Introduction to Model Theory

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Subsection 1

Existentially Closed Structures

Existential Closure: Algebraic Version

- Let L be a first-order language without relation symbols.
- Let $\mathbf{K}$ be a class of L-structures.

  Example: L might be the language of rings and $\mathbf{K}$ the class of fields.

  L might be the language of groups and $\mathbf{K}$ the class of groups.
- We say that a structure A in $\mathbf{K}$ is **existentially closed in $\mathbf{K}$** (or more briefly, **e.c. in $\mathbf{K}$**) if the following holds:

  If E is a finite set of equations and inequations with parameters from A, and E has a simultaneous solution in some extension B of A with B in $\mathbf{K}$, then E has a solution already in A.

Existential Closure: Model-Theoretic Version

- The model-theoretic version of existential closure has the advantage that it covers languages which also have relation symbols.
- A formula is **primitive** if it has the form $\exists\overline{y}\bigwedge_{i<n}\psi_i(\overline{x},\overline{y})$, where n is a positive integer and each formula ψ_i is a literal.
- Note that, in a language without relation symbols, each literal is either an equation or an inequation.

  So, in this case, a primitive formula expresses that a certain finite set of equations and inequations has a solution.
- A structure A in the class **K** of L-structures is **existentially closed in K** (or more briefly, **e.c. in K**) if:

  > For every primitive formula $\phi(\overline{x})$ of L and every tuple $\overline{a}$ in A, if there is a structure B in **K** such that $A\subseteq B$ and $B\models\phi(\overline{a})$, then already $A\models\phi(\overline{a})$.
- This definition agrees with that of the preceding slide.

Existentially Closed Models

- When **K** is the class of fields, an e.c. structure in **K** is known as an **e.c. field**.
- Likewise **e.c. lattice** when **K** is the class of lattices, and so on.
- When **K** is the class of all models of a theory T, we refer to e.c. structures in **K** as **e.c. models of** T.
- Let L be a first-order language.
- Let A, B be L-structures.
- $A \preccurlyeq_1 B$ means that:

  For every existential formula $\phi(\overline{x})$ of L and every tuple $\overline{a}$ in A,

$$B \models \phi(\overline{a}) \quad \text{implies} \quad A \models \phi(\overline{a}).$$

An Equivalent Version

Lemma

Let $\mathbf{K}$ be a class of L-structures and A a structure in $\mathbf{K}$.
Then A is e.c. in $\mathbf{K}$ if and only if the e.c. condition holds with "primitive" replaced by "existential".
In particular, if A, B are structures in $\mathbf{K}$, $A \subseteq B$ and A is e.c., then $A \preccurlyeq_1 B$.

- By disjunctive normal form, every $\exists_1$ formula is logically equivalent to a disjunction of primitive formulas.

Example: Algebraically Closed Fields

Claim: An e.c. field A must be algebraically closed.

Let $F(y)$ be a polynomial of positive degree with coefficients in A.

Using the language of rings, we can rewrite $F(y)$ as $p(\overline{a}, y)$, where:

- $p(\overline{x}, y)$ is a term;
- $\overline{a}$ is a tuple of elements of A.

Replacing F by an irreducible factor if necessary, we have a field $B = A[y]/(F)$ which extends A and contains a root of F.

So $B \models \exists y p(\overline{a}, y) = 0$.

But, by hypothesis, A is an e.c. field.

Hence, $A \models \exists y p(\overline{a}, y) = 0$ also.

So F has a root already in A.

Thus, every e.c. field is algebraically closed.

Example: Algebraically Closed Fields (Cont'd)

Claim: If A is an algebraically closed field, then every finite system of equations and inequations over A which is solvable in some field extending A already has a solution in A.

Let $E(x)$ be a finite system of equations and inequations over A.

We can write the statement "E has a solution" as a primitive formula $\phi(\overline{a})$, where $\overline{a}$ are the coefficients of E in A.

Suppose E has a solution in some field B which extends A.

