Introduction to Model Theory

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LSSU Math 500

1. Automorphisms and Interpretations
   - Automorphisms
   - Relativization
   - Interpreting One Structure in Another

Subsection 1

Automorphisms

The Automorphism Group of a Structure

- Let A be an L-structure.
- Every automorphism of A is a permutation of $\text{dom}(A)$.
- By a previous theorem, the collection of all automorphisms of A is a group under composition.
- This group, regarded as a permutation group on $\text{dom}(A)$, is called the **automorphism group** of A.
- It is denoted by $\text{Aut}(A)$.
- Automorphism groups have traditionally been studied by group theorists and geometers, in settings remote from model theory.
- To exploit past experience, we need some translations between model theory and group theory.

Stabilizers

- For any set Ω, the group of all permutations of Ω is called the **symmetric group** on Ω, in symbols $\mathrm{Sym}(\Omega)$.
- Let G be a subgroup of $\mathrm{Sym}(\Omega)$.
- If X is a subset of Ω, then the **pointwise stabilizer** of X in G is the set
$$G_{(X)} = \{g \in G : g(a) = a, \text{ for all } a \in X\}.$$
- This set forms a subgroup of G.
- We also write $G_{(\overline{a})}$, where $\overline{a}$ is a sequence listing the elements of X.
- The **setwise stabilizer** of X in G is the set
$$G_{\{X\}} = \{g \in G : g(X) = X\}.$$
- It is also a subgroup of G.
- In fact, we have $G_{(X)} \subseteq G_{\{X\}} \subseteq G$.

Orbits and Transitivity

- Let Ω be a set.
- Let G be a subgroup of $\mathrm{Sym}(\Omega)$.
- If a is an element of Ω, the **orbit** of a under G is the set

$$\{g(a) : g \in G\}.$$

- The orbits of all elements of Ω under G form a partition of Ω.
- We say G is **transitive on** Ω if the orbit of every element (or, equivalently, the orbit of one element) is the whole of Ω.
- A structure A is **transitive** if $\mathrm{Aut}(A)$ is transitive on $\mathrm{dom}(A)$.
- The opposite occurs when A has no automorphisms except the identity 1_A.
- In this case, we say that A is **rigid**.

Example: Ordinals

- Let the structure A be an ordinal $(\alpha, <)$.

  So $<$ well-orders the elements of A.

  Then A is rigid.

  Suppose f is an automorphism of A which is not the identity.

  Then there is some element a, such that $f(a) \neq a$.

  Replacing f by f^{-1} if necessary, we can suppose that $f(a) < a$.

  Since f is a homomorphism, $f^2(a) = f(f(a)) < f(a)$.

  By induction $f^{n+1}(a) < f^n(a)$, for each $n < \omega$.

  Then $a > f(a) > f^2(a) > \cdots$.

  This contradicts that $<$ is a well-ordering.

Example: Affine Space

- Let D be the direct sum of countably many cyclic groups of order 2.

  Equivalently, let D be a countable-dimensional vector space over the two-element field $\mathbb{F}_2$.

  On D we define a relation

  $$R(x,y,z,w) \quad \text{iff} \quad x+y=z+w.$$

  The structure A consists of the set D with the relation R.

  Fix d in D. Define $e_d : D \to D$ by

  $$e_d(a) = a+d, \quad a \in D.$$

  e_d is a permutation of D.
  - 1-1: $e_d(a) = e_d(b)$ iff $a+d=b+d$ iff $a=b$.
  - Onto: Let $a \in D$. Then $e_d(a-d) = (a-d)+d = a$.

Example: Affine Space (Cont'd)

- e_d is an automorphism of A taking 0 to d.
  For $x, y, z, w \in D$,

$$\begin{array}{rll} R(x,y,z,w) & \text{iff} & x+y=z+w \\ & \text{iff} & (x+d)+(y+d)=(z+d)+(w+d) \\ & \text{iff} & R(x+d,y+d,z+d,w+d). \end{array}$$

  Thus, A is a transitive structure.
- Fix d again. Define an addition operation $+_d$ in terms of R:

$$x +_d y = z \quad \text{iff} \quad R(x,y,z,d).$$

  This makes D into an abelian group with d as the identity.
  A is what remains of D when we forget which element is 0.
  This the **countable-dimensional affine space over** $\mathbb{F}_2$.

Action of Permutations on Cartesian Products

- Let Ω be a set.
- Let G be a group of permutations of Ω.
- We write Ω^n for the set of all ordered n-tuples of elements of Ω.
- Then G acts as a set of permutations of Ω^n by setting

$$g(a_0,\ldots,a_{n-1}) = (g(a_0),\ldots,g(a_{n-1})).$$

- So we can talk about the **orbits** of G on Ω^n.
- When n is greater than 1 and Ω has more than one element, then G is not transitive on Ω^n.

  Suppose $a, b \in \Omega$, $a \neq b$.

  Then, for all $g \in G$, $g(a,a,\ldots) \neq (a,b,\ldots)$.

  So G cannot be transitive on Ω^n.

Oligomorphic Structures

- We say that G is **oligomorphic** (on Ω) if for every positive integer n, the number of orbits of G on Ω^n is finite.
- We say that a structure A is **oligomorphic** if $\mathrm{Aut}(A)$ is oligomorphic on $\mathrm{dom}(A)$.
- We will see that for countable structures, oligomorphic is the same thing as ω-categorical.

Example

- Consider the ordered set $A = (\mathbb{Q}, <)$ of rational numbers.

  Let $\overline{a}$ and $\overline{b}$ be any two n-tuples in A.

  There is an automorphism of A which takes $\overline{a}$ to $\overline{b}$ if and only if the elements of $\overline{a}$ and $\overline{b}$ are in the same relative order in $\mathbb{Q}$.

  Rephrasing, the number of different orbits equals the number of different relative orders that can be imposed on an n-tuple.

  This number is at most, say, $(2n-1)!$:
  - First, place a_0;
  - There are 3 options for placing a_1 ($a_1 < a_0$, $a_1 = a_0$ or $a_1 > a_0$);
  - There are at most 5 options for placing a_2;

    ⋮
  - There are at most $2(n-1)+1$ options for placing a_{n-1}.

  So A is oligomorphic.

Closed Subgroups

- Suppose G is a group of permutations of a set Ω.
- Let H be a subgroup of G.
- We say that H is **closed** in G if the following holds:

  If $g \in G$ and, for every tuple $\overline{a}$ of elements of Ω, there is h in H, such that $g(\overline{a}) = h(\overline{a})$, then $g \in H$.
- We say that the group G is **closed** if it is closed in the symmetric group $\mathrm{Sym}(\Omega)$.

  Claim: If G is closed and H is closed in G, then H is closed.

  Let $\sigma \in \mathrm{Sym}(\Omega)$, $\overline{a} \in \Omega^n$ and $h \in H$, such that $h(\overline{a}) = \sigma(\overline{a})$.

  Since G is closed and $h \in G$, $\sigma \in G$.

