Introduction to Model Theory

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LSSU Math 500

1. The Countable Case
   - Fraïssé's Construction
   - Omitting Types
   - Countable Categoricity
   - ω-Categorical Structures by Fraïssé's Method

Subsection 1

Fraïssé's Construction

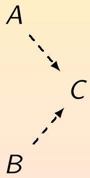
The Age of a Structure

- Let L be a signature.
- Let D be an L-structure.
- The **age** of D is the class $\mathbf{K}$ of all finitely generated structures that can be embedded in D.
- What interests us is not the structures in $\mathbf{K}$ but their isomorphism types.
- So we shall also call a class $\mathbf{J}$ the **age** of D if the structures in $\mathbf{J}$ are, *up to isomorphism*, exactly the finitely generated substructures of D.
- For example, saying that D has "countable age" will mean that D has just countably many isomorphism types of finitely generated substructure.

Ages and Properties

- We call a class an **age** if it is the age of some structure.
- If **K** is an age, then clearly **K** is non-empty and has the following two properties:
  1. (**Hereditary Property**, **HP**) If $A \in \mathbf{K}$ and B is a finitely generated substructure of A then B is isomorphic to some structure in **K**.
  2. (**Joint Embedding Property**, **JEP**)

     If A, B are in **K** then there is C in **K**, such that both A and B are embeddable in C.



Sufficiency of the Condition (Construction)

Theorem

Suppose L is a signature and $\mathbf{K}$ is a non-empty finite or countable set of finitely generated L-structures which has the HP and the JEP. Then $\mathbf{K}$ is the age of some finite or countable structure.

- List the structures in K, possibly with repetitions, as $(A_i : i < \omega)$. Define a chain $(B_i : i < \omega)$ of structures isomorphic to structures in $\mathbf{K}$, as follows:
  - First, put $B_0 = A_0$.
  - Suppose B_i has been chosen.
    Use the joint embedding property to find a structure B' in $\mathbf{K}$ such that both B_i and A_{i+1} are embeddable in B'.
    Take B_{i+1} to be an isomorphic copy of B' which extends B_i.

  Finally, let C be the union $\bigcup_{i<\omega} B_i$.

  C is the union of countably many structures which are at most countable. So C is at most countable.

Sufficiency of the Condition (Propeties)

- We must show that **K** is the age of C.

  By construction every structure in **K** is embeddable in C.

  Let A be any finitely generated substructure of C.

  The finitely many generators of A lie in some B_i.

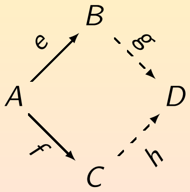
  So A is isomorphic to a structure in **K** (by the hereditary property).

  So **K** is the age of C.
- The theorem holds even if L has function symbols.
- But one way to guarantee that **K** is at most countable is to assume that L is a finite signature with no function symbols.
- When there are no function symbols and only finitely many constant symbols, a finitely generated structure is the same thing as a finite structure.

The Amalgamation Property

- All infinite linear orderings have exactly the same age, namely the finite linear orderings.
- To investigate the sense in which the finite linear orderings "tend to" the rationals rather than, say, the ordering of the integers, Fraïssé singled out the amalgamation property.
- (**Amalgamation Property**, **AP**)

  If A, B, C are in $\mathbf{K}$ and $e : A \to B$, $f : A \to C$ are embeddings, then there are D in $\mathbf{K}$ and embeddings $g : B \to D$ and $h : C \to D$, such that $ge = hf$.



Warning: In general JEP is not a special case of AP.

Think, for instance, of the class of fields.

Linear Orderings and the Amalgamation Property

Claim: The class of all finite linear orderings has the amalgamation property.

The simplest way to see this is to start with the case where:

- A is a substructure of B and C;
- The maps $e : A \to B$ and $f : A \to C$ are inclusions;
- A is exactly the overlap of B and C.

In this case we can form D as an extension of B.

Working by induction on the cardinality of C, we add the elements of C one by one in the appropriate places.

The general case then follows by diagram chasing.

Ultrahomogeneity

- A structure D is called **ultrahomogeneous** if every isomorphism between finitely generated substructures of D extends to an automorphism of D.
- The usually terminology is **homogeneous**, but there are other notions that are known by this name.

Fraïssé's Theorem

Theorem (Fraïssé's Theorem)

Let L be a countable signature and let **K** be a non-empty finite or countable set of finitely generated L-structures which has HP, JEP and AP. Then there is an L-structure D, unique up to isomorphism, such that:

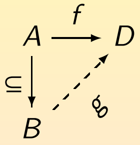
1. D has cardinality $\leq \omega$;
2. **K** is the age of D;
3. D is ultrahomogeneous.

- Sometimes, the structure D of the theorem is referred to as the **universal homogeneous structure of age K**.
- We will call it the **Fraïssé limit** of the class **K**.
- The Fraïssé limit of the class **K** is only determined up to isomorphism.
- The rest of this section is devoted to the proof of Fraïssé's Theorem.

Weak Homogeneity

- A structure D is **weakly homogeneous** if it has the property:

  If A, B are finitely generated substructures of D, $A \subseteq B$ and $f : A \to D$ is an embedding, then there is an embedding $g : B \to D$ which extends f.



- If D is ultrahomogeneous, then clearly D is weakly homogeneous.

Universality Lemma

Lemma

Let C and D be L-structures which are both at most countable. Suppose the age of C is included in the age of D, and D is weakly homogeneous. Then C is embeddable in D. In fact any embedding from a finitely generated substructure of C into D can be extended to an embedding of C into D.

- Let $f_0 : A_0 \to D$ be an embedding of a finitely generated substructure A_0 of C into D. We extend f_0 to an embedding $f_\omega : C \to D$.

  C is at most countable. So it can be written as a union $\bigcup_{n<\omega} A_n$ of a chain of finitely generated substructures, starting with A_0.

  By induction on n we define an increasing chain of embeddings $f_n : A_n \to D$.

Universality Lemma (Cont'd)

- By induction on n we define an increasing chain of embeddings $f_n : A_n \to D$.
  - The first embedding f_0 is given.
  - Suppose f_n has just been defined. The age of D includes that of C. So there is an isomorphism $g : A_{n+1} \to B$, where B is a substructure of D. Then $f_n \cdot g^{-1}$ embeds $g(A_n)$ into D. By weak homogeneity, this embedding extends to an embedding $h : B \to D$.
    Let $f_{n+1} : A_{n+1} \to D$ be hg. Then $f_n \subseteq f_{n+1}$.

  This defines the chain of maps f_n.

  Finally, take f_ω to be the union of the f_n, $n < \omega$.
- Based on the lemma, we say that a countable structure D of age $\mathbf{K}$ is **universal** (for $\mathbf{K}$) if every finite or countable structure of an age that is included in $\mathbf{K}$ is embeddable in D.
- The lemma tells us that countable weakly homogeneous structures are universal for their age.