Extend B to an e.c. field C (see next section).

By assumption, $B \models \phi(\overline{a})$.

So $C \models \phi(\overline{a})$ since $\phi(\overline{a})$ is an $\exists_1$ formula.

By the previous paragraph, C is algebraically closed.

The theory of algebraically closed fields is model-complete (to come).

So, since $A \subseteq C$, it follows that $A \models \phi(\overline{a})$.

Thus E already has a solution in A.

So the e.c. fields are precisely the algebraically closed fields.

Rabinowitsch's Observation About Fields

- Note first that every equation with parameters $\overline{a}$ from a field A can be written as $p(\overline{a},\overline{y})=0$, where $p(\overline{x},\overline{y})$ is a polynomial whose indeterminates are variables from $\overline{x},\overline{y}$, with integer coefficients.
- So we can assume that any primitive formula has the form

$$\exists\overline{y}(p_0(\overline{a},\overline{y})=0\wedge\cdots\wedge p_{k-1}(\overline{a},\overline{y})=0 \\ \wedge q_0(\overline{a},\overline{y})\neq 0\wedge\cdots\wedge q_{m-1}(\overline{a},\overline{y})\neq 0).$$

- In a field, $x\neq 0$ says the same as $\exists zx\cdot z=1$.
- Hence, we can eliminate the inequations, bringing the new existential quantifiers $\exists z$ forward to join $\exists\overline{y}$.
- This reduces the displayed formula to the form

$$\exists\overline{y}(p_0(\overline{a},\overline{y})=0\wedge\cdots\wedge p_{k-1}(\overline{a},\overline{y})=0).$$

- To show that a field A is existentially closed, we only need to consider equations in the definition, not both equations and inequations.

Example: Fraïssé Limits

- Let L be a countable signature. Let $\mathbf{J}$ be a countable set of finitely generated L-structures which has HP, JEP and AP. Let $\mathbf{K}$ be the class of all L-structures with age $\subseteq \mathbf{J}$. Let A be the Fraïssé limit of $\mathbf{K}$. Then A is existentially closed in $\mathbf{K}$.

  Suppose B is in $\mathbf{K}$, $A \subseteq B$, $\overline{a}$ is a tuple of elements of A and $\psi(\overline{x},\overline{y})$ is a conjunction of literals of L such that $B \models \exists \overline{y} \psi(\overline{a},\overline{y})$.

  Take a tuple $\overline{b}$ in B such that $B \models \psi(\overline{a},\overline{b})$. Let:
  - C be the substructure of A generated by $\overline{a}$;
  - D the substructure of B generated by $\overline{a}\overline{b}$.

  Then $C \subseteq D$, and both C and D are in the age $\mathbf{J}$ of A.

  By Fraïssé's Theorem, A is weakly homogeneous.

  Thus, the inclusion map $f : C \to A$ extends to an embedding $g : D \to A$.

  So $A \models \psi(\overline{a}, g(\overline{b}))$. Hence, $A \models \exists \overline{y} \psi(\overline{a},\overline{y})$.

Joint Embedding Property

- Let L be a first-order language.
- Let T be a theory over L.
- We say that T has the **joint embedding property** (**JEP**) if,

$$\begin{tikzcd} A \arrow[dr, dashed] & \\ & C \\ B \arrow[ur, dashed] & \end{tikzcd}$$

given any two models A, B of T, there is a model of T in which both A and B are embeddable.

Joint Embedding Property and $\forall_2$ Sentences

Theorem

Let L be a first-order language and T a theory in L. Suppose that T has JEP, and let A, B be e.c. models of T. Then every $\forall_2$ sentence of L which is true in A is also true in B.

- There is a model C of T in which both A and B are embeddable. Assume that A and B are substructures of C.

  Since A is e.c. in the class of models of T, we have $A \preccurlyeq_1 C$.

  By the Existential Amalgamation Theorem, there is an elementary extension D of A with $C \subseteq D$.