  Since H is closed in G and $h \in H$, $\sigma \in H$.

  Thus, H is closed.

Closed Subgroups and Automorphisms

Theorem

Let Ω be a set. Let G be a subgroup of $\mathrm{Sym}(\Omega)$ and H a subgroup of G. Then the following are equivalent:

(a) H is closed in G.

(b) There is a structure A with $\mathrm{dom}(A) = \Omega$, such that $H = G \cap \mathrm{Aut}(A)$.

In particular a subgroup H of $\mathrm{Sym}(\Omega)$ is of form $\mathrm{Aut}(B)$ for some structure B with domain Ω if and only if H is closed.

(a)⇒(b) For each $n < \omega$ and each orbit Δ of H on Ω^n, choose an n-ary relation symbol R_Δ. Take L to be the signature consisting of all these relation symbols. Make Ω into an L-structure A by putting $R_\Delta^A = \Delta$.

- Every permutation in H takes R_Δ to R_Δ. So $H \subseteq G \cap \mathrm{Aut}(A)$.
- Let $g \in G$ be an automorphism of A. Let $\overline{a}$ be in Ω^n. Then $\overline{a}$ is in some orbit Δ of H. Thus, since $\Delta = R_\Delta^A$, $g(\overline{a})$ must be in the same orbit. Hence, $g(\overline{a}) = h(\overline{a})$, for some h in H. Since H is closed in G, g is in H.

Closed Subgroups and Automorphisms (Converse)

(b)⇒(a) Assuming (b), we show that H is closed in G.

Let g be an element of G, such that for each finite subset W of Ω, there is $h \in H$, with $g|_W = h|_W$.

Let $\phi(\overline{x})$ be an atomic formula of the signature of A, and $\overline{a}$ a tuple of elements of A.

Choose W above so that it contains $\overline{a}$.

Then we have

$$\begin{array}{lll} A \models \phi(\overline{a}) & \text{iff} & A \models \phi(h(\overline{a})) \quad (h \in \text{Aut}(A)) \\ & \text{iff} & A \models \phi(g(\overline{a})). \quad (g|_W = h|_W) \end{array}$$

Thus, g is an automorphism of A.

- When H is closed, the structure A constructed in the proof of (a)⇒(b) is called the **canonical structure** for H.
- By the proof, A can be chosen to be an L-structure with $|L| \leq |\Omega| + \omega$.

Open Subsets of a Symmetric Group

- The word "closed" suggests a topology.
- A subset S of $\mathrm{Sym}(\Omega)$ is called **basic open** if there are tuples $\overline{a}$ and $\overline{b}$ in Ω, such that
$$S = \{g \in \mathrm{Sym}(\Omega) : g(\overline{a}) = \overline{b}\}.$$
- Write this set as $S(\overline{a}, \overline{b})$.
- In particular $\mathrm{Sym}(\Omega)_{(\overline{a})}$ is a basic open set.
- An **open subset** of $\mathrm{Sym}(\Omega)$ is a union of basic open subsets.
- If $\Omega = \mathrm{dom}(A)$, we define a (**basic**) **open subset** of $\mathrm{Aut}(A)$ to be the intersection of $\mathrm{Aut}(A)$ with some (basic) open subset of $\mathrm{Sym}(\Omega)$.

A Topological Group

Lemma

Let A be a structure and write G for $\mathrm{Aut}(A)$.

(a) The definitions above define a topology on G; it is the topology induced by that on $\mathrm{Sym}(\Omega)$. Under this topology, G is a topological group, i.e., multiplication and inverse in G are continuous operations.

(b) A subgroup of G is open if and only if it contains the pointwise stabilizer of some finite set of elements of A.

(c) A subset F of G is closed under this topology if and only if it is closed in the preceding sense (with F for H).

(d) A subgroup H of G is dense in G if and only if H and G have the same orbits on $(\mathrm{dom} A)^n$, for each positive integer n.

(a) A permutation g takes $\overline{a}_1$ to $\overline{b}_1$ and $\overline{a}_2$ to $\overline{b}_2$ if and only if it takes $\overline{a}_1\overline{a}_2$ to $\overline{b}_1\overline{b}_2$. So the intersection of two basic open sets is again basic open. The first sentence of (a) follows at once by general topology.

A Topological Group ((a) and (b))

(a) For the second sentence:
- Note $g \in S(\overline{a},\overline{b})$ if and only if $g^{-1} \in S(\overline{b},\overline{a})$. This proves the continuity of inverse.
- Suppose $gh \in S(\overline{a},\overline{b})$. Write $\overline{c}$ for $h(\overline{a})$. Then $g \in S(\overline{c},\overline{b})$, $h \in S(\overline{a},\overline{c})$, and $S(\overline{c},\overline{b}) \cdot S(\overline{a},\overline{c}) \subseteq S(\overline{a},\overline{b})$. So multiplication is continuous.

(b) For each tuple $\overline{a}$ the pointwise stabilizer $G_{(\overline{a})}$ is $G \cap S(\overline{a},\overline{a})$. This is open. A subgroup of G containing $G_{(\overline{a})}$ is a union of cosets of $G_{(\overline{a})}$. Each of those is basic open. Hence the subgroup is open.

In the other direction, suppose H is an open subgroup containing a non-empty basic open set $G \cap S(\overline{a},\overline{b})$.

Every element of $G_{(\overline{a})}$ can be written as gh with

$$g \in G \cap S(\overline{b},\overline{a}) \subseteq H \quad \text{and} \quad h \in G \cap S(\overline{a},\overline{b}) \subseteq H.$$

Hence H contains $G_{(\overline{a})}$.

A Topological Group ((c) and (c))

(c) Suppose $F \subseteq G$ is topologically closed. Let $g \in G$, such that, for all $\overline{a}$, $g(\overline{a}) = f(\overline{a})$, for some $f \in F$. Thus, for every basic open $S(\overline{a}, \overline{b})$, such that $g \in S(\overline{a}, \overline{b})$, $S(\overline{a}, \overline{b}) \cap F \neq \emptyset$. Since F is closed, $g \in F$. Thus, F is closed in G.

Suppose, conversely, that F is closed in G. Let $g \in G$, such that, for every basic open $S(\overline{a}, \overline{b})$, with $g \in S(\overline{a}, \overline{b})$, $S(\overline{a}, \overline{b}) \cap F \neq \emptyset$. Thus, for all $g \in G$ and all $\overline{a}$, $g(\overline{a}) = \overline{b}$ implies $g(\overline{a}) = \overline{b} = f(\overline{a})$, for some $f \in F$. Since F is closed in G, $g \in F$. Hence, F is topologically closed in G.

(d) H is dense in G iff, for all $g \in G$, every basic open set containing g meets H iff, for all $g \in G$ and all $\overline{a}, \overline{b}$, $g \in S(\overline{a}, \overline{b})$ implies $S(\overline{a}, \overline{b}) \cap H \neq \emptyset$ iff, for all $g \in G$ and all $\overline{a}, \overline{b}$, $g(\overline{a}) = \overline{b}$ implies there exists $h \in H$, such that $h(\overline{a}) = \overline{b}$ iff, for all n, H and G have the same orbits on $(\mathrm{dom} A)^n$.