Uniqueness Proof of Fraïssé's Theorem

Lemma

(a) Let C and D be L-structures with the same age. Suppose that C and D are both at most countable and are both weakly homogeneous. Then C is isomorphic to D.
In fact if A is a finitely generated substructure of C and $f: A \to D$ is an embedding, then f extends to an isomorphism from C to D.

(b) A finite or countable structure is ultrahomogeneous (and hence is the Fraïssé limit of its age) if and only if it is weakly homogeneous.

(a) Express C and D as the unions of chains $(C_n : n < \omega)$ and $(D_n : n < \omega)$ of finitely generated substructures.
Define a chain $(f_n : n < \omega)$ of isomorphisms between finitely generated substructures of C and D, so that, for each n:

- The domain of f_{2n} includes C_n;
- The image of f_{2n+1} includes D_n.

This is done as in the proof of the previous lemma.

Uniqueness Proof of Fraïssé's Theorem (Cont'd)

- Then the union of the f_n is an isomorphism from C to D.

  To get the last sentence of Part (a), take:
  - C_0 to be A;
  - D_0 to be $f(A)$.

  Then proceed with the construction of the chain $(f_n : n < \omega)$.

(b) We have already noted that ultrahomogeneous structures are weakly homogeneous.

The converse follows at once from Part (a), taking $C = D$.

The Uncountable Case: Counterexample

- If C and D are not countable, then Part (a) of the lemma fails.

  Example: Let η be the order type of the rationals.

  Consider:

  - The order type $\eta \cdot \omega_1$ ($= \omega_1$ copies of η laid in a row);
  - Its mirror image ξ.

  Both $\eta \cdot \omega_1$ and ξ are weakly homogeneous.

  Both have the same age, namely the set of all finite linear orderings.

  But clearly they are not isomorphic:

  - In $\eta \cdot \omega_1$ every element has uncountably many successors;
  - This fails in ξ.

The Uncountable Case: A Positive Result

Lemma

Suppose C and D are weakly homogeneous L-structures with the same age. Then C is back-and-forth equivalent to D. So $C \equiv_{\infty,\omega} D$.
If, moreover, $C \subseteq D$, then, for every $\overline{c}$ in C, $(C,\overline{c}) \equiv_{\infty,\omega} (D,\overline{c})$. So $C \preccurlyeq D$.

- The lemma constructs a back-and-forth system from C to D.

  By a previous lemma, C and C are back-and-forth equivalent.

  A previous theorem gives the connection with $L_{\infty,\omega}$.

Existence Proof of Fraïssé's Theorem (Lemma)

Lemma

Let $\mathbf{J}$ be a set of finitely generated L-structures, and $(D_i : i < \alpha)$ a chain of L-structures. If, for each $i < \alpha$, the age of D_i is included in $\mathbf{J}$, then the age of the union $\bigcup_{i<\alpha} D_i$ is also included in $\mathbf{J}$. If each D_i has age $\mathbf{J}$, then $\bigcup_{i<\alpha} D_i$ has age $\mathbf{J}$.

- Let A be in the age of $\bigcup_{i<\alpha} D_i$.
  Then A is a finitely generated substructure of $\bigcup_{i<\alpha} D_i$.
  The set of generators belongs to some D_j, $j < \alpha$.
  Thus, A is in the age of D_j.
  By hypothesis, A is in $\mathbf{J}$.
  For the second statement, let A be in $\mathbf{J}$.
  By hypothesis, A is in the age of D_i.
  Thus, A is in the age of $\bigcup_{i<\alpha} D_i$.

Existence Proof of Fraïssé's Theorem

- We return to the Existence Proof of Fraïssé's Theorem.

  Henceforth we assume that **K** is non-empty, has HP, JEP and AP, and contains at most countably many isomorphism types of structure.

  We suppose without loss that **K** is closed under isomorphic copies.

  We construct a chain $(D_i : i < \omega)$ of structures in **K**, such that:

  > If A and B are structures in **K**, with $A \subseteq B$, and there is an embedding $f : A \to D_i$ for some $i < \omega$, then there are $j > i$ and an embedding $g : B \to D_j$ which extends f. We take D to be the union $\bigcup_{i<\omega} D_i$.

  Then the age of D is included in **K** by the lemma.

  In fact the age of D is exactly **K**. Suppose A is in **K**. Then by JEP, there is B in **K** such that $A \subseteq B$ and D_0 is embeddable in B. By the displayed condition, the identity map on D_0 extends to an embedding of B in some D_j. Thus, B and A lie in the age of D.

  Thus, the condition tells us that D is weakly homogeneous.

  So by a previous lemma, it is ultrahomogeneous of age **K**.

Existence Proof of Fraïssé's Theorem (The Chain)

- It remains to construct the chain.

  Let **P** be a countable set of pairs of structures (A, B) such that:
  - $A, B \in \mathbf{K}$;
  - $A \subseteq B$.

  We can choose **P** so that it includes a representative of each isomorphism type of such pairs.

  Take a bijection $\pi : \omega \times \omega \to \omega$ such that $\pi(i, j) \geq i$, for all i and j.
  - Let D_0 be any structure in $\mathbf{K}$.
  - Suppose D_k has been chosen.
    List as $((f_{kj}, A_{kj}, B_{kj}) : j < \omega)$ the triples (f, A, B) where:
    - $(A, B) \in \mathbf{P}$;
    - $f : A \to D_k$.

    Construct D_{k+1} by the amalgamation property, so that if $k = \pi(i, j)$ then f_{ij} extends to an embedding of B_{ij} into D_{k+1}.

Necessity of the Conditions

Theorem

Let L be a countable signature and D a finite or countable structure which is ultrahomogeneous. Let $\mathbf{K}$ be the age of D.

- $\mathbf{K}$ is non-empty;
- $\mathbf{K}$ has at most countably many isomorphism types of structure;
- $\mathbf{K}$ satisfies HP, JEP and AP.

- We already know everything except that $\mathbf{K}$ satisfies amalgamation.

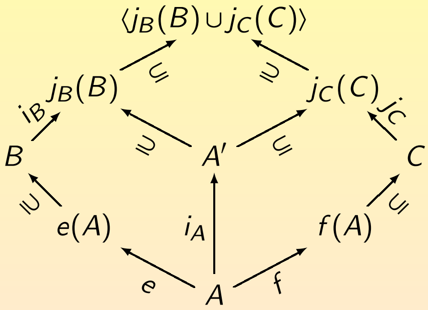
  We may assume $\mathbf{K}$ contains all finitely generated substructures of D.

  Let A, B, C be in $\mathbf{K}$ and $e : A \to B$, $f : A \to C$ be embeddings.

  Then there are isomorphisms $i_A : A \to A'$, $i_B : B \to B'$ and $i_C : C \to C'$ where A', B', C' are substructures of D.

Necessity of the Conditions (Cont'd)

- So $i_A \cdot e^{-1}$ embeds $e(A)$ into D.
  By weak homogeneity there is an embedding $j_B : B \to D$ which extends $i_A \cdot e^{-1}$.
  So the bottom left quadrilateral in the diagram commutes.



Likewise, the bottom right quadrilateral commutes.

The top square also commutes.