  Suppose $A \models \forall \overline{x} \exists \overline{y} \phi(\overline{x}, \overline{y})$, where ϕ is a quantifier-free formula of L, and let $\overline{b}$ be a tuple of elements of B. We must show $B \models \exists \overline{y} \phi(\overline{b}, \overline{y})$.

  But $D \models \phi(\overline{b}, \overline{y})$, since $A \preccurlyeq D$.

  Since B is an e.c. model, $B \preccurlyeq_1 D$.

  Hence, $B \models \exists \overline{y} \phi(\overline{b}, \overline{y})$.

AP Substructures and $\exists_1$ Formulas

Theorem

Let L be a first-order language. Let $\mathbf{K}$ be a class of L-structures which is closed under isomorphic copies. Suppose that the class of all substructures of structures in $\mathbf{K}$ has AP.

- For every $\exists_1$ formula $\phi(\overline{x})$ of L there is a quantifier-free formula $\chi(\overline{x})$ (possibly infinitary) which is equivalent to ϕ in all e.c. structures in $\mathbf{K}$.
- In particular, if there is a first order theory T such that the e.c. structures in $\mathbf{K}$ are exactly the models of T, then ϕ is equivalent modulo T to a quantifier-free formula of L.

Proof

- Let us say that a pair $(A,\overline{a})$ is **good** if:
  - A is an e.c. structure in $\mathbf{K}$;
  - $\overline{a}$ is a tuple of elements of A;
  - $A \models \phi(\overline{a})$.

  For each good pair $(A,\overline{a})$, let

  $$\theta_{(A,\overline{a})}(\overline{x}) := \bigwedge\{\psi(\overline{x}) : \psi \text{ is a literal of } L \text{ and } A \models \psi(\overline{a})\}.$$

  Now set

  $$\chi(\overline{x}) := \bigvee\{\theta_{(A,\overline{a})}(\overline{x}) : (A,\overline{a}) \text{ a good pair}\}.$$

  Let B be any e.c. structure in $\mathbf{K}$.

  Let $\overline{b}$ be a tuple in B such that $B \models \phi(\overline{b})$.

  Then, since $(B,\overline{b})$ is a good pair, $B \models \chi(\overline{b})$.

Proof (Converse)

- Let B be an e.c. structure in $\mathbf{K}$ and $B \models \chi(\overline{b})$. Then there is a good pair $(A, \overline{a})$ and an isomorphism $e : \langle \overline{a} \rangle_A \to B$, with $e(\overline{a}) = \overline{b}$.

  The class of substructures of structures in $\mathbf{K}$ has AP.

  So there is a structure C in $\mathbf{K}$ with embeddings $g : A \to C$ and $h : B \to C$, such that $g(\overline{a}) = h(\overline{b})$. Since $\mathbf{K}$ is closed under isomorphism, we can assume that h is an inclusion.

  By assumption, $A \models \phi(\overline{a})$.

  Since ϕ is an $\exists_1$ formula, $C \models \phi(g(\overline{a}))$.

  Since B is e.c. in $\mathbf{K}$, $B \models \phi(\overline{b})$.

  Finally suppose that the models of T are exactly the e.c. structures in $\mathbf{K}$. Then, by what has been shown, $T \vdash \forall \overline{x}(\phi \leftrightarrow \chi)$.

  Two applications of the Compactness Theorem reduce χ to a first-order quantifier-free formula.

Remark on the Theorem

- The preceding theorem is false if we exclude empty structures.

  Example: Consider the theory

  $$\forall xy(P(x) \leftrightarrow P(y)).$$

  Let ϕ be the formula $\exists x P(x)$.
- If we do exclude empty structures, we can rescue the theorem by requiring ϕ to have at least one free variable.

Application to Fields

- Let **K** be the class of fields. Thanks to the previous example, we already know that the class of e.c. fields is first-order axiomatizable.

  It is a well-known fact of algebra that **K** has the amalgamation property (for example by the uniqueness of algebraic closures).