Automorphism Groups and Structures

- Starting from a structure A, we get by successive abstractions:
  - The permutation group $\text{Aut}(A)$;
  - The topological group $\text{Aut}(A)$;
  - The abstract group $\text{Aut}(A)$.
- At each step some information is discarded.
- How much of this information can be recovered?
  - In some cases, very little, as, e.g., was the case with the ordinals.
  - In general, the larger the automorphism group of a structure, the better the chances of reconstructing the structure from the automorphism group.

Characterization of Open Sets

Theorem

Let G be a closed group of permutations of ω and H a closed subgroup of G. Then the following are equivalent:

(a) H is open in G.

(b) $(G:H) \leq \omega$.

(c) $(G:H) < 2^\omega$.

(a)⇒(b) Suppose (a) holds. Then there is some tuple $\overline{a}$ of elements of ω, such that the stabilizer $G_{(\overline{a})}$ of $\overline{a}$ lies in H. Suppose now that g, j are two elements of G, such that $g(\overline{a}) = j(\overline{a})$. Then $j^{-1}g \in G_{(\overline{a})} \subseteq H$. So the cosets gH, jH are equal. Since there are only countably many possibilities for $g(\overline{a})$, the index $(G:H)$ must be at most countable.

(b)⇒(c) is trivial.

(c)⇒(a) We suppose that H is not open in G.

We construct continuum many left cosets of H in G.

Characterization of Open Sets (Construction)

- We define by induction sequences $(\overline{a}_i : i < \omega)$, $(\overline{b}_i : i < \omega)$ of tuples of elements of ω and a sequence $(g_i : i < \omega)$ of elements of G, such that the following hold for all i.
  1. $\overline{b}_0 = \langle\rangle$; $\overline{b}_{i+1}$ is a concatenation of all the sequences $(k_0 \cdots k_i)(\overline{a}_0 {}^\frown \cdots {}^\frown \overline{a}_i)$, where each k_j is in $\{1, g_0, \ldots, g_i\}$;
  2. $g_i(\overline{b}_i) = \overline{b}_i$;
  3. There is no $h \in H$, such that $h(\overline{a}_i) = g_i(\overline{a}_i)$;
  4. i is an item in $\overline{a}_i$.

  When $\overline{b}_i$ has been chosen, we have, by assumption, that $G_{(\overline{b}_i)} \nsubseteq H$.

  So there is some $g_i \in G$ which fixes $\overline{b}_i$ (giving 2), and is not in H.

  Since H is closed in G, there is a tuple $\overline{a}_i$, such that $h(\overline{a}_i) \neq g_i(\overline{a}_i)$, for all h in H. This ensures 3.

  Adding i to $\overline{a}_i$ if necessary, we obtain 4.

Characterization of Open Sets (Continuum of Subsets)

- For any subset S of $\omega \backslash \{0\}$, define $g_i^S = \begin{cases} g_i, & \text{if } i \in S \\ 1, & \text{if } i \notin S \end{cases}$.

  Let $f_i^S = g_i^S \cdots g_0^S$. For each $j > i$, we have

  $$f_j^S(\overline{a}_i) = g_j^S \cdots g_{i+1}^S g_i^S \cdots g_0^S(\overline{a}_i) \stackrel{1,2}{=} g_i^S \cdots g_0^S(\overline{a}_i) = f_i^S(\overline{a}_i).$$

  So by 4, we can define a map $g_S : \omega \to \omega$ by setting, for each $i < \omega$,

  $$g_S(i) = f_j^S(i), \text{ for all } j \geq i.$$

  g_S is injective: The maps f_i^S are automorphisms.

  g_S is surjective: Consider any $i \in \omega$. Let $j = (f_i^S)^{-1}(i)$.

  - Suppose $j \leq i$. Then $g_S(j) = f_i^S(f_i^S)^{-1}(i) = i$.
  - Suppose $j > i$. Then $g_S(j) = f_j^S(f_i^S)^{-1}(i) = g_j^S \cdots g_{i+1}^S(i) = i$.

  So g_S is a permutation of ω. Note that the f_i^S are in the closed group G. Moreover, for each tuple $\overline{a}$ in ω, g_S agrees on $\overline{a}$ with some f_i^S. Hence, g_S is in G. There are 2^ω distinct subsets S of $\omega \backslash \{0\}$.

Characterization of Open Sets (Continuum of Cosets)

- It remains only to show that the corresponding permutations g_S lie in different right cosets of H.

  Suppose $S \neq T$. Let $i > 0$ be least, say, in S but not in T.

  By 3, there is no element of H which agrees with g_i on $\overline{a}_i$.

  Set $f = f_{i-1}^S \stackrel{i \notin T}{=} f_i^T$. Consider $f^{-1}(\overline{a}_i)$.

  Choose some $j \geq i$, such that all the items in $f^{-1}(\overline{a}_i)$ are $\leq j$.

  We have, for all h in H,

$$\begin{array}{rcl} g_S(f^{-1}(\overline{a}_i)) & \stackrel{\text{choice } j}{=} & f_j^S(f^{-1}(\overline{a}_i)) \stackrel{i \in S}{=} g_j^S \cdots g_{i+1}^S g_i(\overline{a}_i) \\ & = & g_i(\overline{a}_i) \neq h(\overline{a}_i) = hg_j^T \cdots g_{i+1}^T(\overline{a}_i) \\ & = & hf_j^T f^{-1}(\overline{a}_i) = (hg_T)(f^{-1}(\overline{a}_i)). \end{array}$$

  So $g_S \notin Hg_T$, which finishes the proof.

A Model-Theoretic Translation

- Let A be a countable L^+-structure.
- Suppose $L^- \subseteq L^+$ and let B be the L^--reduct $A|_{L^-}$ of A.
- Then $H = \text{Aut}(A)$ is a subgroup of $G = \text{Aut}(B)$.
- Let g be any element of G, and consider the structure gA.
- gA is like A except that for each symbol S of L^+, $S^{gA} = g(S^A)$.
    - The domain of gA is $\text{dom}(A)$;
    - $g(S^A) = S^A$, for each symbol S in L^-.
- So the reduct $(gA)|_{L^-}$ is exactly B again.
- Suppose now that k is another element of G.
- gA is equal to kA when $g(S^A) = k(S^A)$, for each symbol S,
  i.e., when $k^{-1}g$ is an automorphism of A,
  i.e., when the cosets gH and kH in G are equal.
- This shows that the index of $\text{Aut}(A)$ in $\text{Aut}(B)$ is equal to the number of different ways in which the symbols of $L^+ \backslash L^-$ can be interpreted in B so as to give a structure isomorphic to A.