Hence, the outer maps give the needed amalgam.

Subsection 2

Omitting Types

Realizing and Omitting Sets of Formulas

- Let L be a first-order language and T a theory in L.
- Let $\Phi(\overline{x})$ be a set of formulas of L, with $\overline{x} = (x_0, \ldots, x_{n-1})$.
- We say that Φ is **realized in** an L-structure A if there is a tuple $\overline{a}$ of elements of A, such that $A \models \Phi(\overline{a})$.
- We say A **omits** Φ if Φ is not realized in A.
- We look at situations in which T has a model that omits Φ.

Example

- Let L be a first-order language and T a theory in L.

  Let $\Phi(\overline{x})$ be a set of formulas of L, with $\overline{x} = (x_0, \ldots, x_{n-1})$.

  Suppose the following holds.

  There is a formula $\theta(\overline{x})$ of L such that:
  - $T \cup \{\exists \overline{x} \theta\}$ has a model;
  - For every formula $\phi(\overline{x})$ in Φ, $T \vdash \forall \overline{x}(\theta \to \phi)$.

  If T is a complete theory, then the condition implies that $T \vdash \exists \overline{x} \theta$.

  So T certainly has no model that omits Φ.

- The next theorem implies that when the language L is countable, the converse holds too, even if T is not a complete theory.

  If every model of T realizes Φ then the condition is true.

Example: A Type Omitted

- Let L be a first-order language whose signature consists of unary relation symbols P_i, $i < \omega$.

  Let T be the theory in L which consists of all the sentences:
  - $\exists x P_0(x)$;
  - $\exists x \neg P_0(x)$;
  - $\exists x (P_0(x) \wedge P_1(x))$;
  - $\exists x (P_0(x) \wedge \neg P_1(x))$;
  - $\exists x (\neg P_0(x) \wedge P_1(x))$;

    etc. (through all the possible combinations).

  For every $s \subseteq \omega$, define

  $$\Phi_s(x) = \{P_i(x) : i \in s\} \cup \{\neg P_i(x) : i \notin s\}.$$

  Given a structure A, define, for all $s \subseteq \omega$,

  $$|\Phi_s(A)| = |\{a \in \mathrm{dom}(A) : \Phi_s(a)\}|.$$

Example: A Type Omitted (Cont'd)

Claim: If A is a model of T, the A is determined up to isomorphism by $\{|\Phi_s(A)| : s \in \omega\}$.

Suppose A and B are models of T, such that $|\Phi_s(A)| = |\Phi_s(B)|$, $s \subseteq \omega$.

For all $s \subseteq \omega$, let $f_s : \Phi_s(A) \to \Phi_s(B)$ be a bijection.

Then $f = \bigcup_{s \subseteq \omega} f_s$ is an isomorphism from A to B.

Claim: Let $s \subseteq \omega$, A a model of T, with $|A| \leq 2^\omega$. There exists an elementary extension $A \preccurlyeq B$, such that $|B| = 2^\omega$ and $|\Phi_s(B)| = 2^\omega$.

Let B' be a set disjoint from A, such that $|B'| = 2^\omega$.

Construct B as follows:

- $\text{dom}(B) = \text{dom}(A) \cup B'$;
- For all $b \in B$ and all $i < \omega$,

$$P_i^B(b) \quad \text{iff} \quad (b \in A \text{ and } P_i^A(b)) \text{ or } (b \notin A \text{ and } i \in s).$$

Then, by the Elementary Diagram Lemma, it is clear that $A \preccurlyeq B$.

Example: A Type Omitted (Cont'd)

Claim: Let A be a model of T, with $|A| \leq 2^\omega$.

There exists an elementary extension $A \preccurlyeq C$, such that $|C| = 2^\omega$ and $|\Phi_s(C)| = 2^\omega$, for all $s \subseteq \omega$.

By repeated application of the preceding claim.

Claim: T is complete.

Suppose A, B are models of T.

By the Downward Löwenheim-Skolem Theorem, there exist models A' and B', such that $A' \preccurlyeq A$ and $B' \preccurlyeq B$, with

$$|A'| \leq 2^\omega \quad \text{and} \quad |B'| \leq 2^\omega.$$

By the last claim, there exist models A'' and B'', such that:

- $A' \preccurlyeq A''$ and $B' \preccurlyeq B''$;
- $|A''| = 2^\omega$ and $|B''| = 2^\omega$;
- For all $s \subseteq \omega$, $|\Phi_s(A'')| = |\Phi_s(B'')| = 2^\omega$.

By the first claim, it follows that $A'' \cong B''$.

Therefore, $A \equiv B$. So T is complete.

Example: A Type Omitted (Cont'd)

- Recall that, is s is any subset of ω,

$$\Phi_s(x) = \{P_i(x) : i \in s\} \cup \{\neg P_i(x) : i \notin s\}.$$

  T has a countable model A.

  A must omit at least one of the continuum many sets Φ_s, $s \subseteq \omega$.

  By symmetry, if $s \subseteq \omega$, there must be a countable model of T which omits Φ_s.

  However, a model of T cannot omit all the sets Φ_s, or it would be empty.
- Note that it takes infinitely many first-order formulas to specify Φ_s.
- So, if Φ is Φ_s, for some $s \subseteq \omega$, then there does not exist a formula θ, as in the previous example.

Supported and Principal Types

- Let L be a first-order language and T a theory in L.
- Let $\Phi(\overline{x})$ be a set of formulas of L.
- We say that a formula θ of L is a **support** of Φ over T if:
  - $T \cup \{\exists \overline{x} \theta\}$ has a model;
  - For every formula $\phi(\overline{x})$ in Φ, $T \vdash \forall \overline{x}(\theta \to \phi)$.
- If a support θ of Φ is in Φ, we say that θ **generates** Φ over T.
- We say $\Phi(\overline{x})$ is a **supported type over** T if Φ has a support over T.
- We say that Φ is a **principal type over** T if Φ has a generator over T.
- The set Φ is said to be **unsupported** (resp. **non-principal**) over T if it is not a supported (resp. principal) type over T.

Complete Formulas

- Note that if $p(\overline{x})$ is a complete type over the empty set, then a formula $\phi(\overline{x})$ of L is a support of p if and only if it generates p.

  Suppose $p(\overline{x})$ is a complete type over the empty set.

  Let $\phi(\overline{x})$ be a support of $p(\overline{x})$.

  Then $T \cup \{\exists\overline{x}\phi(\overline{x})\}$ is consistent.

  So $T \cup \{\exists\overline{x} \bigwedge \Psi(\overline{x}) : \Psi \subseteq_\omega p(\overline{x})\} \cup \{\exists\overline{x}\phi(\overline{x})\}$ is consistent.

  But $T \cup \{\exists\overline{x} \bigwedge \Psi(\overline{x}) : \Psi \subseteq_\omega p(\overline{x})\}$ is maximally consistent.

  So we must have $\phi(\overline{x}) \in p(\overline{x})$.