  From this we easily deduce that the class of integral domains has the amalgamation property too (by taking fields of fractions).

  But in the signature of rings, a substructure of a field is the same thing as an integral domain. So we have:

Corollary

Let T be the theory of algebraically closed fields. Then T has quantifier elimination.

- By the theorem, the argument above shows that every $\exists_1$ formula is equivalent to a quantifier-free formula modulo T. By a previous lemma, this is sufficient for quantifier elimination.

Subsection 2

Constructing Existentially Closed Structures

Inductive Classes and Local Properties

- Let L be a signature.
- Let $\mathbf{K}$ be a class of L-structures.
- We say that $\mathbf{K}$ is **inductive** if:
  (1) $\mathbf{K}$ is closed under taking unions of chains;
  (2) Every structure isomorphic to a structure in $\mathbf{K}$ is also in $\mathbf{K}$.

  Example: Let T be an $\forall_2$ first-order theory in L.

  Let $\mathbf{K}$ be the class of all models of T.

  By a previous theorem, $\mathbf{K}$ is inductive.

  Thus:
  - The class of all groups is inductive.
  - The class of all fields is inductive.

Inductive Classes and Local Properties

- Let L be a signature.
- Let π be a structural property which an L-structure might have.
- We say that an L-structure A **has π locally** if all the finitely generated substructures of A have property π.
- The class of all L-structures which have π locally is an inductive class.
- A structure is **locally finite** if all its finitely generated substructures are finite.

  Example:
  - The class of all locally finite groups is inductive.
  - The class of all groups without elements of infinite order is inductive.

  Neither of these two classes is first-order axiomatizable.

Inductive Classes and e.c. Structures

Theorem

Let **K** be an inductive class of L-structures and A a structure in **K**. Then there is an e.c. structure B in **K** such that $A \subseteq B$.

Claim: There is a structure A^* in **K**, such that $A \subseteq A^*$, and if $\phi(\overline{x})$ is an $\exists_1$ formula of L, $\overline{a}$ a tuple in A and there is a structure C in **K**, such that $A^* \subseteq C$ and $C \models \phi(\overline{a})$, then $A^* \models \phi(\overline{a})$.

List as $(\phi_i, \overline{a}_i)_{i<\lambda}$ all pairs $(\phi, \overline{a})$, where:

- ϕ is an $\exists_1$ formula of L;
- $\overline{a}$ is a tuple in A.

By induction on i, define a chain of structures $(A_i : i \leq \lambda)$ in **K** by:

- $A_0 = A$;
- If δ is a limit ordinal, $\leq \lambda$, $A_\delta = \bigcup_{i<\delta} A_i$;
- A_{i+1} = some structure C in **K**, such that $A_i \subseteq C$ and $C \models \phi_i(\overline{a}_i)$, if there is such a structure C, and A_i, otherwise.

Put $A^* = A_\lambda$.

Inductive Classes and e.c. Structures (Cont'd)

- To show that A^* is as required, take any $\exists_1$ formula $\phi(\overline{x})$ of L and any tuple $\overline{a}$ of elements of A. Then $(\phi, \overline{a})$ is $(\phi_i, \overline{a}_i)$, for some $i < \lambda$.

  Suppose C is a structure in $\mathbf{K}$ such that $A^* \subseteq C$ and $C \models \phi(\overline{a})$.

  Then $A_i \subseteq C$. So $A_{i+1} \models \phi(\overline{a})$ by definition. Since ϕ is an $\exists_1$ formula and $A_{i+1} \subseteq A^*$, we infer, by a previous theorem, that $A^* \models \phi(\overline{a})$.

  Now define a chain of structures $A^{(n)}$, $n < \omega$, in $\mathbf{K}$ by induction on n.
  - $A^{(0)} = A$;
  - $A^{(n+1)} = A^{(n)*}$.

