Introduction to Model Theory

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LSSU Math 500

1 The First-Order Case: Compactness

- Compactness for First-Order Logic
- Types
- Elementary Amalgamation
- Amalgamation and Preservation
- Expanding the Language

Subsection 1

Compactness for First-Order Logic

The Compactness Theorem

Theorem (Compactness Theorem for First-Order Logic)

Let T be a first-order theory. If every finite subset of T has a model, then T has a model.

- Let L be a first-order language and T a theory in L.
  Assume first that every finite subset of T has a non-empty model.
  We employ the following strategy:
  - We show that T can be extended to a Hintikka set T^+ in a larger first-order language L^+.
  - Then, by a previous theorem, some L^+-structure A is a model of T^+.
  - So the reduct $A^+|_L$ will be a model of T.

  Write κ for the cardinality of L.

  Let c_i, $i < \kappa$, be distinct constants not in L.

  We call these constants **witnesses**.

  Let L^+ be the first-order language got by adding the c_i's to L.

  Then L^+ has κ sentences, say ϕ_i, $i < \kappa$.

The Compactness Theorem (Cont'd)

- We shall define an increasing chain $(T_i : i \leq \kappa)$ of theories in L^+, so that the following hold, where all models are L^+-structures.
  1. For each $i \leq \kappa$, every finite subset of T_i has a model.
  2. For each $i < \kappa$, the number of witnesses c_k which are used in T_i but not in $\bigcup_{j<i} T_j$ is finite.

  The definition is by induction on i.

  We put $T_0 = T$.

  At limit ordinals we take $T_\delta = \bigcup_{i<\delta} T_i$.

  Clearly these definitions respect Conditions 1 and 2.

  Note that Condition 1 is true at T_0 because of our assumption that every finite subset of T has a non-empty model.

The Compactness Theorem (Successor Ordinals)

- For successor ordinals $i+1$ we, first, define

$$T'_{i+1} = \begin{cases} T_i \cup \{\phi_i\}, & \text{if every finite subset of this set has a model} \\ T_i, & \text{otherwise} \end{cases}$$

We, then, define T_{i+1} based on T'_{i+1}.

  - Suppose $\phi_i \in T'_{i+1}$ and ϕ_i has the form $\exists x\psi$, for some formula $\psi(x)$.
    Then, by Condition 2 there is a witness which is not used in T'_{i+1}.
    We choose the earliest such witness c_j.
    We define $T_{i+1} = T'_{i+1} \cup \{\psi(c_j)\}$.
  - Suppose $\phi_i \notin T'_{i+1}$ or ϕ_i is not of the form $\exists x\psi$.
    We define $T_{i+1} = T'_{i+1}$.

These definitions clearly ensure Condition 2.

We must show that Condition 1 remains true when $\phi \in T'_{i+1} \cup \{\psi(c_j)\}$.

The Compactness Theorem (Condition 1)

- Let U be a finite subset of T_{i+1}.

  Let A be any L^+-structure which is a model of $U \cup \{\exists x \psi\}$.

  Then there is an element a of A such that $A \models \psi(a)$.

  Take such an element a, and let B be the L^+ -structure which is exactly like A except that $c_j^B = a$.

  Since the witness c_j never occurs in U, B is still a model of U.

  Since c_j never occurs in $\psi(x)$, $B \models \psi(a)$.

  So $B \models \psi(c_j)$.

  This shows that Condition 1 still holds.

The Compactness Theorem (Conclusion)

Claim: T_κ is a Hintikka set for L^+.

By a previous theorem, it suffices to prove three things:

(a) Every finite subset of T_κ has a model. This holds by Condition 1.

(b) For every sentence ϕ of L^+, either ϕ or $\neg\phi$ is in T_κ.
To prove this, suppose ϕ is ϕ_i and $\neg\phi$ is ϕ_j. If $\phi \notin T_\kappa$, then $\phi_i \notin T_{i+1}$. Thus, there is a finite subset U of T_i, such that $U \cup \{\phi\}$ has no model. By the same argument, if $\neg\phi \notin T_\kappa$, then there is a finite subset U' of T_j, such that $U' \cup \{\neg\phi\}$ has no model. Now $U \cup U'$ is a finite subset of T_κ. So it has a model A. Either $A \models \phi$ or $A \models \neg\phi$. We have a contradiction either way. Thus at least one of $\phi, \neg\phi$ is in T_κ.

(c) For every sentence $\exists x \psi(x)$ in T_κ, there is a closed term t of L^+, such that $\psi(t) \in T_\kappa$.
For this, suppose $\exists x \psi(x)$ is ϕ_i. Since $\phi_i \in T_\kappa$, $\phi_i \in T'_{i+1}$. So T_{i+1} contains a sentence $\psi(c_j)$, where c_j is a witness. Then $\psi(c_j)$ is in T_κ.

Thus T_κ is a Hintikka set T^+ for L^+ and $T \subseteq T^+$. So T has a model.

In the exceptional case when some finite subset of T has only the empty model, the empty L-structure must be a model of all T.

Compactness for First-Order Theories

Corollary

If T is a first-order theory, ψ a first-order sentence and $T \vdash \psi$, then $U \vdash \psi$, for some finite subset U of T.

- Suppose to the contrary that $U \nvdash \psi$, for every finite subset U of T.

  Thus, for every finite subset U of T, there exists a model of U which does not satisfy ψ.

  Equivalently, every finite subset of $T \cup \{\neg\psi\}$ has a model.

  So, by the Compactness Theorem, $T \cup \{\neg\psi\}$ has a model.

  Therefore, $T \nvdash \psi$.

Recursive Enumeration

- A set is **recursively enumerable** (**r.e.** for short) if and only if it can be listed by a Turing machine.

Corollary

Suppose L is a recursive first-order language, and T is a recursively enumerable theory in L. Then the set of consequences of T in L is also recursively enumerable.

- Using one's favorite proof calculus, ne can recursively enumerate all the consequences in L of a finite set of sentences.

  Since T is r. e., we can recursively enumerate its finite subsets.

  The preceding corollary says that every consequence of T is a consequence of one of these finite subsets.

Upward Löwenheim-Skolem Theorem

- First-order logic cannot distinguish between infinite cardinals.
- So every infinite structure has arbitrarily large elementary extensions.

Corollary (Upward Löwenheim-Skolem Theorem)

Let L be a first-order language of cardinality $\leq \lambda$ and A an infinite L-structure of cardinality $\leq \lambda$. Then A has an elementary extension of cardinality λ.

- Name the elements of A.

  Let $\text{eldiag}(A)$ be the elementary diagram of A.

  Let c_i, $i < \lambda$, be λ new constants.

  Define
  $$T = \text{eldiag}(A) \cup \{c_i \neq c_j : i < j < \lambda\}.$$

Upward Löwenheim-Skolem Theorem (Cont'd)

- Claim: Every finite subset of T has a model.

  Suppose U is a finite subset of T. Then for some $n < \omega$, just n of the new constants c_i occur in U. Since A is infinite, we can choose n distinct elements of A. A model of T assigns to each c_i one of these elements.

  By the Compactness Theorem, T has a model B.

  Since B is a model of $\text{eldiag}(A)$, by the Elementary Diagram Lemma, there is an elementary embedding $e : A \to B|_L$.

  Replacing elements of the image of e by the corresponding elements of A, we make $B|_L$ an elementary extension of A.

  Since $B \models T$, we have $c_i^B \neq c_j^B$, whenever $i < j < \lambda$.

  Hence $B|_L$ has at least λ elements.

  To bring the cardinality of $B|_L$ down to exactly λ, we invoke the downward Löwenheim-Skolem theorem.

Compactness in Infinitary Languages?

- The compactness theorem fails for infinitary languages.

  Example: Let c_i, $i < \omega$, be distinct constants.