  Therefore, $\phi(\overline{x})$ generates $p(\overline{x})$.
- A complete type p is principal if and only if it is supported.
- We say that a formula $\phi(\overline{x})$ is **complete** (for T) if it generates a complete type of T.

Countable Omitting Types Theorem

Theorem (Countable Omitting Types Theorem)

Let L be a countable first-order language, T a theory in L which has a model. For each $m < \omega$, let Φ_m be an unsupported set over T in L. Then T has a model which omits all the sets Φ_m.

- The theorem is trivial when T has an empty model.
  So we can assume that T has a non-empty model.
  Let L^+ be the first-order language obtained from L by adding countably many new constants c_i, $i < \omega$, to be known as **witnesses**.
  We define an increasing chain $(T_i : i < \omega)$ of finite sets of sentences of L^+, such that for every i, $T \cup T_i$ has a model.
  Take T_{-1} to be the empty theory.
  Then $T \cup T_{-1} = T$ has a model which is an L^+-structure.
  The intention is that $T^+ = \bigcup_{i<\omega} T_i$, will be a Hintikka set for L^+.
  The canonical model of T^+ will be a model of T omitting all Φ_m.

Countable Omitting Types Theorem (Task List)

- To ensure that T^+ will have these properties, we carry out various tasks as we build the chain.
  - (1) Ensure that for every sentence ϕ of L^+, either ϕ or $\neg\phi$ is in T^+.
  - $(2)_{\psi(x)}$ (For each formula $\psi(x)$ of L^+:) Ensure that if $\exists x\psi(x)$ is in T^+, then there are infinitely many witnesses c such that $\psi(c)$ is in T^+.
  - $(3)_m$ (For each $m<\omega$:) Ensure that for every tuple $\overline{c}$ of distinct witnesses (of appropriate length) there is a formula $\phi(\overline{x})$ in Φ_m, such that the formula $\neg\phi(\overline{c})$ is in T^+.

  If these hold, by a previous theorem, T^+ will be a Hintikka set.

  Write A^+ for the canonical model of the atomic sentences in T^+.

  Then $A^+ \models T^+$ and every element is named by a closed term.

  By the tasks (2), where $\psi(x)$ are the formulas $x=t$ (t a closed term), every element of A^+ is named by infinitely many witnesses.

  So every tuple of elements is named by a tuple of distinct witnesses.

  This, together with (3), ensures that A^+ omits all the types Φ_m.

  The required model of T will be $A^+|_L$.

Countable Omitting Types Theorem (Delegating Tasks)

- There are countably many tasks in the list.
  - Task (1) is one;
  - A Task $(2)_{\psi(x)}$ for each formula $\psi(x)$;
  - A Task $(3)_m$ for each $m < \omega$.

  We have countably many "experts" and give them one task each.

  We partition ω into infinitely many infinite sets.

  We assign one of these sets to each expert.

  Suppose T_{i-1} has been chosen.

  If i is in the set assigned to some expert E, then E will choose T_i.

Countable Omitting Types Theorem (Task (1))

- First consider the expert who handles Task (1).

  Let X be her subset of ω.

  Let her list as $(\phi_i : i \in X)$ all the sentences of L^+.

  Suppose T_{i-1} has been chosen and i is in X.

  Consider whether $T \cup T_{i-1} \cup \{\phi_i\}$ has a model.

  - If it has, she should put $T_i = T_{i-1} \cup \{\phi_i\}$;
  - If not, then every model of $T \cup T_i$ is a model of $\neg\phi_i$.
    We can take T_i to be $T_{i-1} \cup \{\neg\phi_i\}$.

  In this way Task (1) is accomplished by the time the chain is complete.

Countable Omitting Types Theorem (Tasks (2))

- Next consider the expert who deals with Task $(2)_\psi$.

  She waits until she is given a set T_{i-1} which contains $\exists x\psi(x)$.

  Every time this happens, she looks for a witness c not used in T_{i-1}.

  There is such a witness, because T_{i-1} is finite.

  Then a model of $T \cup T_{i-1}$ can be made into a model of $\psi(c)$ by choosing a suitable interpretation for c.

  Let her take T_i to be $T_{i-1} \cup \{\psi(c)\}$.

  Otherwise she should do nothing.

  This strategy works, because her subset of ω contains arbitrarily large numbers.

Countable Omitting Types Theorem (Tasks (3))

- Consider the expert who handles Task $(3)_m$, where Φ_m is a type in n variables.

  Let Y be her assigned subset of ω.

  She lists as $\{\overline{c}_i : i \in Y\}$ all the n-tuples $\overline{c}$ of distinct witnesses.

  Suppose T_{i-1} has been given, with i in Y.

  She writes $\bigwedge T_{i-1}$ as a sentence $\chi(\overline{c}_i, \overline{d})$, where:
  - $\chi(\overline{x}, \overline{y})$ is in L;
  - $\overline{d}$ lists the distinct witnesses which occur in T_{i-1} but not in $\overline{c}_i$.

  By assumption, the theory $T \cup \{\exists\overline{x}\exists\overline{y}\chi(\overline{x}, \overline{y})\}$ has a model.

  The set Φ_m is unsupported over T.

  Hence, there is $\phi(\overline{x})$ in Φ_m, such that $T \nvdash \forall\overline{x}(\exists\overline{y}\chi(\overline{x}, \overline{y}) \to \phi(\overline{x}))$.

  By the Lemma on Constants, $T \nvdash \chi(\overline{c}_i, \overline{d}) \to \phi(\overline{c}_i)$.

  Now, she can put $T_i = T_{i-1} \cup \{\neg\phi(\overline{c}_i)\}$.

  In this way, she also fulfills her task.

Enforceable Properties

- In the proof of the theorem, each expert has to make sure that the theory T^+ has some particular property π.
- The proof shows that the expert can make T^+ have π, provided that she is allowed to choose T_i for infinitely many i.
- We can express this in terms of a game $G(\pi, X)$.
  - There are two players, $\forall$ and $\exists$.
  - X is an infinite subset of ω, with $\omega \backslash X$ infinite and $0 \notin X$.
  - The players have to pick the sets T_i in turn;
    Player $\exists$ makes the choice of T_i if and only if $i \in X$.
  - Player $\exists$ wins if T^+ has property π; otherwise $\forall$ wins.
- We say that π is **enforceable** if player $\exists$ has a winning strategy for this game.

Enforceable Properties: Remarks

- One can show that whether π is enforceable is independent of the choice of X, provided that both X and $\omega\backslash X$ are infinite.
- Some properties of T^+ are really properties of the canonical model A^+, e.g., that every element of A^+ is named by infinitely many witnesses.
- So, we may talk of "enforceable properties" of A^+ (in place of T^+).

Atomic and Prime Models

- A structure A is called **atomic** if for every tuple $\overline{a}$ of elements of A, the complete type $\mathrm{tp}_A(\overline{a})$ of $\overline{a}$ in A is principal.
- A model A of a theory T is said to be **prime** if A can be elementarily embedded in every model of T.
- Recall that $S_n(T)$ is the set of complete first-order types $p(x_0,\ldots,x_{n-1})$ over the empty set with respect to models of T.