The Kueker-Reyes Theorem

Theorem (Kueker-Reyes Theorem)

Let L^- and L^+ be signatures with $L^- \subseteq L^+$. Let A be a countable L^+-structure and let B be the reduct $A|_{L^-}$. Put $G = \text{Aut}(B)$. Then the following are equivalent:

(a) There is a tuple $\overline{a}$ of elements of A, such that $G_{(\overline{a})} \subseteq \text{Aut}(A)$.

(b) There are at most countably many distinct expansions of B which are isomorphic to A.

(c) The number of distinct expansions of B which are isomorphic to A is less than 2^ω.

(d) There is a tuple $\overline{a}$ of elements of A such that for each atomic formula $\phi(x_0,\ldots,x_{n-1})$ of L^+, there is a formula $\psi(x_0,\ldots,x_{n-1},\overline{y})$ of $L^-_{\omega_1\omega}$, such that $A \models \forall\overline{x}(\phi(\overline{x}) \leftrightarrow \psi(\overline{x},\overline{a}))$.

- Our translation gives the equivalence of (a), (b) and (c) at once. It remains to show that (a) is equivalent to (d).

The Kueker-Reyes Theorem ((d)⇒(a))

- Assume, first (d) holds. Let $g \in G_{(\overline{a})}$. We have, for every atomic formula $\phi(x_0,\ldots,x_{n-1})$ of L^+ and every $\overline{b}$ in A,

$$\begin{array}{llll} A \models \phi(\overline{b}) & \text{iff} & A \models \psi(\overline{b},\overline{a}) & \text{(hypothesis)} \\ & \text{iff} & A \models \psi(g(\overline{b}),g(\overline{a})) & (g \in G) \\ & \text{iff} & A \models \psi(g(\overline{b}),\overline{a}) & (g \in G_{(\overline{a})}) \\ & \text{iff} & A \models \phi(g(\overline{b})). & \text{(hypothesis)} \end{array}$$

Therefore, $g \in \mathrm{Aut}(A)$.

The Kueker-Reyes Theorem ((a)⇒(d))

- For the converse, suppose $G_{(\overline{a})} \subseteq \text{Aut}(A)$.

  Let $\phi(x_0,\ldots,x_{n-1})$ be an atomic formula of L^+.

  Without loss we can suppose that ϕ is unnested.

  For simplicity let us assume too that ϕ is $R(x_0,\ldots,x_{n-1})$, where R is some n-ary relation symbol.

  For each n-tuple $\overline{c}$ in $\phi(A^n)$, let $\sigma_{(B,\overline{a},\overline{c})}(\overline{a},\overline{c})$ be the Scott sentence of the structure $(B,\overline{a},\overline{c})$. Note that $\bigvee_{\overline{c}\in\phi(A^n)} \sigma_{(B,\overline{a},\overline{c})}(\overline{a},\overline{x})$ is in $L^-_{\omega_1\omega}$. This is the sentence that plays the role of ψ.

  - Suppose $A \vDash \phi(\overline{d})$. Then $(B,\overline{a},\overline{d}) \vDash \sigma_{(B,\overline{a},\overline{d})}(\overline{a},\overline{d})$. Since $\overline{d} \in \phi(A^n)$, $A \vDash \bigvee_{\overline{c}\in\phi(A^n)} \sigma_{(B,\overline{a},\overline{c})}(\overline{a},\overline{c})$.
  - Assume $\overline{c} \in \phi(A^n)$, i.e., $A \vDash \phi(\overline{c})$, and let $\overline{d}$ such that $A \vDash \sigma_{\overline{c}}(\overline{a},\overline{d})$. Then $(B,\overline{a},\overline{c}) \cong (B,\overline{a},\overline{d})$. So by (a), $(B,\overline{a},\overline{c},R^A) \cong (B,\overline{a},\overline{d},R^A)$. Hence, $A \vDash \phi(\overline{d})$.

  We conclude $A \vDash \forall \overline{x}(\phi(\overline{x}) \leftrightarrow \bigvee_{\overline{c}\in\phi(A^n)} \sigma_{\overline{c}}(\overline{a},\overline{x}))$.

Automorphisms and Rigidity

Corollary

Let A be a countable structure. Then the following are equivalent:

(a) $|\text{Aut}(A)| \leq \omega$.

(b) $|\text{Aut}(A)| < 2^\omega$.

(c) There is a tuple $\overline{a}$ in A such that $(A, \overline{a})$ is rigid.

- In the theorem, take
$$L^- = L, L^+ = (L, \overline{c}),$$
where $\overline{c}$ contains one constant for each element of A.
Then, consider
  - The L^+-structure $(A, \overline{a})$;
  - The L^--reduct $(A, \overline{a}) |_{L^-} = A$.

Then, statements (a),(b) and (c) of the theorem correspond, respectively, to statements (c), (a) and (b) of the corollary.

Subsection 2

Relativization

Relativized Reducts

- Consider two signatures L and L' with $L \subseteq L'$.
- Let C be an L'-structure and B a substructure of the reduct $C|_L$.
- Then we can make the pair C, B into a single structure.
  - Take a new unary relation symbol P.
  - Write L^+ for L' with P added.
  - Expand C to an L^+-structure A by setting $P^A = \text{dom}(B)$.
- We can recover C and B from A by

$$\begin{array}{rcl} C & = & A|_{L'}, \\ B & = & \text{the substructure of } A|_L \text{ whose domain is } P^A. \end{array}$$

- We call B a **relativized reduct** of A.
- The meaning is that to get B from A we have to:
  - "Relativize" the domain to a definable subset of $\text{dom}(A)$;
  - Remove some symbols.
- Forgetting about C, we consider signatures L and L^+, with $L \subseteq L^+$, and a unary relation symbol P in $L^+ \setminus L$.

P-Part and Admissibility Conditions for Relativization

- Let A be an L^+-structure.
- By a previous lemma, the following are necessary and sufficient conditions for P^A to be the domain of a substructure of $A|_L$.
  - For every constant c of L, $c^A \in P^A$;
  - For every $n > 0$, all n-ary F in L and all $\overline{a} \in (P^A)^n$, $F^A(\overline{a}) \in P^A$.
- If the conditions are satisfied, the substructure is uniquely determined.
- We write it A_P, and call it the P-**part** of A.
- From the same lemma one can write these necessary and sufficient conditions as a set of first-order sentences that A must satisfy.
- We call them the **admissibility conditions** for relativization to P.
- A_P depends on the language L as well as A and P.

Relativization Theorem

Theorem (Relativization Theorem)

Let L and L^+ be signatures such that $L \subseteq L^+$, and P a unary relation symbol in $L^+ \setminus L$. Then for every formula $\phi(\overline{x})$ of $L_{\infty\omega}$, there is a formula $\phi^P(\overline{x})$ of $L^+_{\infty\omega}$, such that the following holds:

If A is an L^+-structure such that A_P is defined, and $\overline{a}$ is a sequence of elements from A_P, then $A_P \models \phi(\overline{a})$ if and only if $A \models \phi^P(\overline{a})$.