  Consider the theory T consisting of

$$c_0 \neq c_1, c_0 \neq c_2, c_0 \neq c_3, \ldots,$$
$$\bigvee_{0<i<\omega} c_0 = c_i.$$

  Every proper subset of T has a model.

  But T itself has no model.

Subsection 2

Types

Complete Types

- Let L be a first-order language and A an L-structure.
- Let X be a set of elements of A and $\overline{b}$ a tuple of elements of A.
- Let $\overline{a}$ be a sequence listing the elements of X.
- The **complete type of** $\overline{b}$ **over** X (**with respect to** A, **in the variables** $\overline{x}$) is the set of all formulas $\psi(\overline{x}, \overline{a})$, such that:
  - $\psi(\overline{x}, \overline{y})$ is in L;
  - $A \models \psi(\overline{b}, \overline{a})$.
- More loosely, the complete type of $\overline{b}$ over X is everything we can say about $\overline{b}$ in terms of X.
- The tuple $\overline{a}$ may be infinite, but, since each formula $\psi(\overline{x}, \overline{y})$ of L has only finitely many free variables, only a finite part of X is mentioned in $\psi(\overline{x}, \overline{a})$.

Notation on Complete Types

- We denote the complete type of $\overline{b}$ over X with respect to A by $\mathrm{tp}_A(\overline{b}/X)$, or $\mathrm{tp}_A(\overline{b}/\overline{a})$, where $\overline{a}$ lists the elements of X.
- The elements of X are called the **parameters** of the complete type.
- Complete types are written p, q, r etc.
- One writes $p(\overline{x})$ if one wants to show that the variables of the type are $\overline{x}$.
- We write $\mathrm{tp}_A(\overline{b})$ for $\mathrm{tp}_A(\overline{b}/\varnothing)$, the type of $\overline{b}$ over the empty set of parameters.
- Note that if B is an elementary extension of A, then

$$\mathrm{tp}_B(\overline{b}/X) = \mathrm{tp}_A(\overline{b}/X).$$

Complete Types over a Set of Elements

- Let $p(\overline{x})$ be a set of formulas of L with parameters from X.
- We say that $p(\overline{x})$ is a **complete type over** X (with respect to A, in the variables $\overline{x}$) if it is the complete type of some tuple $\overline{b}$ over X with respect to some elementary extension of A.
- Putting it loosely again, a complete type over X is everything we can say in terms of X about some possible tuple $\overline{b}$ of elements that are in A or, perhaps, in an elementary extension of A.

Types and Realizability

- A **type** over X (with respect to A, in the variables $\overline{x}$) is a subset of a complete type over X.
- We shall write $\Phi, \Psi, \Phi(\overline{x})$ etc. for types.
- A type is called an *n*-**type**, $n < \omega$, if it has just n free variables.
- We say that a type $\Phi(\overline{x})$ over X is **realized** by a tuple $\overline{b}$ in A if $\Phi \subseteq \mathrm{tp}_A(\overline{b}/X)$.
- If Φ is not realized by any tuple in A, we say that A **omits** Φ.
- We say that a set $\Phi(\overline{x})$ of formulas of L, with parameters in A, is **finitely realized** in A if for every finite subset Ψ of Φ,

$$A \models \exists \overline{x} \bigwedge \Psi.$$

Characterization of Types and Complete Types

Theorem

Let L be a first-order language, A an L-structure, X a set of elements of A and $\Phi(x_0, \ldots, x_{n-1})$ a set of formulas of L with parameters from X. Then, writing $\overline{x}$ for $(x_0, \ldots, x_{n-1})$,

(a) $\Phi(\overline{x})$ is a type over X with respect to A if and only if Φ is finitely realized in A;

(b) $\Phi(\overline{x})$ is a complete type over X with respect to A if and only if $\Phi(\overline{x})$ is a set of formulas of L with parameters from X, which is maximal with the property that it is finitely realized in A.

In particular, if Φ is finitely realized in A, then it can be extended to a complete type over X with respect to A.

Characterization of Types (Proof)

(a) Suppose Φ is a type over X with respect to A.

Then, there are an elementary extension B of A and an n-tuple $\overline{b}$ in B, such that $B \models \bigwedge \Phi(\overline{b})$.

Let Ψ be a finite subset of Φ.

Then $B \models \bigwedge \Psi(\overline{b})$. Hence, $B \models \exists \overline{x} \bigwedge \Psi(\overline{x})$.

But $A \preccurlyeq B$ and the sentence is first-order. So $A \models \exists \overline{x} \bigwedge \Psi(\overline{x})$.

For the converse, we use again elementary diagrams.

Suppose Φ is finitely realized in A.

Form eldiag(A).

Take an n-tuple of distinct new constants $\overline{c} = (c_0, \ldots, c_{n-1})$.

Define T to be the theory

$$T = \text{eldiag}(A) \cup \Phi(\overline{c}).$$

Characterization of Types (Cont'd)

- Claim: Every finite subset of T has a model.

  Let U be a finite subset of T.

  Let Ψ be the set of formulas $\psi(\overline{x})$ of Φ, such that $\psi(\overline{c}) \in U$.

  By assumption $A \models \exists \overline{x} \bigwedge \Psi$. Hence, for some $\overline{a}$ in A, $A \models \bigwedge \Psi(\overline{a})$.

  By interpreting the constants $\overline{c}$ as names of the elements $\overline{a}$, we make A into a model of U. This proves the claim.

  By the Compactness Theorem, T has a model C.

  Since $C \models \text{eldiag}(A)$, by the Elementary Diagram Lemma, there exists an elementary embedding $e : A \to C|_L$.

  By making the usual replacements, we can assume that $A \preccurlyeq C|_L$.

  Let $\overline{b}$ be the tuple $\overline{c}^C$. Since $C \models T$, $C \models \bigwedge \Phi(\overline{b})$.

  So $\overline{b}$ satisfies $\Phi(\overline{x})$ in some elementary extension of A.

  We conclude that Φ is a type over X with respect to A.

Characterization of Complete Types

(b) Suppose Φ is a complete type over X.

Then Φ contains either ϕ or $\neg\phi$, for each formula $\phi(\overline{x})$ of L with parameters from X.

This implies that Φ is a maximal type over X with respect to A.

Suppose, now, that Φ is a maximal type over X with respect to A.

Then for some $\overline{b}$ in some elementary extension B of A, $B \models \bigwedge\Phi(\overline{b})$.

So Φ is included in the complete type of $\overline{b}$ over X.

By maximality, it must equal this complete type.

Types of First-Order Theories

- By the Characterization Theorem, if X is the empty set of parameters, then the question whether Φ is a type over X with respect to A depends only on $\mathrm{Th}(A)$.
- Types over the empty set with respect to A are also known as the **types of** $\mathrm{Th}(A)$.
- More generally, let T be any theory in a first-order language.
- A **type of** T is a set $\Phi(\overline{x})$ of formulas of L such that $T \cup \{\exists \overline{x} \bigwedge \Psi\}$ is consistent for every finite subset $\Psi(\overline{x})$ of Φ.
- A **complete type of** T is a maximal type of T.
- If T happens to be a complete theory, then we can replace "$T \cup \{\exists \overline{x} \bigwedge \Psi\}$ is consistent" by the equivalent "$T \vdash \exists \overline{x} \bigwedge \Psi$".

Stone Spaces of a Structure

- Let A be an L-structure.
- Let X be a set of elements of A.
- Let n be a positive integer.
- Denote $S_n(X;A)$ the set of complete n-types over X with respect to A.
- When A is fixed we write simply $S_n(X)$.
- When T is a complete theory, we write $S_n(T)$ for the set of complete types of T.
- The sets $S_n(X;A)$ are known as the **Stone spaces** of A.