  Put $B = \bigcup_{n<\omega} A^{(n)}$. Then B is in $\mathbf{K}$, since $\mathbf{K}$ is inductive. Also $A \subseteq B$.

  Suppose $\phi(\overline{x})$ is an $\exists_1$ formula of L and $\overline{a}$ is a tuple from B, such that $C \models \phi(\overline{a})$, for some C which is in $\mathbf{K}$ and extends B.

  Since $\overline{a}$ is finite, it lies within $A^{(n)}$, for some $n < \omega$.

  Since $A^{(n+1)}$ is $A^{(n)*}$, $A^{(n+1)} \models \phi(\overline{a})$.

  But ϕ is an $\exists_1$ formula and $A^{(n+1)} \subseteq B$. So again $B \models \phi(\overline{a})$.

The Size of e.c. Structures in Inductive Classes

Corollary

Let $\mathbf{K}$ be an inductive class, A a structure in $\mathbf{K}$ and λ an infinite cardinal $\geq |A|$. Suppose also that for every structure C in $\mathbf{K}$ and every set X of $\leq \lambda$ elements of C, there is a structure B in $\mathbf{K}$, such that $X \subseteq \mathrm{dom}(B)$, $B \preccurlyeq C$ and $|B| \leq \lambda$. (For example, suppose $\mathbf{K}$ is the class of all models of some $\forall_2$ theory in a first-order language of cardinality $\leq \lambda$.) Then there is an e.c. structure B in $\mathbf{K}$, such that $A \subseteq B$ and $|B| \leq \lambda$.

- Counting the number of pairs $(\phi, \overline{a})$, we find that we can use λ as the cardinal λ in the proof of the theorem. In that proof, choose each structure A_{i+1} so that it has cardinality $\leq \lambda$. For example if there is a structure C in $\mathbf{K}$ such that $A_i \subseteq C$ and $C \models \phi_i(\overline{a}_i)$, choose A_{i+1} in $\mathbf{K}$ so that $A_{i+1} \preccurlyeq C$, $\mathrm{dom}(A_i) \subseteq \mathrm{dom}(A_{i+1})$ and $|A_{i+1}| \leq \lambda$. Then $A_i \subseteq A_{i+1}$ and $A_{i+1} \models \phi_i(\overline{a}_i)$ as required. With these choices, A^* also has cardinality $\leq \lambda$. Hence, so does B in the theorem.

E.c. Models of $\forall_2$ Theories

- $T_\forall$ was defined to be the set of all $\forall_1$ sentences ϕ of L such that $T \vdash \phi$.
- By a previous corollary, the models of $T_\forall$ are precisely the substructures of T.
- So every model of $T_\forall$ can be extended to a model of T.

Corollary

Let L be a first-order language, T an $\forall_2$ theory in L and A an infinite model of $T_\forall$. Then there is an e.c. model B of T such that $A \subseteq B$ and $|B| = \max(|A|, |L|)$.

- By a previous corollary, there is a model C of T, such that $A \subseteq C$.

  By the Downward Löwenheim-Skolem theorem, we can take C to be of cardinality $\max(|A|, |L|)$.

  Finally, we apply the preceding corollary.

Unsatisfiability of $\exists_1$ Formulas in e.c. Structures

Theorem

Let L be a first-order language, T an $\forall_2$ theory in L and A a model of T. Then the following are equivalent:

(a) A is an e.c. model of T.

(b) A is a model of $T_\forall$, and for every $\exists_1$ formula $\phi(\overline{x})$ of L and every tuple $\overline{a}$ in A, if $A \vDash \neg\phi(\overline{a})$, then there is an $\exists_1$ formula $\chi(\overline{x})$ of L, such that $A \vDash \chi(\overline{a})$ and $T \vdash \forall\overline{x}(\chi \to \neg\phi)$.

(c) A is an e.c. model of $T_\forall$.