Complete Types and Atomic and Prime Models

Theorem

Let L be a countable first-order language and T a complete theory in L which has infinite models.

(a) If for every $n < \omega$, $S_n(T)$ is at most countable, then T has a countable atomic model.

(b) If A is a countable atomic L-structure which is a model of T, then A is a prime model of T.

(a) There are only countably many non-principal complete types.

By the theorem, we can omit all of them in some model A of T.

By hypothesis, T is complete and has infinite models.

So A can be found with cardinality ω.

Atomic and Prime Models (Part (b))

(b) Let B be any model of T.

Claim: If $\overline{a}, \overline{b}$ are n-tuples realizing the same complete type in A, B, respectively, and d is any element of B, then there is c of A, such that $\overline{a}c, \overline{b}d$ realize the same complete $(n+1)$-type in A, B respectively.

Since the complete type of $\overline{b}d$ is principal by assumption, it has a generator $\psi(\overline{x}, y)$. Since $\overline{a}$ and $\overline{b}$ realize the same complete type, and $B \models \exists y \psi(\overline{b}, y)$, we infer that $A \models \exists y \psi(\overline{a}, y)$. Hence there is an element c in A, such that $A \models \psi(\overline{a}, c)$. Then $\overline{a}c$ realizes the same complete type as $\overline{b}d$. This proves the claim.

Now let $b_0, b_1, \ldots$ list all the elements of B. Work by induction on n. Using the claim we find $a_0, a_1, \ldots$ of A so that, for all n,

$$(A, a_0, \ldots, a_{n-1}) \equiv (B, b_0, \ldots, b_{n-1}).$$

By the Elementary Diagram Lemma, $b_i \mapsto a_i$ is an elementary embedding of B into A.

Atomic Elementarily Equivalent Structures

- In short, if T is complete and all the sets $S_n(T)$ are countable then there is a "smallest" countable model of T.
- The proof of the theorem can be adapted to prove another useful result which has nothing to do with countable structures.

Theorem

Let L be a countable first-order language. Let A and B be two elementarily equivalent L-structures, both of which are atomic. Then A and B are back-and-forth equivalent.

- Let $\overline{a}$ and $\overline{b}$ be tuples in A and B respectively, such that $(A, \overline{a}) \equiv (B, \overline{b})$. With an argument exactly the same as in the theorem, we show that:
  - For every c of A, there is d of B, such that $(A, \overline{a}, c) \equiv (B, \overline{b}, d)$;
  - For every d of B, there is c of A, such that $(A, \overline{a}, c) \equiv (B, \overline{b}, d)$.

Subsection 3

Countable Categoricity

ω-Categoricity

- A complete theory which has exactly one countable model up to isomorphism is said to be ω-**categorical**.
- A structure A is ω-**categorical** if $\mathrm{Th}(A)$ is ω-categorical.

Theorem of Engeler, Ryll-Nardzewski and Svenonius

Theorem (Theorem of Engeler, Ryll-Nardzewski and Svenonius)

Let L be a countable first-order language and T a complete theory in L which has infinite models. Then the following are equivalent:

(a) Any two countable models of T are isomorphic.

(b) If A is any countable model of T, then $\mathrm{Aut}(A)$ is oligomorphic (i.e., for every $n < \omega$, $\mathrm{Aut}(A)$ has only finitely many orbits in its action on n-tuples of elements of A).

(c) T has a countable model A such that $\mathrm{Aut}(A)$ is oligomorphic.

(d) Some countable model of T realizes only finitely many complete n-types for each $n < \omega$.

(e) For each $n < \omega$, $S_n(T)$ is finite.

(f) For each $x = (x_0, \ldots, x_{n-1})$, there are only finitely many pairwise non-equivalent formulas $\phi(\overline{x})$ of L modulo T.

(g) For each $n < \omega$, every type in $S_n(T)$ is principal.

Proof ((b)⇒(c)⇒(d)⇒(e))

(b)⇒(c): T has a countable model by the Downward Löwenheim Skolem Theorem.

(c)⇒(d): Let A be an oligomorphic countable model of T.

Suppose for some $n < \omega$, A realizes infinitely many complete n-types.

Then $\text{Aut}(A)$ has infinitely many orbits on n-tuples of A, contradiction.

(d)⇒(e): Let A be a countable model of T realizing only finitely many complete n-types for each $n < \omega$. For a fixed n, let $p_0, \ldots, p_{k-1}$ be the distinct types in $S_n(T)$ which are realized in A.

For each p_i there exists $\phi_i(\overline{x})$ in L in p_i but not in p_j, $j \neq i$.

Now $A \models \forall \overline{x} \bigvee_{i<k} \phi_i(\overline{x})$ and T is a complete theory.

So sentence $\forall \overline{x} \bigvee_{i<k} \phi_i(\overline{x})$ is a consequence of T.

If $\psi(\overline{x})$ is in L and $i < k$, $A \models \forall \overline{xy}(\phi_i(\overline{x}) \wedge \phi_i(\overline{y}) \rightarrow (\psi(\overline{x}) \leftrightarrow \psi(\overline{y})))$.

So $T \vdash \forall \overline{xy}(\phi_i(\overline{x}) \wedge \phi_i(\overline{y}) \rightarrow (\psi(\overline{x}) \leftrightarrow \psi(\overline{y})))$.

It follows that $p_0, \ldots, p_{k-1}$ are the only types in $S_n(T)$.

Proof ((e)⇒(f)⇒(g))

(e)⇒(f): Suppose two formulas $\phi(\overline{x})$ and $\psi(\overline{x})$ of L lie in exactly the same types $\in S_n(T)$. Then ϕ and ψ must be equivalent modulo T.

So if $S_n(T)$ has finite cardinality k, there are at most 2^k non-equivalent formulas $\phi(\overline{x})$ of L modulo T.

(f)⇒(g): For any $n < \omega$ and $\overline{x} = (x_0, \ldots, x_{n-1})$, take a maximal family of pairwise non-equivalent formulas $\phi(\overline{x})$ of L modulo T. Assuming (f), this family is finite.

Let p be any type $\in S_n(T)$.

Let θ be the conjunction of all formulas of the family which are in p.

Then θ is a support of p.

Proof ((a)⇔(g))

(a)⇒(g): Suppose (g) fails.

Then for some $n < \omega$, there is a non-principal type q in $S_n(T)$.

By the omitting types theorem, T has a model A which omits q.

By the definition of types, T also has a model B which realizes q.

Since T is complete and has infinite models, both A and B are infinite.

By the Downward Löwenheim-Skolem theorem we can suppose that both A and B are countable.

Hence T has two countable models which are not isomorphic.

Thus (a) fails.

(g)⇒(a): By (g) all models of T are atomic.

Hence, they are back-and-forth equivalent by a previous theorem.

By a previous theorem, all countable models of T are isomorphic.

Proof ((g)⇔(b))

(g)⇒(b): Again we deduce from (g) that all models of T are atomic.