- We define ϕ^P by induction on the complexity of ϕ:
  1. If ϕ is atomic, $\phi^P = \phi$;
  2. $(\bigwedge_{i\in I} \psi_i)^P = \bigwedge_{i\in I}(\psi_i^P)$ and $(\bigvee_{i\in I} \psi_i)^P = \bigvee_{i\in I}(\psi_i^P)$;
  3. $(\neg\phi)^P$ is $\neg(\phi^P)$;
  4. $(\forall y\psi(\overline{x},y))^P = \forall y(Py \to \psi^P(\overline{x},y))$; $(\exists y\psi(\overline{x},y))^P = \exists y(Py \wedge \psi^P(\overline{x},y))$.

  Then the condition follows by induction on the complexity of ϕ.
- The formula ϕ^P in this theorem is called the **relativization** of ϕ to P.
- Note that, if ϕ is first-order, then so is ϕ^P.

Property of the Relativization

Corollary

Let L and L^+ be signatures with $L \subseteq L^+$ and P a unary relation symbol in $L^+ \backslash L$. If A and B are L^+-structures such that $A \preccurlyeq B$ and A_P is defined, then B_P is defined and $A_P \preccurlyeq B_P$.

- First, we show that B_P is defined.
  - For all constants c in L, $c^B = c^A \in A_P \subseteq B_P$.
  - Let $n > 0$, F an n-ary function symbol of L and $\overline{b}$ in P^B.
    Since A_P is defined, $A \models (\forall \overline{x})(\bigwedge_{i=1}^n P(x_i) \to P(F(\overline{x})))$.
    Since $A \preccurlyeq B$, $B \models (\forall \overline{x})(\bigwedge_{i=1}^n P(x_i) \to P(F(\overline{x})))$.
    By hypothesis, $B \models P(x_i)[b_i]$, for all $i < n$.
    So $B \models P(F(\overline{x}))[\overline{b}]$, i.e., $F^B(\overline{b}) \in P^B$.

  Now we show that $A_P \preccurlyeq B_P$. Using the notation of the Relativization Theorem, for every ϕ in L and all $\overline{a}$ in A_P,

$$A_P \models \phi(\overline{a}) \quad \text{iff} \quad A \models \phi^P(\overline{a}) \quad \text{iff} \quad B \models \phi^P(\overline{a}) \quad \text{iff} \quad B_P \models \phi(\overline{a}).$$

Example: Linear Groups

- Suppose G is a group of $n \times n$ matrices over a field F.
  We can make G and F into a single structure A as follows.
  The signature of A has:
  - Two unary relation symbols group and field;
  - Two ternary relation symbols add and mult;
  - n^2 binary relation symbols coeff_{ij}, $1 \leq i, j \leq n$.

  The sets group^A and field^A consist of the elements of G and F, resp.
  The relations add^A and mult^A express addition and multiplication in F.
  For each matrix $g \in G$, the ij-th entry in g is the unique element f, such that $\text{coeff}_{ij}(g, f)$ holds.
- Note that multiplication in G can be defined in terms of the field operations, using the symbols coeff_{ij}.
- Note, also, there are no function or constant symbols.
- So B_P and B_Q are automatically defined for any structure B of the same signature as A.

Relativization Using a First-Order Formula

- Sometimes a structure B is picked out inside a structure A, not by a unary relation symbol P, but by a formula $\theta(x)$.
- When θ is in the first-order language of A, then again we call B a **relativized reduct** of A.
- The case in which $\theta(x)$ is $P(x)$, becomes a special case.
- If θ also contains parameters from A, we call B a **relativized reduct with parameters**.
- One can adapt the Relativization Theorem straightforwardly by putting θ in place of P everywhere.

Example: ω as a Relativized Reduct

- Suppose A is a transitive model of Zermelo-Fraenkel set theory.

  Let $\theta(x)$ be the formula "$x \in \omega$".

  The ordering $<$ on ω coincides with $\in$.

  We can write set-theoretic formulas that define $+$ and $\cdot$.

  Note that ω satisfies a rather strong form of the Peano axioms:
  - 0 is not of the form $x+1$;
    If $x, y \in \omega$ and $x+1 = y+1$, then $x = y$;
  - For every formula $\phi(x)$ of the first-order language of A, possibly with parameters from A, if $\phi(0)$ and $\forall x(x \in \omega \wedge \phi(x) \rightarrow \phi(x+1))$ both hold in A, then $\forall x(x \in \omega \rightarrow \phi(x))$ holds in A.
- The latter is the induction axiom schema for subsets of ω which are first-order definable (with parameters) in A.
- Of course this includes the subsets of ω which are first-order definable in the structure $(\omega, <)$ itself, by the relativization theorem.

Example: Relativized Reducts of Rationals as Ordered Set

- Let A be the following structure:
  - The domain of A is the set $\mathbb{Q}$ of rational numbers;
  - The relations of A are all those which are $\emptyset$-definable from the usual ordering $<$ of the rationals.

  We find the relativized reducts of A (without parameters).
  - First note that $\text{Aut}(A)$ is exactly $\text{Aut}(\mathbb{Q},<)$ since A is a definitional expansion of $(\mathbb{Q},<)$.
  - Next, $\text{Aut}(A)$ is transitive on $\mathbb{Q}$. It follows that any subset of $\mathbb{Q}$ which is definable without parameters is either empty or the whole of $\mathbb{Q}$. So we can forget the relativization.
  - Thirdly, if B is any reduct of A, then $\text{Aut}(A) \subseteq \text{Aut}(B) \subseteq \text{Sym}(Q)$, and $\text{Aut}(B)$ is closed in $\text{Sym}(Q)$ by a previous theorem.

Relativized Reducts of Rationals as Ordered Set (Cont'd)

- And finally, $\mathrm{Aut}(\mathbb{Q},<)$ is oligomorphic and its orbits on n-tuples are all $\emptyset$-definable. So every orbit of $\mathrm{Aut}(B)$ on n-tuples is a union of finitely many orbits of $\mathrm{Aut}(\mathbb{Q},<)$. Hence it is defined by some relation of A.

So, up to definitional equivalence, the relativized reducts of A correspond exactly to the closed groups lying between $\mathrm{Aut}(A)$ and $\mathrm{Sym}(\mathbb{Q})$.

It can be shown that apart from $\mathrm{Aut}(A)$ and $\mathrm{Sym}(\mathbb{Q})$, there are just three such groups.

- The first is the group of all permutations of A which either preserve the order or reverse it.
- The second is the group of all permutations which preserve the cyclic relation "$x<y<z$ or $y<z<x$ or $z<x<y$"; This corresponds to taking an initial segment of $\mathbb{Q}$ and moving it to the end.
- The third is the group generated by these other two.
  It consists of those permutations which preserve the relation "exactly one of x,y lies between z and w".