(a)$\Rightarrow$(b): Assume (a). Then A is a model of $T_\forall$.
Let $\phi(\overline{x})$ be an $\exists_1$ formula of L and $\overline{a}$ in A, such that $A \vDash \neg\phi(\overline{a})$.
Now A is an e.c. model of T. So there is no model C of T such that $A \subseteq C$ and $C \vDash \phi(\overline{a})$. Add distinct new constants $\overline{c}$ to the language as names for the elements of $\overline{a}$. Since there is no model C as described, by the Diagram Lemma, the theory $\mathrm{diag}(A) \cup T \cup \{\phi(\overline{c})\}$ has no model.

Unsatisfiability of $\exists_1$ Formulas in e.c. Structures (Cont'd)

So by the Compactness Theorem, there are a tuple $\overline{d}$ of distinct elements of A and a quantifier-free formula $\theta(\overline{x},\overline{y})$ of L, such that $A \models \theta(\overline{c},\overline{d})$ and $T \vdash \theta(\overline{c},\overline{d}) \to \neg\phi(\overline{c})$.

Now we use the Lemma on Constants, noting that even if the tuple $\overline{a}$ contains repetitions, we had the foresight to introduce a tuple $\overline{c}$ of distinct constants. The result is that $T \vdash \forall\overline{x}(\exists y\theta(\overline{x},\overline{y}) \to \neg\phi(\overline{x}))$.

To infer (b), let χ be $\exists\overline{y}\theta$.

(b)⇒(c): Assume (b). Then A is a model of $T_\forall$. Suppose $\phi(\overline{x})$ is an $\exists_1$ formula of L. Let $\overline{a}$ be in A, such that for some model C of $T_\forall$, $A \subseteq C$ and $C \models \phi(\overline{a})$. We must show that $A \models \phi(\overline{a})$.

Suppose not. By (b), there is an $\exists_1$ formula $\chi(\overline{x})$, such that $A \models \chi(\overline{a})$ and $T \vdash \forall\overline{x}(\chi \to \neg\phi)$. Now χ is $\exists_1$ and $A \subseteq C$. Hence, $C \models \chi(\overline{a})$.

The sentence $\forall\overline{x}(\chi \to \neg\phi)$ is $\forall_1$ (after some trivial rearrangement). So it lies in $T_\forall$. Since C is a model of $T_\forall$, we infer that $C \models \neg\phi(\overline{a})$, a contradiction.

Unsatisfiability of $\exists_1$ Formulas in e.c. Structures (Cont'd)

(c)$\Rightarrow$(a): Assume (c).

First, we must show that A is a model of T.

Since T is an $\forall_2$ theory, a typical sentence in T can be written $\forall\overline{x}\exists\overline{y}\psi(\overline{x},\overline{y})$ with ψ quantifier-free.

Let $\overline{a}$ be any tuple in A. We must show that $A \vDash \exists\overline{y}\psi(\overline{a},\overline{y})$.

By a previous corollary, since $A \vDash T_\forall$, there is a model C of T, such that $A \subseteq C$. Then $C \vDash \exists\overline{y}\psi(\overline{a},\overline{y})$. But A is an e.c. model of $T_\forall$. So $A \vDash \exists\overline{y}\psi(\overline{a},\overline{y})$. Thus A is a model of T.

Finally, every model of T extending A is also a model of $T_\forall$.

Hence, A is an e.c. model of T.

Resultants

- Let T be an $\forall_2$ theory in a first-order language L.
- Let $\phi(\overline{x})$ be an $\exists_1$ formula of L.
- The **resultant** $\text{Res}_\phi(\overline{x})$ **of** ϕ is the set of all $\forall_1$ formulas $\psi(\overline{x})$ of L, such that $T \vdash \forall\overline{x}(\phi \to \psi)$.

Lemma

Let L be a first-order language, T an $\forall_2$ theory in L and A an L-structure. Suppose $\phi(\overline{x})$ is an $\exists_1$ formula of L and $\overline{a}$ is a tuple from A. Then the following are equivalent:

(a) There is a model B of T, such that $A \subseteq B$ and $B \models \phi(\overline{a})$.