Claim: If A, B are countable models of T and $\overline{a}, \overline{b}$ are n-tuples in A, B, respectively, such that $(A, \overline{a}) \equiv (B, \overline{b})$, then there is an isomorphism from A to B which takes $\overline{a}$ to $\overline{b}$.

This follows by the last theorem of the preceding section.

Let A be a countable model of T.

Let $\overline{a}, \overline{b}$ be n-tuples which realize the same complete type in A.

By the claim, $\overline{a}$ and $\overline{b}$ lie in the same orbit of $\text{Aut}(A)$.

To deduce (b), we need only show that (g) implies (e).

Proof ((g)⇔(b) Cont'd)

- To deduce (b), we need only show that (g) implies (e).
  Assume $S_n(T)$ is infinite. Suppose $S_n(T)$ contains λ principal types.
  Let $\theta_i(\overline{x})$, $i < \lambda$, be supports of these types.
  Take an n-tuple $\overline{c}$ of distinct new constants and define
  $$T' := T \cup \{\neg\theta_i(\overline{c}) : i < \lambda\}.$$
  If $(A, \overline{a})$ is a model of T', then A is a model of T in which $\overline{a}$ realizes a non-principal type.
  So it suffices to prove that T' has a model.
  Let $\Phi(\overline{x})$ be a finite subset of $\{\theta_i(\overline{x}) : i < \lambda\}$.
  Since $S_n(T)$ is infinite, there is a type $p(\overline{x})$ in $S_n(T)$ distinct from the types generated by the formulas in Φ.
  Hence every finite subset of T' has a model.
  By the Compactness Theorem, T' has a model.

ω-Categoricity and Local Finiteness

Corollary

If A is an ω-categorical structure, then A is locally finite.
In fact there is a (unique) function $f:\omega\to\omega$, depending only on $\mathrm{Th}(A)$, with the property that, for each $n<\omega$, $f(n)$ is the least number m, such that every n-generator substructure of A has at most m elements.

- Let $\overline{a}$ be an n-tuple of elements of A.

  Let $c\neq d$ be elements of the substructure $\langle\overline{a}\rangle_A$ generated by $\overline{a}$.

  The complete types of $\overline{a}c$ and $\overline{a}d$ over the empty set say how c and d are generated. So $\mathrm{tp}_A(\overline{a}c)\neq\mathrm{tp}_A(\overline{a}d)$.

  So by Part (e) of the theorem for $n+1$, $\langle\overline{a}\rangle_A$ is finite.

  This proves the first sentence.

ω-Categoricity and Local Finiteness (Cont'd)

- Let B be the unique countable structure elementarily equivalent to A.

  By Part (b) of the theorem, for each $n < \omega$, there are finitely many orbits of n-tuples in B.

  Let $\overline{b}_0, \ldots, \overline{b}_{k-1}$ be representatives of these orbits.

  Write m_i for the number of elements of the substructure $\langle \overline{b}_i \rangle_B$ of B generated by $\overline{b}_i$.

  Then define

$$f(n) = \max(m_i : i < k).$$

  A and B realize exactly the same types in $S_n(T)$, namely all of them.

  So this choice of f works for A as well as B.

Example: ω-Categorical Groups

- By the corollary, every countable ω-categorical group is locally finite and has finite exponent.

  For abelian groups, this provides a good description.

  Any abelian group A of finite exponent is a direct sum of finite cyclic groups.

  We can write down a first-order theory which says how often each cyclic group occurs in the sum (where the number of times is either 0, 1, 2, . . . or infinity).

  So an infinite abelian group is ω-categorical if and only if it has finite exponent.

  For groups in general the situation is much more complicated.

n-Tuples in the Same Orbit

Corollary

Let L be a countable first-order language. Let A be an L-structure which is either finite, or countable and ω-categorical. Then for any positive integer n, a pair $\overline{a}, \overline{b}$ of n-tuples from A are in the same orbit under $\text{Aut}(A)$ if and only if they satisfy the same formulas of L.

- This is almost the claim in (g)⇒(b) of the Theorem of Engeler, Ryll-Nardzewski and Svenonius, except that A may be finite.

  As in that proof, it suffices to show that A is atomic.

  If A is countable and ω-categorical, we get it by (g) of the theorem.

  If A is finite, we deduce it by the argument of (d)⇒(e) in the proof of the theorem.

n-Tuples in the Same Orbit (Rephrasing)

- Recall that a formula is **complete** if it generates a complete type.

Corollary

Let L be a countable first-order language. Let A be an L-structure which is either finite, or countable and ω-categorical. Then for each n,

- There are finitely many complete formulas $\phi_i(x_0,\dots,x_{n-1})$, $i < k_n$, of L for $\mathrm{Th}(A)$;
- The orbits of $\mathrm{Aut}(A)$ on $(\mathrm{dom}A)^n$ are exactly the sets $\phi_i(A^n)$, $i < k_n$.

- There are finitely many types of $S_n(T)$.

  All of them are principal.

  The conclusion follows by the preceding corollary.

Automorphism Groups and Definitional Equivalence

- We can almost recover A from the permutation group Aut(A).

Theorem

Let A be a countable ω-categorical L-structure with domain Ω, and let B be the canonical structure for Aut(A) on Ω. Then the relations on Ω which are first-order definable in A without parameters are exactly the same as those definable in B without parameters. In other words, A and B are definitionally equivalent.

- By definition of the canonical structure B, it has the same automorphism group as A, say G.
  Write L' for the language of B, assuming it is disjoint from L.
  Let R be an n-ary relation symbol of L.
  Then R^A is a union of finitely many orbits of G on Ω.
  So R can be defined by a disjunction of formulas of L' which define these orbits. The same argument works in the other direction.

Interpretations and ω-Categoricity

- We note that interpretations always preserve ω-categoricity.

Theorem

Let K and L be countable first-order languages, Γ an interpretation of L in K, and A an ω-categorical K-structure. Then ΓA is ω-categorical.

- Let A' be a countable structure which is elementarily equivalent to A. Then $\Gamma A' = \Gamma A$ by the reduction theorem. So it suffices to show that $\Gamma A'$ is ω-categorical. By the construction in a previous theorem, every element of $\Gamma A'$ is an equivalence class of the relation $=_\Gamma$ on $\text{dom}(A')$. Write $\overline{a}^=$ for the equivalence class containing the tuple $\overline{a}$.

  Each automorphism α of A' induces an automorphism $\Gamma(\alpha)$ of $\Gamma A'$, by the rule $\Gamma(\alpha)(\overline{a}^=) = (\alpha\overline{a})^=$. $\text{Aut}(A')$ is oligomorphic. So $\text{Aut}(\Gamma A')$ is oligomorphic too.
- In particular, relativized reducts of ω-categorical structures are ω-categorical.

Subsection 4

ω-Categorical Structures by Fraïssé's Method

Uniform Local Finiteness

- Fraïssé's construction has proved to be a very versatile way of building ω-categorical structures.
- The trick is to make sure that if **K** is the class whose Fraïssé limit we are taking, the sizes of the structures in **K** are kept under control by the number of generators.
- We say that a structure A is **uniformly locally finite** if there is a function $f : \omega \to \omega$, such that:

  For every substructure B of A, if B has a generator set of cardinality at most n, then B itself has cardinality at most $f(n)$.