Example: Orderable Groups

- An **ordered group** is a group G which carries a linear ordering $<$ such that if g, h and k are any elements of G, then

$$g < h \quad \text{implies} \quad k \cdot g < k \cdot h \quad \text{and} \quad g \cdot k < h \cdot k.$$

- A group is **orderable** if a linear ordering can be added so as to make it into an ordered group.
- Clearly an orderable group cannot have elements $\neq 1$ of finite order.
  Suppose, to the contrary that $g^n = 1$.
  - Suppose $1 < g$. Then $g^i < g^{i+1}$, for all $i = 0, \ldots, n$. Thus, $1 < g < g^2 < \cdots < g^n = 1$, a contradiction.
  - If $g < 1$, we argue similarly.
- This is not a sufficient condition for orderability (unless the group happens to be abelian).

Pseudo-Elementary Classes and PC_Δ Classes

- Let L be a first-order language.
- A **pseudo-elementary class** (for short, a **PC class**) of L-structures is a class of structures of the form $\{A|_L : A \models \phi\}$ for some sentence ϕ in a first-order language $L^+ \supseteq L$.
- A $\mathbf{PC}_\Delta$**-class** of L-structures is a class of the form $\{A|_L : A \models U\}$, for some theory U in a first-order language $L^+ \supseteq L$.

  Example: The class of orderable groups is a PC class.

Example: Ordered Abelian Groups

- Let L be the first-order language of linear orderings (with symbol $<$).

  Let U be the theory of ordered abelian groups.

  Then the class $\mathbf{K} = \{A|_L : A \models U\}$ is the class of all linear orderings which are orderings of abelian groups.

  This is a PC class, since U can be written as a finite theory and, hence, as a single sentence.

PC'_Δ Classes of Structures

- One can generalize these notions, using relativized reducts A_P.
- We define a **PC'_Δ class** of L-structures to be a class of the form

$$\{A_P : A \models U \text{ and } A_P \text{ is defined}\},$$

  for some theory U in a language $L^+ \supseteq L \cup \{P\}$.
- By the admissibility conditions, every PC'_Δ class can be written as $\{A_P : A \models U'\}$, for some theory U' in L^+.

  Example: A natural example of a PC'_Δ class is the class of multiplicative groups of fields.
  - L has only the symbol for multiplication.
  - The unary symbol P picks out the non-zero elements.
  - U is the theory of fields.

  One can show that this class is not first-order axiomatizable.

PC_Δ and PC'_Δ Classes

- PC'_Δ appears to be a generalization of PC_Δ.
- However, the two notions are exactly the same.

Theorem

The PC'_Δ classes are exactly the PC_Δ classes. More precisely, let $\mathbf{K}$ be a class of L-structures.

(a) If $\mathbf{K}$ is a PC'_Δ class $\{A_P : A \models U \text{ and } A_P \text{ is defined}\}$ for some theory U in a first-order language L^+, then $\mathbf{K}$ is also a PC_Δ class $\{A|_L : A \models U^*\}$ for some theory U^* in a first-order language L^* with $|L^*| \leq |L^+|$.

(b) If $\mathbf{K}$ is a PC' class and all structures in $\mathbf{K}$ are infinite, then $\mathbf{K}$ is a PC class.

Subsection 3

Interpreting One Structure in Another

Interpretations

- Let K and L be signatures.
- Let A be a K-structure and B an L-structure.
- For a positive integer n, an (n-**dimensional**) **interpretation** Γ **of** B **in** A is defined to consist of three items:
  1. A formula $\partial_\Gamma(x_0,\ldots,x_{n-1})$ of signature K;
  2. For each unnested atomic formula $\phi(y_0,\ldots,y_{m-1})$ of L, a formula $\phi_\Gamma(\overline{x}_0,\ldots,\overline{x}_{m-1})$ of signature K in which the $\overline{x}_i$ are disjoint n-tuples of distinct variables;
  3. A surjective map $f_\Gamma : \partial_\Gamma(A^n) \to \text{dom}(B)$, such that for all unnested atomic formulas ϕ of L and all $\overline{a}_i \in \partial_\Gamma(A^n)$,

$$B \vDash \phi(f_\Gamma(\overline{a}_0),\ldots,f_\Gamma(\overline{a}_{m-1})) \quad \text{iff} \quad A \vDash \phi_\Gamma(\overline{a}_0,\ldots,\overline{a}_{m-1}).$$

- ∂_Γ and ϕ_Γ (for all unnested atomic ϕ) are the **defining formulas** of Γ.
- ∂_Γ is the **domain formula** of Γ;
- The map f_Γ is the **coordinate map** of Γ.
  - It assigns to each element $f_\Gamma(\overline{a})$ of B the "coordinates" $\overline{a}$ in A;
  - An element may have several different tuples of coordinates.

Interpretability and Conventions

- Unless anything is said to the contrary, we assume that the defining formulas of Γ are all first-order.
- For example, we say that B is **interpretable** in A if there is an interpretation of B in A with all its defining formulas first-order.
- We say that B is **interpretable in** A **with parameters** if there is a sequence $\overline{a}$ of elements of A, such that B is interpretable in $(A, \overline{a})$.
- We shall write $=_\Gamma$ for ϕ_Γ when ϕ is the formula $y_0 = y_1$.
- Wherever possible we shall abbreviate $(\overline{a}_0, \ldots, \overline{a}_{m-1})$ and $(f(\overline{a}_0), \ldots, f(\overline{a}_{m-1}))$ to $\overline{a}$ and $f(\overline{a})$, respectively.

Example: Relativized Reductions

- Suppose B is the relativized reduct A_P.

  Then there is a one-dimensional interpretation Γ of B in A.

  The defining formulas of Γ are as follows.
  - $\partial_\Gamma(x) := P(x)$;
  - $\phi_\Gamma := \phi(\overline{x})$, for each unnested atomic formula $\phi(\overline{y})$.

  The coordinate map $f_\Gamma : P^A \to \text{dom}(A)$ is simply the inclusion map.

  We call the interpretation Γ a **relativized reduction**.

Example: Rationals and Integers

- The familiar interpretation of the rationals in the integers is a two-dimensional interpretation Γ.
  The domain formula is
  $$\partial_\Gamma(x_0,x_1) := x_1 \neq 0.$$
  The other defining formulas are:
  - $=_\Gamma(x_{00},x_{01};x_{10},x_{11}) := x_{00}\cdot x_{11} = x_{01}\cdot x_{10}$;
  - $\text{plus}_\Gamma(x_{00},x_{01};x_{10},x_{11};x_{20},x_{21}) := x_{21}\cdot(x_{00}\cdot x_{11}+x_{01}\cdot x_{10}) = x_{01}\cdot x_{11}\cdot x_{20}$;
  - $\text{times}_\Gamma(x_{00},x_{01};x_{10},x_{11};x_{20},x_{21}) := x_{00}\cdot x_{10}\cdot x_{21} = x_{01}\cdot x_{11}\cdot x_{20}$.