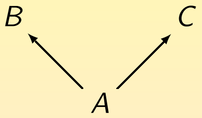
Subsection 3

Elementary Amalgamation

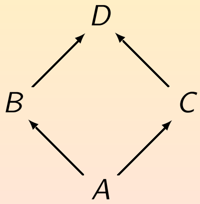
Amalgamation Theorems

- An **amalgamation theorem** is a theorem of the following shape:

  We are given two models B, C of some theory T, and a structure A (not necessarily a model of T), which is embedded into both B and C.

  The theorem states that there is a third model D of T, such that both B and C are embeddable into D by embeddings which agree on A. The embeddings may be required to preserve certain formulas.

Construction and Classification

- There are two ways of using amalgamation.
  - One is to build up a structure M by:
    - Taking smaller structures;
    - Extending them;
    - Amalgamating the extensions.
  - The second way is not to construct but to classify.
    - We classify all the ways of extending the bottom structure A;
    - Then we classify the ways of amalgamating these extensions.

  In favorable cases this leads to a structural classification of all the models of a theory.

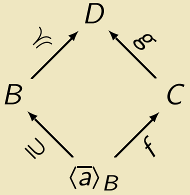
  Stability theory is an example that follows this path.

Elementary Amalgamation Theorem

Theorem (Elementary Amalgamation Theorem)

Let L be a first-order language. Let B and C be L-structures and $\overline{a},\overline{c}$ sequences of elements of B,C, respectively, such that $(B,\overline{a})\equiv(C,\overline{c})$.

Then there exist an elementary extension D of B and an elementary embedding $g: C \to D$, such that $g(\overline{c}) = \overline{a}$. In a picture, where $f : \langle\overline{a}\rangle \to C$ is the unique embedding which takes $\overline{a}$ to $\overline{c}$ (by the Diagram Lemma).



- Replacing C by an isomorphic copy if necessary, we can assume that $\overline{a}=\overline{c}$, and otherwise B and C have no elements in common. Consider the theory

$$T = \mathrm{eldiag}(B)\cup\mathrm{eldiag}(C),$$

where each element names itself.

Elementary Amalgamation Theorem (Lemma)

Claim: T has a model.

By the Compactness Theorem, it suffices to show that every finite subset of T has a model.

Let T_0 be a finite subset of T.

T_0 contains just finitely many sentences from $\text{eldiag}(C)$.

Let their conjunction be $\phi(\overline{a},\overline{d})$, where:

- $\phi(\overline{x},\overline{y})$ is a formula of L;
- $\overline{d}$ consists of pairwise distinct elements in C but not in $\overline{a}$.

Of course only finitely many variables in $\overline{x}$ occur free in ϕ.

If T_0 has no model then $\text{eldiag}(B) \vdash \neg\phi(\overline{a},\overline{d})$.

But the elements $\overline{d}$ are distinct and they are not in B.

So, by the Lemma on Constants, $\text{eldiag}(B) \vdash \forall\overline{y}\neg\phi(\overline{a},\overline{y})$.

But then $(B,\overline{a}) \vDash \forall\overline{y}\neg\phi(\overline{a},\overline{y})$. So $(C,\overline{c}) \vDash \forall\overline{y}\neg\phi(\overline{c},\overline{y})$ by hypothesis.

This contradicts that $\phi(\overline{a},\overline{d})$ is in $\text{eldiag}(C)$.

Elementary Amalgamation Theorem (Conclusion)

- Let D^+ be a model of T.

  Let D be the reduct $D^+|_L$.

  Now $D^+ \models \mathrm{eldiag}(B)$.

  By the Elementary Diagram Lemma, we can assume that:
  - D is an elementary extension of B;
  - $b^{D^+} = b$, for all elements b of B.

  Define $g(d) = d^{D^+}$, for each element d of C.

  Now $D^+ \models \mathrm{eldiag}(C)$.

  By the Elementary Diagram Lemma again, g is an elementary embedding of C into D.

  Finally

$$\begin{array}{rcll} g(\overline{c}) & = & g(\overline{a}) & (\overline{a} = \overline{c}) \\ & = & \overline{a}^{D^+} & (\text{definition of } g) \\ & = & \overline{a}. & (\overline{a} \text{ in } B) \end{array}$$

Consequences

- In the theorem $\overline{a}$ can be empty.

  In this case the theorem says that any two elementarily equivalent structures can be elementarily embedded together into some structure.
- The theorem can be rephrased as follows:

  If $(B, \overline{a}) \equiv (C, \overline{c})$ and $\overline{d}$ is any sequence of elements of C, then there is an elementary extension B' of B containing elements $\overline{b}$ such that $(B', \overline{a}, \overline{b}) \equiv (C, \overline{c}, \overline{d})$.
- One of the most important consequences is the following:

  If A is any structure, we can simultaneously realize all the complete types with respect to A in a single elementary extension of A.

  This is discussed in the following result.

Realization of Types in Elementary Extensions

Corollary

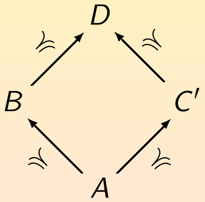
Let L be a first-order language and A an L-structure. Then there is an elementary extension B of A, such that every type over $\text{dom}(A)$ with respect to A is realized in B.

- It suffices to realize all maximal types over $\text{dom}(A)$ with respect to A.
  Let these be P_i, $i < \lambda$, with λ a cardinal.
  For $i < \lambda$, let $A \preccurlyeq A_i$ and $\overline{a}_i$ in A_i, such that $p_i = \text{tp}_{A_i}(\overline{a}_i / \text{dom} A)$.
  Define an elementary chain $(B_i : i \leq \lambda)$ by induction as follows:
  - B_0 is A;
  - For each limit ordinal $\delta \leq \lambda$, $B_\delta = \bigcup_{i<\delta} B_i$ (which is an elementary extension of each B_i by a previous theorem).
  - When B_i has been defined and $i < \lambda$, use the theorem to choose B_{i+1} to be an elementary extension of B_i, such that there is an elementary embedding $e_i : A_i \to B_{i+1}$ which is the identity on A.

  Put $B = B_\lambda$. For each $i < \lambda$, $e_i(\overline{a}_i)$ is a tuple in B_λ realizing p_i.

The Case of Elementary Extensions

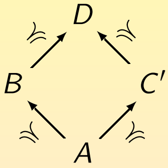
- Consider the case of the theorem where $\overline{a}$ lists the elements of an elementary substructure A of B.
- In this case the theorem tells us the following.
- If A, B and C are L-structures and $A \preccurlyeq B$ and $A \preccurlyeq C$,



then there are an elementary extension D of B and an elementary embedding $g : C \to D$, such that, putting $C' = g(C)$, the shown diagram of elementary inclusions commutes.

Heir-Coheir Amalgams

- Consider again the diagram



- We call it an **heir-coheir amalgam** if:

  For every first-order formula $\psi(\overline{x},\overline{y})$ of L and all tuples $\overline{b},\overline{c}$ from B, C', respectively, if $D \models \psi(\overline{b},\overline{c})$, then there is $\overline{a}$ in A, such that $B \models \psi(\overline{b},\overline{a})$.
- We say also that it is an **heir-coheir amalgam of** B **and** C **over** A.
- It is an **heir-coheir amalgam of** B'' **and** C'' **over** A whenever B'' and C'' are elementary extensions of A, such that there are isomorphisms $i : B'' \to B$ and $j : C'' \to C'$ which are the identity on A.

