive integer.

  By an m-**term** $t(\overline{x})$ we shall mean a term $\sum_{i<m} s_i$, where each s_i is either 0 or 1 or x_j or $-x_j$ for some $j<n$.

  Let $\overline{a}=(a_0,\ldots,a_{n-1})$ and $\overline{b}=(b_0,\ldots,b_{n-1})$ be two n-tuples in $\mathbb{Z}$.

  We say that $\overline{a}$ is m-**equivalent** to $\overline{b}$ if, for every m-term $t(\overline{x})$, the following hold in $\mathbb{Z}$:

  1. $t(\overline{a})>0$ if and only if $t(\overline{b})>0$;
  2. $t(\overline{a})$ is congruent to $t(\overline{b})$ (mod q), for each integer q, $1\leq q\leq m$.

  Note that if $\overline{a}$ is m-equivalent to $\overline{b}$, then $\overline{a}$ is m'-equivalent to $\overline{b}$, for all $m'<m$.

3-Equivalence and Unnested Formulas

Lemma

Suppose $\overline{a}$ and $\overline{b}$ are n-tuples of elements of $\mathbb{Z}$ which are 3-equivalent. Then, for every unnested atomic formula $\phi(\overline{x})$ of L,

$$\mathbb{Z} \models \phi(\overline{a}) \quad \text{iff} \quad \mathbb{Z} \models \phi(\overline{b}).$$

- For example,

$$\begin{aligned} \mathbb{Z} \models a_0 + a_1 = a_2 \quad &\text{iff} \quad \mathbb{Z} \not\models (a_0 + a_1 - a_2 > 0 \vee -a_0 - a_1 + a_2 > 0) \\ &\text{iff} \quad \mathbb{Z} \not\models (b_0 + b_1 - b_2 > 0 \vee -b_0 - b_1 + b_2 > 0) \\ &\text{iff} \quad \mathbb{Z} \models b_0 + b_1 = b_2. \end{aligned}$$

m^{2m}-Equivalence and m-Equivalence

Lemma

Suppose m is a positive integer, and $\overline{a}$ and $\overline{b}$ are n-tuples of elements of $\mathbb{Z}$ which are m^{2m}-equivalent. Then, for every element c of $\mathbb{Z}$, there is an element d of $\mathbb{Z}$, such that the tuples $\overline{a}c, \overline{b}d$ are m-equivalent.

- Suppose $c \in \mathbb{Z}$. Consider all the true sentences of the form

$$t(\overline{a}) + ic \equiv j \pmod{q},$$

where $t(\overline{x})$ is an $(m-1)$-term, $0 < i < m$ and $j < q \leq m$. $\overline{a}$ and $\overline{b}$ are m^{2m}-equivalent. So $t(\overline{a})$ and $t(\overline{b})$ are certainly congruent modulo $m!$. Let α be the remainder when c is divided by $m!$. Let d be any element of $\mathbb{Z}$ congruent to α modulo $m!$. Then

$$t(\overline{b}) + id \equiv j \pmod{q}$$

whenever $t(\overline{a}) + ic \equiv j \pmod{q}$. This tells us how to find a d to take care of Condition(s) 2 in the definition.

m^{2m}-Equivalence and m-Equivalence (Condition 1)

- Turning to Condition 1, consider the set of all true statements of the forms
$$t(\overline{a}) + ic > 0, \quad t(\overline{a}) + ic \leq 0,$$
where $t(\overline{x})$ is an $(m-1)$-term and $0 < i < m$. After multiplying by suitable integers, we can bring these inequalities to the forms
$$t(\overline{a}) + m!c > 0, \quad t(\overline{a}) + m!c \leq 0,$$
where $t(\overline{x})$ is an $m!(m-1)$-term. Taking greatest and least values in the obvious way, we can reduce these to a condition of the form $-t_1(\overline{a}) < m!c \leq -t_2(\overline{a})$, together with a set of inequalities $\Phi(\overline{a})$ which do not mention c (possibly we reach a single inequality if $m!c$ is bounded only on one side). So there is a number x in $\mathbb{Z}$, such that $-t_1(\overline{a}) < x \leq -t_2(\overline{a})$, and x is congruent to $m!\alpha \pmod{(m!)^2}$.

m^{2m}-Equivalence and m-Equivalence (Cont'd)

- Now $-t_1(\overline{a})$ is at most an $m!(m-1)$-term. By assumption it is congruent modulo $(m!)^2$ to $-t_1(\overline{b})$. Similarly with $-t_2(\overline{a})$.

  Hence, there is also a number y in $\mathbb{Z}$, such that $-t_1(\overline{b}) < y \leq -t_2(\overline{b})$, and y is congruent to $m!\alpha$ (mod $(m!)^2$).

  Put $d = \frac{y}{m!}$. Then d is congruent to α modulo $m!$. We have:
  - $-t_1(\overline{b}) < m!d \leq -t_2(\overline{b})$;
  - The inequalities $\Phi(\overline{b})$ hold since they use at worst $m! \cdot 2(m-1)$-terms.

  Tracing backwards, we have all the corresponding

$$t(\overline{a}) + ic > 0 \quad \text{iff} \quad t(\overline{b}) + id > 0,$$

  where $t(\overline{x})$ is an $(m-1)$-term and $0 < i < m$.

  Thus, d serves for the lemma.