  The formulas ψ_Γ for the remaining unnested atomic formulas ψ express addition and multiplication of rationals in terms of addition and multiplication of integers, just as in the algebra texts.
  The coordinate map is
  $$f_\Gamma((m,n)) = \frac{m}{n}, \quad n \neq 0.$$

Example: Algebraic Extensions (Outline)

- Let A be a field.

  Let $p(X)$ an irreducible polynomial of degree n over A.

  Let ξ be a root of $p(X)$ in some field extending A.

  Given an n-tuple $\overline{a} = (a_0, \ldots, a_{n-1})$ of elements of A, we write

$$q_{\overline{a}}(X) := X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0.$$

  Then there is an n-dimensional interpretation Γ of $A[\xi]$ in A.
  - $\partial_\Gamma(A^n)$ is the whole of A^n;
  - The remaining defining formulas are:
    - $=_\Gamma(\overline{a}, \overline{b})$ says that $p(X)$ divides $(q_{\overline{a}}(X) - q_{\overline{b}}(X))$;
    - $(y_0 + y_1 = y_2)_\Gamma$ and $(y_0 \cdot y_1 = y_2)_\Gamma$ follow the usual definitions of addition and multiplication of polynomials.

    All can be written as positive primitive formulas.
  - $f_\Gamma(\overline{a}) = q_{\overline{a}}(\xi)$.

  ∂_Γ is quantifier-free and ϕ_Γ is p.p. for every unnested atomic ϕ.

Admissibility Conditions

- Let Γ be an interpretation of an L-structure B in a K-structure A.
- There are certain sentences of signature K which must be true in A just because Γ is an interpretation, regardless of what A and B are:
  (i) $=_\Gamma$ defines an equivalence relation on $\partial_\Gamma(A^n)$;
  (ii) For each unnested atomic formula ϕ of L, if $A \models \phi_\Gamma(\overline{a}_0,\ldots,\overline{a}_{n-1})$ with $\overline{a}_0,\ldots,\overline{a}_{n-1}$ in $\partial_\Gamma(A^n)$, then also $A \models \phi_\Gamma(\overline{b}_0,\ldots,\overline{b}_{n-1})$ for each element $\overline{b}_i$ of $\partial_\Gamma(A^n)$ which is $=_\Gamma$-equivalent to $\overline{a}_i$;
  (iii) If $\phi(y_0)$ is a formula of L of form $c = y_0$, then there is $\overline{a}$ in $\partial_\Gamma(A^n)$, such that for all $\overline{b}$ in $\partial_\Gamma(A^n)$, $A \models \phi_\Gamma(\overline{b})$ if and only if $\overline{b}$ is $=_\Gamma$-equivalent to $\overline{a}$;
  (iv) A clause like (iii) for each function symbol.
- These sentences are called the **admissibility conditions** of Γ.
- They generalize the admissibility conditions for a relativized reduct.
- They depend only on on the defining formulas, but not on the coordinate map, of Γ.

The Reduction Theorem

Theorem (Reduction Theorem)

Let A be a K-structure, B an L-structure and Γ an n-dimensional interpretation of B in A. For every formula $\phi(\overline{y})$ of the language $L_{\infty\omega}$, there is a formula $\phi_\Gamma(\overline{x})$ of the language $K_{\infty\omega}$, such that for all $\overline{a}$ from $\partial_\Gamma(A^n)$,

$$B \models \phi(f_\Gamma(\overline{a})) \quad \text{iff} \quad A \models \phi_\Gamma(\overline{a}).$$

- By a previous corollary, every formula of $L_{\infty\omega}$ is equivalent to a formula of $L_{\infty\omega}$ in which all atomic subformulas are unnested.
  We prove the theorem by induction on the complexity of formulas.
  - Atomic formulas are handled by the definition of interpretation.
  - $(\neg\phi)_\Gamma = \neg(\phi_\Gamma)$;
  - $(\bigwedge_{i\in I}\phi_i)_\Gamma = \bigwedge_{i\in I}(\phi_i)_\Gamma$ and $(\bigvee_{i\in I}\phi_i)_\Gamma = \bigvee_{i\in I}(\phi_i)_\Gamma$;
  - $(\forall y\phi)_\Gamma = \forall x_0\dots x_{n-1}(\partial_\Gamma(x_0,\dots,x_{n-1}) \to \phi_\Gamma)$;
  - $(\exists y\phi)_\Gamma = \exists x_0\dots x_{n-1}(\partial_\Gamma(x_0,\dots,x_{n-1}) \wedge \phi_\Gamma)$.

Interpretations and Reduction Maps

- The map $\phi \mapsto \phi_\Gamma$ of the Reduction Theorem depends only on the defining formulas of Γ, and not at all on the coordinate map f_Γ.
- The defining formulas of Γ form an **interpretation of L in K**.
- The map $\phi \mapsto \phi_\Gamma$ of the Reduction Theorem is the **reduction map** of this interpretation.

The Associated Functor (Domain)

Theorem

Let Γ be an n-dimensional interpretation of a signature L in a signature K, and let $\text{Admis}(\Gamma)$ be the set of admissibility conditions of Γ. For every K-structure A which is a model of $\text{Admis}(\Gamma)$, there are an L-structure B and a map $f : \partial_\Gamma(A^n) \to \text{dom}(B)$, such that:

(a) Γ with f forms an interpretation of B in A;

(b) If g and C are such that Γ and g form an interpretation of C in A, then there is an isomorphism $i : B \to C$, such that $i(f(\overline{a})) = g(\overline{a})$, for all $a \in \partial_\Gamma(A^n)$.

- Let A be a model of Γ. Then we build an L-structure B as follows. Define a relation $\sim$ on $\partial_\Gamma(A^n)$ by $\overline{a} \sim \overline{a}'$ iff $A \vDash =_\Gamma(\overline{a}, \overline{a}')$. By (i) of the admissibility conditions, $\sim$ is an equivalence relation. Write $\overline{a}^\sim$ for the equivalence class of $\overline{a}$. $\text{dom}(B)$ is the set of all equivalence classes $\overline{a}^\sim$ with $\overline{a}$ in $\partial_\Gamma(A^n)$.

The Associated Functor (Condition (a))

- We now defined the relations, constants and functions of B.
  - For every relation symbol R of L, we define the relation R^B by
  $$(\overline{a}_0^{\sim},\ldots,\overline{a}_{m-1}^{\sim}) \in R^B \quad \text{iff} \quad A \models \phi_\Gamma(\overline{a}_0,\ldots,\overline{a}_{m-1}),$$
  where $\phi(y_0,\ldots,y_{m-1})$ is $R(y_0,\ldots,y_{m-1})$.
  By (ii) of the admissibility conditions, this is a sound definition.
  - The definitions of c^B and F^B are defined similarly.
  We rely on (iii) and (iv) of the admissibility conditions.

This defines the L-structure B.

We define $f : \partial_\Gamma(A^n) \to \text{dom}(B)$ by $f(\overline{a}) = \overline{a}^{\sim}$.