(b) $A \models \bigwedge \text{Res}_\phi(\overline{a})$.

(a)⇒(b): Suppose (a) holds. Let $\psi(\overline{x})$ be a formula in $\text{Res}_\phi(\overline{x})$.

Then $B \models \psi(\overline{a})$, since B is a model of T.

But ψ is an $\forall_1$ formula and $A \subseteq B$. So $A \models \psi(\overline{a})$.

Resultants (Cont'd)

(b)⇒(a): Assuming that (a) fails, we shall contradict (b).

Let $\overline{c}$ be a tuple of distinct new constants naming the elements $\overline{a}$.

By the Diagram Lemma, $T \cup \text{diag}(A) \cup \{\phi(\overline{c})\}$ has no model.

By the Compactness Theorem, there are a quantifier-free formula $\theta(\overline{x}, \overline{y})$ of L and distinct elements $\overline{d}$ of A such that

$$A \models \theta(\overline{a}, \overline{d}) \quad \text{and} \quad T \vdash \phi(\overline{c}) \to \neg\theta(\overline{c}, \overline{d}).$$

By the Lemma on Constants, we find $T \vdash \forall\overline{x}(\phi(\overline{x}) \to \forall\overline{y}\neg\theta(\overline{x}, \overline{y}))$.

So $\forall\overline{y}\neg\theta(\overline{x}, \overline{y})$ is a formula in Res_ϕ.

But $A \models \exists\overline{y}\theta(\overline{a}, \overline{y})$.

So $A \not\models \bigwedge \text{Res}_\phi(\overline{a})$.

Example: Nilpotent Elements in Commutative Rings

- Let A be a commutative ring and a an element of A.

  We ask when there is a commutative group $B \supseteq A$ containing a non-zero idempotent b (i.e., $b^2 = b$) which is divisible by a?

  I.e., when A can be extended to a commutative ring B in which it holds that

  $$\exists z(az \neq 0 \wedge (az)^2 = az).$$

  Claim: There is such a ring B if and only if a is not nilpotent (i.e., if and only if there is no $n < \omega$ such that $a^n = 0$).

  Suppose, first, $ab \neq 0$ and $(ab)^2 = ab$.

  Then, for all n,

  $$0 \neq ab = (ab)^n = a^n b^n.$$

  So $a^n \neq 0$.

Nilpotent Elements in Commutative Rings (Converse)

- In the other direction, suppose a is not nilpotent.
  Consider the ring $A[x]/I$, where I is the ideal generated by $a^2x^2 - ax$.
  To show that $A[x]/I$ will serve for B with x/I as b, we need to check that:
  - I does not contain ax;
  - I does not contain any non-zero element of A.

  Suppose for example that $ax = (\sum_{i<n} c_i x^i)(a^2x^2 - ax)$.
  Then $(-c_0 a - 1)x + \sum_{2 \leq i \leq n+1}(c_{i-2}a^2 - c_{i-1}a)x^i + c_n a^2 x^{n+2} = 0$.
  So $c_0 a = -1$, $c_{i-2}a^2 = c_{i-1}a$, $2 \leq i \leq n+1$, and $c_n a^2 = 0$.
  Then $0 = c_n a^2 = c_{n-1}a^3 = \cdots = c_0 a^{n+2} = -a^{n+1}$, a contradiction.
  The argument to show $A \cap I = \{0\}$ is similar but easier.
- Thus in commutative rings, the resultant of the formula $\exists z(xz \neq 0 \wedge (xz)^2 = xz)$ is a set of $\forall_1$ formulas which is equivalent (modulo the theory of commutative rings) to the set $\{x^n \neq 0 : n > \omega\}$.
- There is no harm in identifying the resultant with this set, or with the infinitary formula $\bigwedge_{n<\omega} x^n \neq 0$.