Example: Vector Spaces

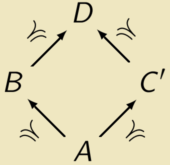
- Suppose A is an infinite vector space over a field K.
  Let B and C be vector spaces with A as subspace.
  Put $B = B_1 \oplus A$ and $C = C_1 \oplus A$.
  We can amalgamate B and C over A by putting $D = B_1 \oplus C_1 \oplus A$.
  Suppose some equation
  $$\sum_{i<m} \lambda_i b_i = \sum_{j<n} \mu_j c_j$$
  holds in D, where the b_i are in B and the c_j are in C.
  Let $\pi : D \to B_1 \oplus A$ be the projection along C_1.
  Then $\sum_{i<m} \lambda_i \pi(b_i) = \sum_{j<n} \mu_j \pi(c_j)$.
  But $\pi(b_i) = b_i$ and $\pi(c_j)$ lies in A.
  Thus, the heir-coheir condition holds for $\psi := \sum_{i<m} \lambda_i x_i = \sum_{j<n} \mu_j y_j$.
  In fact, since A is infinite, one can show that the condition holds whenever ψ is quantifier free. Then, by quantifier elimination, it follows that D forms an heir-coheir amalgam of B and C over A.

Amalgam of Elementary Extensions

- The next theorem says that heir-coheir amalgams always exist when B and C are elementary extensions of A.

Theorem

Let A, B and C be L-structures such that $A \preccurlyeq B$ and $A \preccurlyeq C$. Then there exist an elementary extension D of B and an elementary embedding $g : C \to D$ such that the diagram (with $C' = g(C)$) is an heir-coheir amalgam.



- We assume that $(\operatorname{dom} B) \cap (\operatorname{dom} C) = \operatorname{dom}(A)$, so that constants behave properly in diagrams. Then we take T to be the theory

$$\operatorname{eldiag}(B) \cup \operatorname{eldiag}(C) \cup \{\neg\psi(\overline{b}, \overline{c}) : \psi \text{ is a first-order formula of } L \text{ and } \overline{b} \text{ is a tuple in } B, \text{ such that } B \models \neg\psi(\overline{b}, \overline{a}) \text{ for all } \overline{a} \text{ in } A\}.$$

Amalgam of Elementary Extensions (Cont'd)

- Suppose T has no model. By the Compactness Theorem, there are:
  - A tuple $\overline{a}$ from A;
  - A tuple $\overline{d}$ of distinct elements in C but not in A;
  - A tuple $\overline{b}$ of elements of B;
  - A sentence $\theta(\overline{a},\overline{d})$ in $\text{eldiag}(C)$;
  - Sentences $\psi_i(\overline{b},\overline{a},\overline{d})$, $i<k$;

  such that:
  - $B \models \neg\psi_i(\overline{b},\overline{a}',\overline{a}'')$, for all $\overline{a}',\overline{a}''$ in A;
  - $\text{eldiag}(B) \vdash \theta(\overline{a},\overline{d}) \to \psi_0(\overline{b},\overline{a},\overline{d}) \vee \cdots \vee \psi_{k-1}(\overline{b},\overline{a},\overline{d})$.

  Quantifying out the constants $\overline{d}$, by the Lemma on Constants, we get

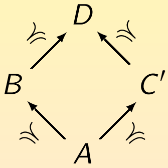
$$B \models \forall\overline{y}(\theta(\overline{a},\overline{y}) \to \psi_0(\overline{b},\overline{a},\overline{y}) \vee \cdots \vee \psi_{k-1}(\overline{b},\overline{a},\overline{y})).$$

  We also have $C \models \exists\overline{y}\theta(\overline{a},\overline{y})$. So $A \models \exists\overline{y}\theta(\overline{a},\overline{y})$. Hence, $A \models \theta(\overline{a},\overline{a}'')$, for some $\overline{a}''$ in A. So $B \models \theta(\overline{a},\overline{a}'')$. Thus, $B \models \psi_i(\overline{b},\overline{a},\overline{a}'')$, for some $i<k$. This is a contradiction.

  The rest of the proof is as in the Elementary Amalgamation Theorem.

The Strong Elementary Amalgamation Property

- If the diagram is an heir-coheir amalgam, then the overlap of B and C in D is precisely A.



  Suppose $b = g(c)$, for some b in B and some c in C.

  By the heir-coheir property, $b = a$, for some a in A.
- Amalgams with this minimum-overlap property are said to be **strong**.
- In this terminology we have just shown that first-order logic has the **strong elementary amalgamation property**.

Example: Vector Spaces (Cont'd)

- We present a more abstract proof that $D = B_1 \oplus C_1 \oplus A$ is an heir-coheir amalgam of $B = B_1 \oplus A$ and $C = C_1 \oplus A$ over A.

  Since A is infinite, $A \preccurlyeq B$ and $A \preccurlyeq C$ by quantifier elimination.

  By the theorem, some vector space D' forms an heir-coheir amalgam of B and C over A.

  Identifying B and C with their images in D', we may suppose that B and C generate D'.

  If D'' is the subspace of D' generated by B and C, then, by quantifier elimination $D'' \preccurlyeq D'$.

  Now D' is a strong amalgam of B and C over A.

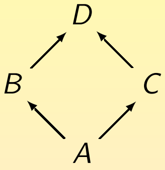
  This means precisely that $D' = B_1 \oplus C_1 \oplus A$.

  So D' is D.

  Thus, D is an heir-coheir amalgam of B and C over A.

Example: Algebraically Closed Fields

- Suppose that in the figure:
  - Both B and C are the field of complex numbers;
  - A is the field of reals;
  - D is some algebraically closed field which amalgamates B and C over A.



  Let:
  - $i, -i$ be the square roots of -1 regarded as elements of B;
  - $j, -j$ be the square roots of -1 regarded as elements of C.

  Then in D, i must be identified with either j or $-j$.

  So the amalgam is not strong.
- This example shows that, if $\langle \overline{a} \rangle_B$ in the Elementary Amalgamation Theorem is not algebraically closed in B, then, in general, there is no hope of making the amalgam D strong.

Algebraic Elements and Algebraically Closed Subsets

- Let B be an L-structure.
- Let X be a set of elements of B.
- We say that an element b of B is **algebraic over** X if there are a first-order formula $\phi(x, \overline{y})$ of L and a tuple $\overline{a}$ in X, such that

$$B \models \phi(b, \overline{a}) \wedge \exists_{\leq n} x \phi(x, \overline{a}),$$

  for some finite n.
- We write $\text{acl}_B(X)$ for the set of all elements of B algebraic over X.
- If $\overline{a}$ lists the elements of X, we also write $\text{acl}_B(\overline{a})$, for $\text{acl}_B(X)$.

Algebraically Closed Subsets

- Let B be an L-structure.
- Let X be a set of elements of B.
- The operator acl satisfies the following properties.
    1. $X \subseteq \text{acl}_B(X)$;
    2. $Y \subseteq \text{acl}_B(X)$ implies $\text{acl}_B(Y) \subseteq \text{acl}_B(X)$;
    3. If $B \preccurlyeq C$, then $\text{acl}_B(X) = \text{acl}_C(X)$.
- By Property 3, we can often write $\text{acl}(X)$ for $\text{acl}_B(X)$ without danger of confusion.
- We say that a tuple $\overline{b}$ is **algebraic over** X if every element in $\overline{b}$ is algebraic over X.
- We say that a type $\Phi(\overline{x})$ over a set X with respect to B is **algebraic** if every tuple realizing it is algebraic over X.

Non-algebraic Elements and Elementary Extensions

Lemma

Let B be an L-structure, X a set of elements of B listed as $\overline{a}$, and b an element of B. Suppose $b \notin \text{acl}_B(X)$.

(a) There is an elementary extension A of B with an element $c \notin \text{dom}(B)$, such that $(B, \overline{a}, b) \equiv (A, \overline{a}, c)$.

(b) There is an elementary extension D of B, with an elementary substructure C containing X, such that $b \notin \text{dom}(C)$.

(a) Let c be a new constant.

Let $p(x)$ be the complete type of b over X.

It suffices to show

$$\text{eldiag}(B) \cup p(c) \cup \{c \neq d : d \in \text{dom}(B)\}$$

has a model.