Then f is surjective. Moreover, B has been defined so as to ensure

$$B \models \phi(f(\overline{a}_0),\ldots,f(\overline{a}_{m-1})) \quad \text{iff} \quad A \models \phi_\Gamma(\overline{a}_0,\ldots,\overline{a}_{m-1}).$$

Hence, Γ and f are an interpretation of B in A. This proves (a).

The Associated Functor ((Condition (b))

- To prove (b), suppose Γ and g are an interpretation of C in A.

  For each tuple $\overline{a} \in \partial_\Gamma(A^n)$, define $i(f(\overline{a}))$ to be $g\overline{a}$.

  Claim: This is a sound definition of an isomorphism $i : B \to C$.

  Suppose $f(\overline{a}) = f(\overline{a}')$. Then $A \models =_\Gamma(\overline{a}, \overline{a}')$. Since g is an interpretation, $g(\overline{a}) = g(\overline{a}')$. Thus, the definition of i is sound.

  A similar argument in the other direction shows that i is injective.

  i is surjective since g is surjective, being an interpretation.

  Clause 3 for f and g show that i is an embedding.

  This proves the claim, and with it the theorem.
- We write ΓA for the structure B of the theorem.
- The Reduction Theorem applies to ΓA as follows:

  For all formulas $\phi(\overline{y})$ of L, all K-structures A satisfying the admissibility conditions of Γ, and all tuples $\overline{a} \in \partial_\Gamma(A^n)$, $\Gamma A \models \phi(\overline{a}^\sim)$ iff $A \models \phi_\Gamma(\overline{a})$.

The Action of Γ on Elementary Embeddings

- Let Γ be an n-dimensional interpretation of a signature L in a signature K.
- Let A and A' be models of the admissibility conditions of Γ.
- Let $e : A \to A'$ be an elementary embedding.
- For every tuple $\overline{a} \in \partial_\Gamma(A^n)$, $e(\overline{a})$ is in $\partial_\Gamma(A'^n)$.

  We have

$$\begin{array}{rl} \overline{a} \in \partial_\Gamma(A^n) & \text{iff} \quad A \vDash \partial_\Gamma(\overline{a}) \\ & \text{iff} \quad A' \vDash \partial_\Gamma(e(\overline{a})) \\ & \text{iff} \quad e(\overline{a}) \in \partial_\Gamma(A'^n). \end{array}$$

- Similarly, if $\overline{c} \in \partial_\Gamma(A^n)$ satisfies $A \vDash =_\Gamma(\overline{a},\overline{c})$, then $A' \vDash =_\Gamma(e(\overline{a}),e(\overline{c}))$.
- Hence, there is a well-defined map $\Gamma e : \text{dom}(\Gamma A) \to \text{dom}(\Gamma A')$, given by

$$(\Gamma e)(\overline{a}^\sim) = (e(\overline{a}))^\sim.$$

The Action of Γ on Elementary Embeddings (Properties)

- We defined $\Gamma e : \text{dom}(\Gamma A) \to \text{dom}(\Gamma A')$, given by
$$(\Gamma e)(\overline{a}^{\sim}) = (e(\overline{a}))^{\sim}.$$
- It can be shown that:
  - $\Gamma 1_A = 1_{\Gamma A}$;
  - If $e_1 : A \to A'$ and $e_2 : A' \to A''$ are elementary embeddings, then
$$\Gamma(e_2 e_1) = (\Gamma e_2)(\Gamma e_1).$$

For the first, we have
$$\Gamma 1_A(\overline{a}^{\sim}) = (1_A(\overline{a}))^{\sim} = \overline{a}^{\sim} = 1_{\Gamma A}(\overline{a}^{\sim}).$$

And for the second
$$\begin{array}{rcl}(\Gamma e_2)((\Gamma e_1)(\overline{a}^{\sim})) & = & (\Gamma e_2)((e_1(\overline{a}))^{\sim}) \\ & = & (e_2(e_1(\overline{a})))^{\sim} \\ & = & \Gamma(e_2 e_1)(\overline{a}^{\sim}).\end{array}$$

The Action of Γ on Elementary Embeddings (Conclusion)

Claim: Γe is an elementary embedding of ΓA into $\Gamma A'$.

Let $\overline{a}$ be a sequence of tuples from $\partial_\Gamma(A^n)$ and ϕ a formula of L.

Then we have

$$\begin{array}{rcll} \Gamma A \models \phi(\overline{a}^\sim) & \text{iff} & A \models \phi_\Gamma(\overline{a}) & \text{(Reduction Theorem)} \\ & \text{implies} & A' \models \phi_\Gamma(e\overline{a}) & \text{(e elementary)} \\ & \text{iff} & \Gamma A' \models \phi((\Gamma e)(\overline{a}^\sim)). & \text{(Reduction Theorem)} \end{array}$$

- The definition of Γe makes sense whenever:
  - A, A' are models of the admissibility conditions of Γ;
  - $e : A \to A'$ is any homomorphism which preserves ∂_Γ and $=_\Gamma$.
- If e also preserves all the formulas ϕ_Γ for unnested atomic formulas ϕ of L, then Γe is a homomorphism from ΓA to $\Gamma A'$.

The Associated Functor of an Interpretation

Theorem

Let Γ be an interpretation of a signature L in a signature K, with admissibility conditions Admis(Γ).

(a) Γ induces a functor, written Func(Γ), from the category of models of Admis(Γ) and elementary embeddings, to the category of L-structures and elementary embeddings.

(b) If the formulas ∂_Γ and ϕ_Γ (for unnested atomic ϕ) are $\exists_1^+$ formulas, then we can extend the functor Func(Γ) in (a), replacing "elementary embeddings" by "homomorphisms".

- We call the functor Func(Γ) in either the (a) or the (b) version, the **associated functor** of the interpretation Γ.
- Usually we shall write it just Γ since there is little danger of confusing the interpretation with the functor.

Interpretations and Maps Between Automorphism Groups

- Suppose Γ is the associated functor of an interpretation of L in K.
- Then whenever ΓA is defined, we have a group homomorphism $\alpha \mapsto \Gamma\alpha$ from $\text{Aut}(A)$ to $\text{Aut}(\Gamma A)$.

Theorem

Let Γ be an interpretation of L in K, and let A be an L-structure such that ΓA is defined. Then the induced homomorphism $h : \text{Aut}(A) \to \text{Aut}(B)$ is continuous.

- It suffices to show that if F is a basic open subgroup of $\text{Aut}(B)$, then there is an open subgroup E of $\text{Aut}(A)$ such that $h(E) \subseteq F$.

  Let F be $\text{Aut}(B)_{(\overline{b})}$, for some tuple $\overline{b}$ of elements of B.

  Let X be a finite set of elements of A such that each element in $\overline{b}$ is of form $f_\Gamma(\overline{a})$, for some tuple $\overline{a}$ of elements of X.

  Then by the definition of h, $h(\text{Aut}(A)_{(X)}) \subseteq \text{Aut}(B)_{(\overline{b})}$.