- We say that a class **K** of structures is **uniformly locally finite** if there is a function $f : \omega \to \omega$, such that the displayed condition holds for every structure A in **K**.
- If the signature of **K** is finite and has no function symbols, then **K** is uniformly locally finite.

ω-Categorical Structures by Fraïssé's Method

Theorem

Suppose that the signature L is finite and $\mathbf{K}$ is a countable uniformly locally finite set of finitely generated L-structures with HP, JEP and AP. Let M be the Fraïssé limit of $\mathbf{K}$ and T the first-order theory $\mathrm{Th}(M)$ of M.

(a) T is ω-categorical;

(b) T has quantifier elimination.

- First we show that there is an $\forall_2$ theory U in L whose models are precisely the weakly homogeneous structures of age $\mathbf{K}$.

  We discuss, next, two crucial points for the construction of U.

ω-Categorical Structures by Fraïssé (Preparation)

- Two crucial points for the construction of U.
- Let A is any finite L-structure with n generators $\overline{a}$.

  By our assumption on L, there exists a quantifier-free formula $\psi = \psi_{A,\overline{a}}(x_0,\ldots,x_{n-1})$, such that:

  > For any L-structure B and n-tuple $\overline{b}$ of elements of B, $B \models \psi(\overline{b})$ if and only if there is an isomorphism from A to $\langle \overline{b} \rangle_B$ which takes $\overline{a}$ to $\overline{b}$.

  In fact $\psi_{A,\overline{a}}$ is a conjunction of literals satisfied by $\overline{a}$ in A.
- By the uniform local finiteness, for each $n < \omega$, there are only finitely many isomorphism types of structures in $\mathbf{K}$ with n generators.

ω-Categorical Structures by Fraïssé (Constructing U)

- Take U_0 to be the set of all sentences of the form
$$\forall \overline{x}(\psi_{A,\overline{a}}(\overline{x}) \to \exists y \psi_{B,\overline{a}b}(\overline{x}, y)),$$
where:
  - B is a structure in **K** generated by a tuple $\overline{a}b$ of distinct elements;
  - A is the substructure generated by $\overline{a}$.

In case $\overline{a}$ is empty, the sentence reduces to $\exists y \psi_{B,b}(y)$.
Take U_1 to be the set of all sentences of the form
$$\forall \overline{x} \bigvee_{A,\overline{a}} \psi_{A,\overline{a}}(\overline{x}),$$
where the disjunction ranges over all pairs $A, \overline{a}$, such that:
  - A is in **K**;
  - $\overline{a}$ is a tuple of the same length as $\overline{x}$ which generates A.

Uniform local finiteness implies that this is a finite disjunction.
Write U for the union $U_0 \cup U_1$.
Clearly M is a model of U.

ω-Categorical Structures by Fraïssé (ω-Categoricity)

- Suppose D is any countable model of U.

  When $\overline{a}$ is empty, the sentences in U_0 say that every one-generator structure in **K** is embeddable in D.

  In general the sentences in U_0 say that if A, B are finitely generated substructures of D, $A \subseteq B$, B comes from A by adding one more generator, and $f : A \to D$ is an embedding, then there is an embedding $g : B \to D$ which extends f.

  It is not hard to see, using induction on the number of generators, that these two facts imply that every structure in **K** is embeddable in D.

  So, taken with U_1, they tell us that the age of D is exactly **K**.

  Using the sentences U_0 again, an induction on the size of $\text{dom}(B) \backslash \text{dom}(A)$ tells us that D is weakly homogeneous.

  By a previous lemma, D is isomorphic to M.

  Hence, U is ω-categorical, and U is a set of axioms for T.

ω-Categorical Structures by Fraïssé (Quantifier Elimination)

- Suppose now that $\phi(\overline{x})$ is a formula of L with $\overline{x}$ non-empty.

  Let X be the set of all tuples $\overline{a}$ in M, such that $M \models \phi(\overline{a})$.

  If $\overline{a}$ is in X, and $\overline{b}$ is such that there is an isomorphism $e : \langle\overline{a}\rangle_M \to \langle\overline{b}\rangle_M$ taking $\overline{a}$ to $\overline{b}$, then e extends to an automorphism of M.

  So $\overline{b}$ is in X too.

  It follows that ϕ is equivalent modulo T to the disjunction of all the formulas $\psi_{\langle\overline{a}\rangle,\overline{a}}(\overline{x})$ with $\overline{a}$ in X.

  This is a finite disjunction of quantifier-free formulas.

  Finally, let ϕ is a sentence of L.

  Since T is complete, ϕ is equivalent modulo T to either $\neg\bot$ or $\bot$.

  So T has quantifier elimination.

Homogeneity, Finiteness, Categoricity, Quantifier Elimination

Corollary

Let L be a finite signature and M a countable L-structure. Then the following are equivalent:

(a) M is ultrahomogeneous and uniformly locally finite.

(b) $\mathrm{Th}(M)$ is ω-categorical and has quantifier elimination.

(a)⇒(b) By the preceding theorem.

(b)⇒(a) By a previous corollary, if $\mathrm{Th}(M)$ is ω-categorical, then it is uniformly locally finite.

By another corollary, if M is countable and ω-categorical, then, for every n, a pair $\overline{a}$, $\overline{b}$ of n-tuples M are in the same orbit under $\mathrm{Aut}(M)$ iff they satisfy the same L-formulas.

Thus, if $\mathrm{Th}(M)$ has quantifier elimination, it is ultrahomogeneous.

- We describe two applications of the theorem in detail.

First Application: The Random Graph

- A **graph** is a structure consisting of:
  - A set X;
  - An irreflexive symmetric binary relation R defined on X.
- The elements of X are called the **vertices**.
- An **edge** is a pair of vertices $\{a, b\}$ such that aRb.
- We say that two vertices a, b are **adjacent** if $\{a, b\}$ is an edge.
- A **path of length** n is a sequence of edges

$$\{a_0, a_1\}, \{a_1, a_2\}, \ldots, \{a_{n-2}, a_{n-1}\}, \{a_{n-1}, a_n\}.$$

- The path is a **cycle** if $a_n = a_0$.
- A **subgraph** of a graph G is simply a substructure of G.
- We write L for the first-order language appropriate for graphs
- Its signature consists of just one binary relation symbol R.

Properties of Finite Graphs and the Random Graph

Lemma

Let $\mathbf{K}$ be the class of all finite graphs.

- The signature of $\mathbf{K}$ contains only finitely many symbols.
- $\mathbf{K}$ contains arbitrarily large finite structures.
- $\mathbf{K}$ is uniformly locally finite.
- The class $\mathbf{K}$ has HP, JEP and AP.

- All of these facts are relatively obvious.
- So by previous theorems:
  - $\mathbf{K}$ has a Fraïssé limit A;
  - $\text{Th}(A)$ is ω-categorical and has quantifier elimination.
- The structure A is a countable graph.
- It is known as the **random graph** and denoted by Γ.