Non-algebraic Elements and Elementary Extensions (b)

- Suppose that $\text{eldiag}(B) \cup p(c) \cup \{c \neq d : d \in \text{dom}(B)\}$ has no model.

  By the Compactness Theorem and the Lemma on Constants, there are finitely many $d_0, \ldots, d_{n-1}$ in B and a formula $\phi(x)$ of $p(x)$ (note $p(x)$ is closed under $\wedge$), such that

  $$\text{eldiag}(B) \vdash \forall x(\phi(x) \to x = d_0 \vee \cdots \vee x = d_{n-1}).$$

  Hence $B \models \phi(b) \wedge \exists_{\leq n} x \phi(x)$.

  We conclude that $b \in \text{acl}_B(X)$, a contradiction.

(b) Take A and c as in Part (a).

Since $(A, \overline{a}, b) \equiv (A, \overline{a}, c)$, the Amalgamation Theorem gives us an elementary extension D of A and an elementary embedding $g : A \to D$, such that $g(\overline{a}) = \overline{a}$ and $g(b) = c$.

Then D is an elementary extension of $g(B)$ and $g(b) = c \notin \text{dom}(B)$.

So the lemma holds if $g(B)$ and B are taken for B and C, respectively.

Strong Amalgamation over Algebraically Closed Sets

- We can make the amalgam strong in the amalgamation theorem whenever $\langle \overline{a} \rangle_B$ is algebraically closed in B (or in C, by symmetry).

Theorem (Strong Elementary Amalgamation over Algebraically Closed Sets)

Let B and C be L-structures and $\overline{a}$ a sequence of elements in both B and C such that $(B, \overline{a}) \equiv (C, \overline{a})$. Then there exist an elementary extension D of B and an elementary embedding $g : C \to D$, such that $g(\overline{a}) = \overline{a}$ and $(\text{dom} B) \cap g(\text{dom} C) = \text{acl}_B(\overline{a})$.

- Replacing C by an isomorphic copy if necessary, we can assume that B and C have no elements in common other than those in $\overline{a}$.

  Consider the theories

$$\begin{aligned} T &= \text{eldiag}(B) \cup \text{eldiag}(C); \\ T^+ &= T \cup \{b \neq c : b \in \text{dom}(B) \setminus \text{acl}_B(\overline{a}) \text{ and } c \in \text{dom}(C) \setminus \text{acl}_C(\overline{a})\}, \end{aligned}$$

  where each element names itself.

Amalgamation over Algebraically Closed Sets (Cont'd)

- Suppose we have shown that T^+ has a model.

  Suppose D and g are defined, as in the Amalgamation Theorem, using T^+ in place of T.

  Then $g(\overline{a}) = \overline{a}$.

  It easily follows that g maps $\mathrm{acl}_C(\overline{a})$ onto $\mathrm{acl}_B(\overline{a})$.

  Thus, we have $\mathrm{acl}_B(\overline{a}) \subseteq (\mathrm{dom} B) \cap g(\mathrm{dom} C)$.

  The sentences "$b \neq c$" guarantee the opposite inclusion.

  It remains only to show that T^+ has a model.

  Assume for contradiction that T^+ has no model.

  By compactness, there are finite subsets Y of $\mathrm{dom}(B) \setminus \mathrm{acl}_B(\overline{a})$ and Z of $\mathrm{dom}(C) \setminus \mathrm{acl}_C(\overline{a})$, such that for every elementary extension D of B and elementary embedding $g : C \to D$, with $g(\overline{a}) = \overline{a}$, $Y \cap g(Z) \neq \varnothing$.

  Choose D and g to make $Y \cap g(Z)$ as small as possible.

  To save notation we can assume that g is the identity so that $C \preccurlyeq D$.

Amalgamation over Algebraically Closed Sets (Cont'd)

- Since $Y \cap Z \neq \emptyset$, there is some $b \in Y \cap Z$.

  By the lemma, there is an elementary extension D' of D, with an elementary substructure C' containing $\overline{a}$, such that $b \notin \text{dom}(C')$.

  Applying the preceding theorem to the elementary embedding $C' \preccurlyeq D'$ (the same embedding twice over), we find:
  - An elementary extension E of D';
  - An elementary embedding $e: D' \to E$ which is the identity on C', such that
  $$(\text{dom}D') \cap e(\text{dom}D') = \text{dom}(C').$$

  Now we finish the proof by showing that $Y \cap e(Z) \subsetneqq Y \cap Z$.
  - $Y \cap e(Z) \subseteq Y \cap Z$.
    Suppose $d \in Y \cap e(Z)$. Then d is in C'. Hence, $e(d) = d$.
  - b is in $(Y \cap Z) \backslash (Y \cap e(Z))$.
    b is in D' but not in C'. So $b \notin e(\text{dom}D')$. Hence, $b \notin e(Z)$.

  Thus, e contradicts the choice of $Y \cap g(Z)$ as minimal.

Subsection 4

Amalgamation and Preservation

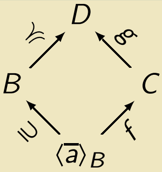
Preservation of Existential Sentences

- Let L be a language.
- Let A and B be L-structures.
- We write $A \Rrightarrow_1 B$ to mean that:
  For every first-order existential sentence ϕ of L,
$$A \models \phi \quad \text{implies} \quad B \models \phi.$$
- Likewise we write $A \Rrightarrow_1^+ B$ to mean that:
  For every first-order $\exists_1^+$ sentence of L,
$$A \models \phi \quad \text{implies} \quad B \models \phi.$$
- Note that $\Rrightarrow_1$ implies $\Rrightarrow_1^+$.
- Note, also, that if $f : \langle \overline{a} \rangle_B \to C$ is a homomorphism, then
$$(C, f(\overline{a})) \Rrightarrow_1^+ (B, \overline{a}) \quad \text{implies} \quad f \text{ is an embedding.}$$

Existential Amalgamation Theorem

Theorem (Existential Amalgamation Theorem)

Let B and C be L-structures, $\overline{a}$ a sequence of elements of B and $f : \langle \overline{a} \rangle \to C$ a homomorphism such that $(C, f(\overline{a})) \Rrightarrow_1 (B, \overline{a})$. Then there exist an elementary extension D of B and an embedding $g : C \to D$, such that $g(f(\overline{a})) = \overline{a}$. In a picture, where $(C, f(\overline{a})) \Rrightarrow_1 (B, \overline{a})$.



- The assumptions imply that f is an embedding.

  So we can replace C by an isomorphic copy and assume that f is the identity on $\langle \overline{a} \rangle_B$, and that $\langle \overline{a} \rangle_B$ is the overlap of $\text{dom}(B)$ and $\text{dom}(C)$.

  As in the Amalgamation Theorem, it suffices to show that the theory $T = \text{eldiag}(B) \cup \text{diag}(C)$ has a model.

  If T has no model, by compactness, there is a conjunction $\phi(\overline{a}, \overline{d})$ of finitely many sentences in $\text{diag}(C)$, such that $(B, \overline{a}) \models \neg \exists \overline{y} \phi(\overline{a}, \overline{y})$.

  Since $\phi(\overline{a}, \overline{y})$ is quantifier-free and $(C, \overline{a}) \Rrightarrow_1 (B, \overline{a})$, we infer that $(C, \overline{a}) \models \neg \exists \overline{y} \phi(\overline{a}, \overline{y})$. This contradicts that $\phi(\overline{a}, \overline{d})$ is true in C.

The case of Empty Tuple

- Since we allow structures to be empty, the tuple $\overline{a}$ in the theorem can be the empty tuple.

Corollary

Let B and C be L-structures such that $C \Rrightarrow_1 B$. Then C is embeddable in some elementary extension of B.