Elimination Set for Th($\mathbb{Z}$)

- We define $m_0, m_1, \ldots$ inductively by $m_0 = 3$, $m_{i+1} = m_i^{2m_i}$.
- We define the equivalence relations E_k by

$$(\mathbb{Z}, \overline{a})\ E_k\ (\mathbb{Z}, \overline{b}) \quad \text{if} \quad \overline{a} \text{ is } m_k\text{-equivalent to } \overline{b}.$$

- By previous lemmas, $(E_k : k < \omega)$ is a graded back-and-forth system for $\{\mathbb{Z}\}$.
- So by a previous lemma, we have an elimination set for Th($\mathbb{Z}$).
- The formulas in the elimination set are of two fairly simple forms.
  - An inequality;
  - A congruence to some fixed modulus.

Decidability of Th($\mathbb{Z}$)

Theorem

Th($\mathbb{Z}$) is decidable.

Claim: For any tuple $\overline{a}$ in $\mathbb{Z}$ and any $k < \omega$ we can compute a bound $\delta(\overline{a}, k)$, such that, for every c, there is d, with $|d| < \delta(\overline{a}, k)$, such that $(\mathbb{Z}, \overline{a}c) E_k (\mathbb{Z}, \overline{a}d)$.

By the proof of the preceding lemma, $\delta(\overline{a}, k)$ can be chosen to be $m^{2m} \cdot \mu$, where m is m_k and μ is $\max\{|a_i| : a_i \text{ occurs in } \overline{a}\}$.

It follows by induction on k that if $\phi(\overline{x})$ is a formula of L of quantifier rank k, and $\overline{a}$ is a tuple of elements of $\mathbb{Z}$, then we can compute in a bounded number of steps whether or not $\mathbb{Z} \vdash \phi(\overline{a})$.

Decidability of Th($\mathbb{Z}$) (Cont'd)

- Suppose, e.g., that ϕ is $\exists y \psi(\overline{x}, y)$, where ψ has quantifier rank $k-1$.

  If there is an element c, such that $\mathbb{Z} \models \psi(\overline{a}, c)$, then there is such an element $c < \delta(\overline{a}, k-1)$.

  So we only need check the truth of $\mathbb{Z} \models \psi(\overline{a}, c)$ for finitely many c.

  By the induction hypothesis, this takes only a finite number of steps.

Application: Replacements Preserving $\equiv$

Theorem

Let G_1, G_2 and H be groups. Assume $G_1 \equiv G_2$. Then $G_1 \times H \equiv G_2 \times H$.

- By a previous corollary, it suffices to show that, if $k < \omega$ and $G_1 \approx_k G_2$, then $G_1 \times H \approx_k G_2 \times H$. Assume, henceforth, that $G_1 \approx_k G_2$.

  Then player $\exists$ has a winning strategy σ for the game $\text{EF}_k[G_1, G_2]$.

  Let the two players meet to play the game $\text{EF}_k[G_1 \times H, G_2 \times H]$.

  Player $\exists$ will guide her choices by playing another game on the side.

  The side game will in fact be $\text{EF}_k[G_1, G_2]$.

  - Suppose player $\forall$ offers an element, say the element $a \in G_1 \times H$;
  - Player $\exists$ first splits it into a product $a = g \cdot h$, with $g \in G_1$ and $h \in H$;
  - Pretending g was the choice of $\forall$ in the side game, $\exists$ uses her strategy σ to choose a reply $g' \in G_2$ in the side game;
  - Her public reply to the element a is the element $b = g' \cdot h \in G_2 \times H$.

  Similarly, if player $\forall$ chooses from $G_2 \times H$.

Elementary Equivalence and Products of Groups

- At the end let the play be $(g_0 \cdot h_0, \ldots, g_{k-1} \cdot h_{k-1}; g_0' \cdot h_0, \ldots, g_{k-1}' \cdot h_{k-1})$. Player $\exists$ has won the side game. The unnested atomic formulas of the language L of groups are of the form $x = y$, $1 = y$, $x_0 \cdot x_1 = y$ and $x^{-1} = y$.
  So for all $i, j, \ell < k$, we have
  $$g_i = g_j \text{ iff } g_i' = g_j', \qquad 1 = g_i \text{ iff } 1 = g_i',$$
  $$g_i \cdot g_j = g_\ell \text{ iff } g_i' \cdot g_j' = g_\ell', \quad g_i^{-1} = g_j \text{ iff } g_i'^{-1} = g_j'.$$
  By the definition of cartesian products, for all $i, j, \ell < k$,
  $$\begin{array}{rcl} g_i \cdot h_i = g_j \cdot h_j & \text{iff} & g_i' \cdot h_i = g_j' \cdot h_j, \\ 1 = g_i \cdot h_i & \text{iff} & 1 = g_i' \cdot h_i, \\ g_i \cdot h_i \cdot g_j \cdot h_j = g_\ell \cdot h_\ell & \text{iff} & g_i' \cdot h_i \cdot g_j' \cdot h_j = g_\ell' \cdot h_\ell, \\ (g_i \cdot h_i)^{-1} = g_j \cdot h_j & \text{iff} & (g_i' \cdot h_i)^{-1} = g_j' \cdot h_j. \end{array}$$
  So player $\exists$ wins the public game too.

Generalizing

- The proof of the preceding theorem uses very few facts about groups.
- It would work equally well in any case where a part of a structure can be isolated and replaced.

  Example: Suppose we want to compare two linear orderings.

  Assume that one is obtained from the other by replacing an interval by an elementarily equivalent linear ordering.

  Then the two original orderings are elementarily equivalent.

  Example: We could take an infinite product of groups and make replacements at all factors simultaneously.