Characterization of the Random Graph

Theorem

Let A be a countable graph. The following are equivalent:

(a) A is the random graph Γ.

(b) For all disjoint finite sets X and Y of vertices of A, there is a vertex not in $X \cup Y$, adjacent to all vertices in X and to no vertices in Y.

(a)⇒(b): Let A be the random graph Γ.

Let X, Y be disjoint finite sets of vertices of Γ.

Construct a finite graph G as follows:

- The vertices of G are the vertices in $X \cup Y$ together with one new vertex u;
- Vertices in $X \cup Y$ are adjacent in G if they are adjacent in Γ;
  u is adjacent to all the vertices in X and none of the vertices in Y.

Characterization of the Random Graph (Cont'd)

- Γ is the Fraïssé limit of $\mathbf{K}$.

  So there is an embedding $f : G \to \Gamma$.

  The restriction of f to $X \cup Y$ is an isomorphism between finite substructures of Γ.

  So it extends to an automorphism g of Γ.

  Then $g^{-1}(f(u))$ is the element described in (b).

Characterization of the Random Graph (Converse)

(b)⇒(a): Assume (b).

Claim: Suppose $G \subseteq H$ are finite graphs and $f : G \to A$ is an embedding. Then f extends to an embedding $g : H \to A$.

By induction on the number n of vertices in H but not in G.

Clearly we only need worry about the case $n = 1$.

Let w be the vertex which is in H but not in G.

- Let X be the set of vertices $f(x)$, with x in G and adjacent to w in H.
- Let Y be the set of vertices $f(y)$, with y in G but not adjacent to w.

By (b) there exists u in A which is adjacent to all of X and none of Y.

We extend f to g by putting $g(w) = u$.

Taking G to be the empty structure, it follows that every finite graph is embeddable in A. So the age of A is $\mathbf{K}$. Taking G to be a substructure of A, it follows that A is weakly homogeneous.

So by a previous lemma, A is the Fraïssé limit of $\mathbf{K}$.

Second Application: The Random Structure

- Let L be a non-empty finite signature.
- Let $\mathbf{K}$ be the class of all finite L-structures.
- Then clearly $\mathbf{K}$ has HP, JEP and AP, and there are just countably many isomorphism types of structures in $\mathbf{K}$.
- So $\mathbf{K}$ has a countable Fraïssé limit.
- It is known as the **random structure** of signature L.
- It is denoted by $\mathrm{Ran}(L)$.

Axioms and Completeness

- Let T be the set of all sentences of L of the form

$$\forall \overline{x}(\psi(\overline{x}) \rightarrow \exists y \chi(\overline{x}, y)),$$

  such that for some finite L-structure B and some listing of the elements of B without repetition as $\overline{b}c$:
  - The formula $\psi(\overline{x})$ lists the literals satisfied by $\overline{b}$ in B;
  - The formula $\chi(\overline{x}, y)$ lists the literals satisfied by $\overline{b}c$ in B.
- Inspection shows that T consists of exactly the sentences U_0 seen in the proof of a previous theorem.
- The sentences U_1 of the same proof are trivially satisfied in this case.
- So T is a set of axioms for the theory of $\mathrm{Ran}(L)$.
- In particular T is complete.

$\mu_n(\phi)$

- Let $n < \omega$.
- Let $\phi(x_0,\ldots,x_{n-1})$ be a formula of L.
- Let $\overline{a}$ be a tuple of objects a_i, $i < n$.
- We write $\kappa_n(\phi)$ for the number of non-isomorphic L-structures B whose distinct elements are $a_0,\ldots,a_{n-1}$, such that $B \models \phi(\overline{a})$.
- We write $\mu_n(\phi(\overline{a}))$ for the ratio

$$\frac{\kappa_n(\phi(\overline{a}))}{\kappa_n(\forall x x = x)}.$$

- This is the proportion of those L-structures with elements $a_0,\ldots,a_{n-1}$ for which $\phi(\overline{a})$ is true.

Calculating $\lim_{\to\infty} \mu_n(\phi)$

Lemma

Let ϕ be any sentence in T. Then $\lim_{n\to\infty} \mu_n(\phi) = 1$.

- Let ϕ be the sentence $\forall \overline{x}(\psi(\overline{x}) \to \exists y \chi(\overline{x}, y))$.

  We show $\lim_{n\to\infty} \mu_n(\neg\phi) = 0$.

  Since $\mu_n(\neg\phi) = 1 - \mu_n(\phi)$, this will prove the lemma.

  Suppose $\overline{x}$ is $(x_0, \ldots, x_{m-1})$, and $n > m$.

  Consider those structures B whose distinct elements are $a_0, \ldots, a_{n-1}$, such that

$$B \models \psi(a_0, \ldots, a_{m-1}).$$

  We determine the probability p that

$$B \models \forall y \neg\chi(a_0, \ldots, a_{m-1}, y).$$

Calculating $\lim_{\to\infty}\mu_n(\phi)$ (Cont'd)

- The $n-m$ elements $a_m,\ldots,a_{n-1}$ have equal and independent chances of serving for y.

  So the probability p must be the $(n-m)$-th power of the probability that
  $$B \models \neg\chi(a_0,\ldots,a_{m-1},a_m).$$
  But, the signature of L is not empty.

  So there is some positive real $k<1$, such that
  $$B \models \neg\chi(a_0,\ldots,a_{m-1},a_m) \text{ with probability } k.$$
  So $p=k^{n-m}$.

  Next, consider those L-structures C whose distinct elements are $a_0,\ldots,a_{n-1}$.

  We estimate the probability $q=\mu_n(\neg\phi)$ that $C\models\neg\phi$.

Calculating $\lim_{\to\infty}\mu_n(\phi)$ (Cont'd)

- q is at most the probability that $C \models \psi(\overline{c}) \wedge \forall y \neg\chi(\overline{c}, y)$ for a tuple $\overline{c}$ of distinct elements of C, times the number of ways of choosing $\overline{c}$ in C.

  So
  $$\mu_n(\neg\phi) \leq n^m \cdot k^{n-m} = \gamma \cdot n^m \cdot k^n, \text{ where } \gamma = k^{-m}.$$

  Since $0 < k < 1$, we have $n^m \cdot k^n \stackrel{n\to\infty}{\to} 0$.

  It follows that $\lim_{n\to\infty}(\neg\phi) = 0$.

Theorem (Zero-One Law)

Let ϕ be any first-order sentence of a finite relational signature. Then $\lim_{n\to\infty}\mu_n(\phi)$ is either 0 or 1.

- We have already seen that T is a complete theory.
  - If ϕ is a consequence of T, by the lemma $\lim_{n\to\infty}\mu_n(\phi)$ is 1.
  - If ϕ is not a consequence of T, then T implies $\neg\phi$.
    So $\lim_{n\to\infty}(\neg\phi)$ is 1. Thus, $\lim_{n\to\infty}\mu_n(\phi)$ is 0.