- Amalgamation theorems like the preceding theorem tend to spawn offspring of the following kinds:
  (i) Criteria for a structure to be expandable or extendable in certain ways;
  (ii) Syntactic criteria for a formula or set of formulas to be preserved under certain model theoretic operations (results of this kind are called **preservation theorems**);
  (iii) Interpolation theorems.

  We provide examples.

Extendability of a Structure to a Model

- Let T be a theory in a first-order language L.
- $T_\forall$ is the set of all $\forall_1$ sentences of L which are consequences of T.

Corollary

Let T be a theory in a first-order language L. Then the models of $T_\forall$ are precisely the substructures of models of T.

- Any substructure of a model of T is certainly a model of $T_\forall$ by a previous result.

  Conversely, let C be a model of $T_\forall$.

  We must show that C is a substructure of a model of T.

  By the corollary, it suffices to find a model B of T such that $C \Rrightarrow_1 B$.

  We find B as follows.

Extendability of a Structure to a Model (Cont'd)

- Let U be the set of all $\exists_1$ sentences ϕ of L, such that $C \models \phi$.

  Claim: $T \cup U$ has a model.

  If not, then by the compactness theorem, there is some finite set $\{\phi_0, \ldots, \phi_{k-1}\}$ of sentences in U, such that $T \vdash \neg\phi_0 \vee \cdots \vee \neg\phi_{k-1}$.

  $\neg\phi_0 \vee \cdots \neg\phi_{k-1}$ is logically equivalent to an $\forall_1$ sentence θ.

  Moreover, $T \vdash \theta$. So $\theta \in T_\forall$.

  Hence, $C \models \theta$.

  This is absurd, since $C \models \phi_i$, for each $i < k$.

  So $T \cup U$ has a model as claimed.

  Let B^+ be any model of $T \cup U$.

  Let B the L-reduct of B^+.

Characterization of Formulas Preserved by Substructures

Theorem (Łoś-Tarski Theorem)

Let T be a theory in a first-order language L and $\Phi(\overline{x})$ a set of formulas of L. (The sequence of variables $\overline{x}$ need not be finite.) Then the following are equivalent:

(a) If A and B are models of T, $A \subseteq B$, $\overline{a}$ is a sequence of elements of A and $B \models \bigwedge \Phi(\overline{a})$, then $A \models \bigwedge \Phi(\overline{a})$. ($\Phi$ is preserved in substructures for models of T.)

(b) Φ is equivalent modulo T to a set $\Psi(\overline{x})$ of $\forall_1$ formulas of L.

(b)⇒(a) By a previous corollary.

(a)⇒(b) Suppose (a) holds.

We first prove (b) under the assumption that Φ is a set of sentences. Define

$$\Psi := (T \cup \Phi)_\forall.$$

Formulas Preserved by Substructures (Cont'd)

- By the corollary, among models of T, the models of Ψ are precisely the substructures of models of Φ.
  By (a), every such substructure is itself a model of Φ.
  So Φ and Ψ are equivalent modulo T.
- We turn to the case where $\overline{x}$ is not empty.
  Form the language $L(\overline{c})$ by adding new constants $\overline{c}$ to L.
  Suppose $\Phi(\overline{x})$ is preserved in substructures for L-structures which are models of T. Then it is not hard to see that $\Phi(\overline{c})$ must be preserved in substructures for $L(\overline{c})$-structures which are models of T.
  But $\Phi(\overline{c})$ is a set of sentences.
  So the previous argument shows that $\Phi(\overline{c})$ is equivalent modulo T to a set $\Psi(\overline{c})$ of $\forall_1$ sentences of $L(\overline{c})$.
  By the Lemma on Constants, $T \vdash \forall\overline{x}(\bigwedge\Phi(\overline{x}) \leftrightarrow \bigwedge\Psi(\overline{x}))$.
  Thus, $\Phi(\overline{x})$ is equivalent to $\Psi(\overline{x})$ modulo T, in the language $L(\overline{c})$.
  Hence, they are also equivalent in the language L.

Preservation Theorems for Single Formulas

- If Φ in the Łoś-Tarski Theorem is a single formula, then one more application of compactness boils Ψ down to a single $\forall_1$ formula.
- In short, modulo any first-order theory T, the formulas preserved in substructures are precisely the $\forall_1$ formulas.
- Note that $\exists_1$ formulas are up to logical equivalence just the negations of $\forall_1$ formulas.

Corollary

If T is a theory in a first-order language L and ϕ is a formula of L, then the following are equivalent:

(a) ϕ is preserved by embeddings between models of T;

(b) ϕ is equivalent modulo T to an $\exists_1$ formula of L.

- The full dual of the Łoś-Tarski Theorem is also true, with sets of $\exists_1$ formulas rather than single $\exists_1$ formulas.

An Interpolation Theorem

- The Interpolation Theorem associated with the Existential Amalgamation Theorem is an elaboration of the Łoś-Tarski Theorem (which it obviously implies, in the case of single formulas).

Theorem

Let T be a theory in a first-order language L and let $\phi(\overline{x})$, $\chi(\overline{x})$ be formulas of L. Then the following are equivalent:

(a) Whenever $A \subseteq B$, A and B are models of T, $\overline{a}$ is a tuple in A and $B \models \phi(\overline{a})$, then $A \models \chi(\overline{a})$.

(b) There is an $\forall_1$ formula $\psi(\overline{x})$ of L such that $T \vdash \forall \overline{x}(\phi \to \psi) \wedge \forall \overline{x}(\psi \to \chi)$. ($\psi$ is an "interpolant" between ϕ and χ.)

- The proof is an adaptation of the Łoś-Tarski Theorem.

Variants of Existential Amalgamation

- The Existential Amalgamation Theorem has infinitely many variants for different classes of formulas, with only trivial changes in the proof.
- Each of these variants has its own preservation and interpolation theorems.
- Two variants are given without proof.

Theorem

Let L be a first-order language, and let B and C be L-structures, $\overline{a}$ a sequence of elements of C and $f : \langle\overline{a}\rangle_C \to B$ a homomorphism such that $(C, \overline{a}) \Rrightarrow_1^+ (B, f(\overline{a}))$. Then there exist an elementary extension D of B and a homomorphism $g : C \to D$ which extends f.

Variants of Existential Amalgamation (Cont'd)

- Let L be a first-order language.
- Let A, B be L-structures.
- We write $A \Rrightarrow_2 B$ to mean that:

  For every $\exists_2$ sentence ϕ of L,

  $$A \models \phi \quad \text{implies} \quad B \models \phi.$$

- Equivalently, $A \Rrightarrow_2 B$ if and only if:

  For every $\forall_2$ sentence ϕ of L,

  $$B \models \phi \quad \text{implies} \quad A \models \phi.$$

Theorem

Let L be a first-order language, B and C L-structures, $\overline{a}$ a sequence of elements of B and $f : \langle \overline{a} \rangle_B \to C$ an embedding such that $(C, f(\overline{a})) \Rrightarrow_2 (B, \overline{a})$. Then there exist an elementary extension D of B and an embedding $g : C \to D$, such that g preserves all $\forall_1$ formulas of L.

Formulas Preserved by Unions of Chains

Theorem (Chang-Łoś-Suszko Theorem)

Let T be a theory in a first-order language L, and $\Phi(\overline{x})$ a set of formulas of L. Then the following are equivalent:

(a) $\bigwedge\Phi$ is preserved in unions of chains $(A_i : i < \gamma)$ whenever $\bigcup_{i<\gamma} A_i$ and all the A_i, $i < \gamma$, are models of T.

(b) Φ is equivalent modulo T to a set of $\forall_2$ formulas of L.

(b)⇒(a) By a previous theorem.

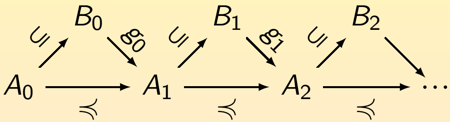
(a)⇒(b) Assume (a) holds. Just as in the proof of the Łoś-Tarski Theorem, we can assume that Φ is a set of sentences.

Let Ψ be the set of all $\forall_2$ sentences of L which are consequences of $T \cup \Phi$. We must show that $T \cup \Psi \vdash \Phi$.

For this it will be enough to prove that every model of $T \cup \Psi$ is elementarily equivalent to a union of some chain of models of $T \cup \Phi$ which is itself a model of T.

Formulas Preserved by Unions of Chains (Cont'd)

- Let A_0 be any model of $T \cup \Psi$. We construct an elementary chain $(A_i : i < \omega)$, extensions $B_i \supseteq A_i$ and embeddings $g_i : B_i \to A_{i+1}$ so that the following diagram commutes:



  Requirement: For each $i < \omega$, $B_i \models T \cup \Phi$ and $(B_i, \overline{a}_i) \Rrightarrow_1 (A_i, \overline{a}_i)$ when $\overline{a}_i$ lists all the elements of A_i.

  The diagram is constructed in two steps, assuming that A_i has already been chosen.

  - One first extends A_i to a structure B_i satisfying the Requirement;
  - Then, having A_i and B_i, we construct A_{i+1}.

Formulas Preserved by Unions of Chains (Cont'd)

- Given A_i, we construct B_i.
  We know $A_0 \preccurlyeq A_i$. So $A_i \models (T \cup \Phi)_{\forall_2}$, where $(T \cup \Phi)_{\forall_2}$ denotes the set of all $\forall_2$ consequences of $T \cup \Phi$.
  We want to find B_i so that:
  - $B_i \models T \cup \Phi$;
  - $A_i \subseteq B_i$;
  - $(B_i, \overline{a}_i) \Rrightarrow_1 (A_i, \overline{a}_i)$.

  By the preceding Variant of Existential Amalgamation, it suffices to find a model C of $T \cup \Phi$, such that $A_i \Rrightarrow_2 C$.
  Let U be the set of all $\exists_2$ sentences ϕ, such that $A_i \models \phi$.
  We must show that $T \cup \Phi \cup U$ has a model.
  If not, by compactness, there exists finite $\{\phi_0, \ldots, \phi_{k-1}\} \subseteq U$, such that $T \cup \Phi \vdash \neg\phi_1 \vee \cdots \vee \neg\phi_{k-1}$. Since all ϕ_i's are $\exists_2$, the sentence $\neg\phi_1 \vee \cdots \neg\phi_{k-1}$ is a equivalent to a $\forall_2$ sentence θ. But $T \cup \Phi \vdash \theta$.
  Hence, $\theta \in (T \cup \Phi)_{\forall_2}$. Therefore, $A_i \models \theta$.
  This contradicts $A \models \phi_i$, for all $i < k$.

Formulas Preserved by Unions of Chains (Cont'd)

- Now, by the Existential Amalgamation Theorem and the second part of the Requirement, there are an elementary extension A_{i+1} of A_i and an embedding $g_i : B_i \to A_{i+1}$, such that g is the identity on A_i.

  In the diagram we can replace each B_i by its image under g_i.

  Thus, we assume that all the maps are inclusions.

  Then $\bigcup_{i<\omega} A_i$ and $\bigcup_{i<\omega} B_i$ are the same structure C.

  By the Tarski- Vaught Elementary Chain Theorem $A_0 \preccurlyeq C$.

  So C is a model of T and the union of a chain of models B_i of $T \cup \Phi$, and A_0 is elementarily equivalent to C, as required.
- Just as with the Łoś-Tarski Theorem, using compactness, we get:

  A formula ϕ of L is preserved in unions of chains (where all the structures are models of T) if and only if ϕ is equivalent modulo T to a $\forall_2$ formula of L.

Subsection 5

Expanding the Language

Expansion Theorem

Theorem

Let L_1 and L_2 be first-order languages, $L = L_1 \cap L_2$, B an L_1-structure, C an L_2-structure, and $\overline{a}$ a sequence of elements of B and of C, such that $(B|_L, \overline{a}) \equiv (C|_L, \overline{a})$. Then there are an $(L_1 \cup L_2)$-structure D such that $B \preccurlyeq D|_{L_1}$, and an elementary embedding $g: C \to D|_{L_2}$, such that $g(\overline{a}) = \overline{a}$.

- Note that an almost invisible alteration of the proof of the Elementary Amalgamation gives a weak version of the theorem we want.

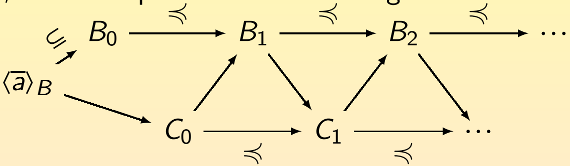
  Under our hypotheses, there are an elementary extension D of B and an elementary embedding $g: C|_L \to D|_L$, such that $g(\overline{a}) = \overline{a}$.

  (It suffices to show that $\text{eldiag}(B) \cup \text{eldiag}(C|_L)$ has a model.)

Expansion Theorem (Cont'd)

- Put $B_0 = B$, $C_0 = C$.

  Use the weak version of the theorem, alternately from this side and from that, to build up a commutative diagram



  where the maps from B_i to C_i and from C_i to B_{i+1} are elementary embeddings of the L-reducts.

  The diagram induces an isomorphism $e : \bigcup_{i<\omega} B_i|_L \to \bigcup_{i<\omega} C_i|_L$.

  $\bigcup_{i<\omega} B_i|_L$ is an L_1-structure.

  $\bigcup_{i<\omega} C_i|_L$ is an L_2-structure.

  Use e and $\bigcup_{i<\omega} C_i|_L$ to expand $\bigcup_{i<\omega} B_i|_L$ to an $(L_1 \cup L_2)$-structure D.

  By the elementary chain theorem, D is as required.

A Characterization Theorem

- The theorem generates characterization and interpolation theorems.
- Suppose L and L^+ are first-order languages with $L \subseteq L^+$.
- If T is an L^+-theory, T_L denotes the set of all consequences of T in L.

Corollary

Let L and L^+ be first-order languages with $L \subseteq L^+$ and T a theory in L^+. Let A be an L-structure. Then $A \models T_L$ if and only if for some model B of T, $A \preccurlyeq B|_L$.

- First, suppose $B \models T$ and $A \preccurlyeq B|_L$.

  Then $B|_L \models T_L$. Hence, $A \models T_L$.

  Assume, conversely, that $A \models T_L$.

  We show, there exists a model B of T, such that $(B|_L, \overline{a}) \equiv (A, \overline{a})$.

A Characterization Theorem (Cont'd)

- We find a model B of T, such that $(B|_L, \overline{a}) \equiv (A, \overline{a})$.

  Consider the theory $\text{eldiag}(A) \cup T$.

  It suffices to show that it has a model.

  If not, by compactness, there exists finite $\{\phi_0(\overline{a}), \ldots, \phi_{k-1}(\overline{a})\} \subseteq \text{eldiag}(A)$, such that

  $$T \vdash \neg\phi_0(\overline{a}) \vee \cdots \vee \neg\phi_{k-1}(\overline{a}).$$

  Thus, by definition, $\neg\phi_0(\overline{a}) \vee \cdots \vee \neg\phi_{k-1}(\overline{a}) \in T_L$.

  By hypothesis, $A \models T_L$.

  Hence, $A \models \neg\phi_0(\overline{a}) \vee \cdots \vee \neg\phi_{k-1}(\overline{a})$.

  This contradicts $\phi_i(\overline{a}) \in \text{eldiag}(A)$, for all $i < k$.

An Interpolation Theorem

Theorem

Let L_1, L_2 be first-order languages, $L = L_1 \cap L_2$ and T_1, T_2 theories in L_1, L_2, respectively, such that $T_1 \cup T_2$ has no model. Then there is some sentence ψ of L, such that $T_1 \vdash \psi$ and $T_2 \vdash \neg\psi$.

- Take $\Psi = (T_1)_L$.
  By compactness, it suffices to show that $\Psi \cup T_2$ has no model.
  Towards a contradiction, let C be a model of $\Psi \cup T_2$.
  By the preceding corollary there is an L_1-structure B, such that $C|_L \preccurlyeq B|_L$ and $B \models T_1$. Then $B|_L \equiv C|_L$.
  By the preceding theorem, there are an $(L_1 \cup L_2)$-structure D, such that $B \preccurlyeq D|_{L_1}$ and an elementary embedding $g : C \to D|_{L_2}$.
  Now, on the one hand, $B \preccurlyeq D|_{L_1}$. So $D \models T_1$.
  On the other hand, g is elementary. So $D \models T_2$.
  Thus $T_1 \cup T_2$ does have a model, a contradiction.

Craig's Interpolation Theorem

Corollary (Craig's Interpolation Theorem)

Let L_1, L_2 be first-order languages, $L = L_1 \cap L_2$ and ϕ, χ sentences of L_1, L_2, respectively, such that $\phi \vdash \chi$. Then there is a sentence ψ of $L_1 \cap L_2$, such that $\phi \vdash \psi$ and $\psi \vdash \chi$.

- By hypothesis, $\{\phi, \neg\chi\}$ is inconsistent.

  Thus, by the theorem, there exists a sentence ψ of L, such that

  $$\phi \vdash \psi \quad \text{and} \quad \neg\chi \vdash \neg\psi.$$

  The second is equivalent to $\psi \vdash \chi$.

A Preservation Theorem

- This preservation theorem talks about formulas which are preserved under taking off symbols and putting them back on again.

Theorem

Let L and L^+ be first-order languages with $L \subseteq L^+$, let T be a theory in L^+ and $\phi(\overline{x})$ a formula of L^+. Then the following are equivalent:

(a) If A and B are models of T and $A|_L = B|_L$, then for all tuples $\overline{a}$ in A, $A \models \phi(\overline{a})$ if and only if $B \models \phi(\overline{a})$.

(b) $\phi(\overline{x})$ is equivalent modulo T to a formula $\psi(\overline{x})$ of L.

- From the Expansion Theorem, as a corresponding result followed from the Existential Amalgamation Theorem.
- The implication (a)⇒(b) in the case where ϕ is an unnested atomic formula $R(x_0,\ldots,x_{n-1})$ or $F(x_0,\ldots,x_{n-1}) = x_n$ is known as **Beth's Definability Theorem**.

Beth's Definability Theorem

- Let L and L^+ be first-order languages with $L \subseteq L^+$.
- Let T be a theory in L^+.
- A relation symbol R of T^+ is **implicitly defined by T in terms of L** if whenever A and B are models of T with $A|_L = B|_L$, then $R^A = R^B$.
- A function symbol F of T^+ is **implicitly defined by T in terms of L** if whenever A and B are models of T with $A|_L = B|_L$, then, for all $\overline{a}$ in A, $F^A(\overline{a}) = F^B(\overline{a})$.
- R is **explicitly defined by T in terms of L** if T has some consequence of the form $\forall\overline{x}(R\overline{x} \leftrightarrow \psi)$, where $\psi(\overline{x})$ is a formula of L.
- F is **explicitly defined by T in terms of L** if T has some consequence of the form $\forall\overline{x}y(F(\overline{x}) = y \leftrightarrow \psi)$, where $\psi(\overline{x}, y)$ is in L.
- It is immediate that, if R (or F) is explicitly defined by T in terms of L, then it is implicitly defined by T in terms of L.
- Beth's Theorem states the converse: Relative to a first-order theory, implicit definability equals explicit definability.

Padoa's Method

- The notion of implicit definability makes sense in a broader context.
- Let L and L^+ be languages (not necessarily first-order), with $L \subseteq L^+$.
- Let T a theory in L^+.
- Let R a relation symbol of L^+.
- We say that R is **implicitly defined by T in terms of L** if, whenever A and B are models of T with $A|_L = B|_L$, then $R^A = R^B$.
- **Padoa's method** for proving the undefinability of R by T in terms of L involves producing models A and B of T, such that
  - $A|_L = B|_L$;
  - $R^A \neq R^B$.
- If L and L^+ are not first-order, Beth's Theorem may fail.

A Refinement of the Łoś-Tarski Theorem

- **Local theorems** are theorems of the following form.

  If "enough" finitely generated substructures of a structure A belong to a certain class $\mathbf{K}$, then A also belongs to $\mathbf{K}$.

Theorem

Let L be a first-order language and $\mathbf{K}$ a $\mathbf{PC}'_\Delta$ class of L-structures. Suppose that $\mathbf{K}$ is closed under taking substructures. Then $\mathbf{K}$ is axiomatized by a set of $\forall_1$ sentences of L.

- The theorem refines the Łoś-Tarski Theorem.
- Its proof is a refinement of the earlier proof.
- Let $\mathbf{K}$ be the PC'_Δ class $\{B_P : B \models U\}$.

  Define

$$T^* = \{\phi \ \forall_1 \text{ sentence in } L : B \models U \text{ implies } B_P \models \phi\}.$$

  Every structure in $\mathbf{K}$ is a model of T^*.

A Refinement of the Łoś-Tarski Theorem (Cont'd)

Claim: Every model A of T^* is in $\mathbf{K}$.

Consider the theory $\text{diag}(A) \cup \{P(a) : a \in \text{dom} A\} \cup U$.

Claim: This theory has a model.

If not, then by the Compactness Theorem, there are a conjunction $\psi(\overline{x})$ of literals of L, and a tuple $\overline{a}$ of distinct elements $a_0, \ldots, a_{m-1}$ of A, such that $A \vDash \psi(\overline{a})$ and $U \vdash P(a_0) \wedge \cdots \wedge P(a_{m-1}) \to \neg\psi(\overline{a})$.

By the lemma on constants, $U \vdash \forall \overline{x}(Px_0 \wedge \cdots \wedge Px_{m-1} \to \neg\psi(\overline{x}))$.

Hence the sentence $\forall \overline{x} \neg\psi(\overline{x})$ is in T^*. So it must be true in A.

This contradicts the fact that $A \vDash \psi(\overline{a})$ and proves the claim.

By the claim there is a model D of the theory.

By the Diagram Lemma, A is embeddable in D_P.

But D_P is in $\mathbf{K}$ and $\mathbf{K}$ is closed under substructures.

Since $\mathbf{K}$ is closed under isomorphic copies, it follows that A is in $\mathbf{K}$.

Example: Faithful Linear Representations of Groups

- Let n be a positive integer and G a group. We say that G has a **faithful n-dimensional linear representation** if G is embeddable in $\mathrm{GL}_n(F)$, the group of invertible n-by-n matrices over some field F.

Corollary

Let n be a positive integer and G a group. Suppose that every finitely generated subgroup of G has a faithful n-dimensional linear representation. Then G has a faithful n-dimensional linear representation.

- Let $\mathbf{K}$ be the class of groups with faithful n-dimensional linear representations. We note that $\mathbf{K}$ is closed under substructures. There is a theory U in a suitable first-order language, such that $\mathbf{K}$ is precisely the class $\{B_P : B \models U\}$. By the theorem, $\mathbf{K}$ is axiomatized by an $\forall_1$ theory T. If G is not in $\mathbf{K}$, then there is some sentence $\forall \overline{x} \psi(\overline{x})$ in T, with ψ quantifier-free, such that $G \models \exists \overline{x} \neg \psi(\overline{x})$. Find a tuple $\overline{a}$ in G so that $G \models \neg \psi(\overline{a})$. Then the subgroup $\langle \overline{a} \rangle_G$ is not in $\mathbf{K}$.