Existentially Closed Models of $\forall_2$ Theories

Theorem

Let L be a first-order language, T an $\forall_2$ theory in L and A a model of T. The following are equivalent:

(a) A is an e.c. model of T.

(b) For every $\exists_1$ formula $\phi(\overline{x})$ of L, $A \models \forall \overline{x}(\phi(\overline{x}) \to \bigwedge \mathrm{Res}_\phi(\overline{x}))$.

- By definition of Res_ϕ, every model of T satisfies the implication

$$\forall \overline{x}(\phi(\overline{x}) \to \bigwedge \mathrm{Res}_\phi(\overline{x})).$$

The implication in the other direction is just a rewrite of clause (b) in the theorem.

Example (Cont'd)

- By the theorem, if A is an e.c. commutative ring, then an element of A is nilpotent if and only if it does not divide any nonzero idempotent.

  It follows that the condition "x is nilpotent" can be expressed in A by a first-order formula.

  Moreover the same first-order formula works for any other e.c. commutative ring.

  This condition certainly is not first-order for commutative rings in general.

Subsection 3

Model Completeness

Characterization of Model Complete Theories

- We have defined a theory T in a first-order language L to be **model-complete** if every embedding between L-structures which are models of T is elementary.

Theorem

Let T be a theory in a first-order language L. The following are equivalent:

(a) T is model-complete.

(b) Every model of T is an e.c. model of T.

(c) If L-structures A, B are models of T and $e : A \to B$ is an embedding, then there are an elementary extension D of A and an embedding $g : B \to D$, such that ge is the identity on A.

(d) If $\phi(\overline{x}, \overline{y})$ is a formula of L which is a conjunction of literals, then $\exists \overline{y} \phi$ is equivalent modulo T to an $\forall_1$ formula $\psi(\overline{x})$ of L.

(e) Every formula $\phi(\overline{x})$ of L is equivalent modulo T to an $\forall_1$ formula $\psi(\overline{x})$ of L.

Characterization of Model Complete Theories (Proof)

(a)⇒(b): This is immediate from the definition of e.c. model.

(b)⇒(c): Assume (b). Let $e : A \to B$ be an embedding between models of T. Let $\overline{a}$ be a sequence listing all the elements of A.

Then $(B, e(\overline{a})) \Rrightarrow_1 (A, \overline{a})$ since A is an e.c. model of T.

The conclusion follows by the Existential Amalgamation Theorem.

(c)⇒(d): We first claim that if (c) holds, then every embedding between models of T preserves $\forall_1$ formulas of L.

Let $e : A \to B$ be an embedding between models of T.

Let $\overline{a}$ be in A and $\phi(\overline{x})$ an $\forall_1$ formula of L, such that $A \models \phi(\overline{a})$.

Take D and g as in (c). Then $D \models \phi(g(e(\overline{a})))$.

But ϕ is an $\forall_1$ formula. So $B \models \phi(e\overline{a})$.

By a previous corollary, every $\forall_1$ formula of L is equivalent to an $\exists_1$ formula of L. Taking negations, every $\exists_1$ formula $\phi(\overline{x})$ of L is equivalent to an $\forall_1$ formula $\psi(\overline{x})$ of L.

Characterization (Proof Cont'd)

(d)⇒(e): Assume (d). Let $\phi(\overline{x})$ be any formula of L. Assume ϕ is in prenex form $\exists\overline{x}_0\forall\overline{x}_1\cdots\forall\overline{x}_{n-2}\exists\overline{x}_{n-1}\theta_n(\overline{x}_0,\overline{x}_1,\ldots,\overline{x}_{n-1},\overline{x})$, with θ_n quantifier-free.

By (d), $\exists\overline{x}_{n-1}\theta_n(\overline{x}_0,\overline{x}_1,\ldots,\overline{x}_{n-1},\overline{x})$ is equivalent modulo T to a formula $\forall\overline{z}_{n-1}\theta_{n-1}(\overline{x}_0,\overline{x}_1,\ldots,\overline{x}_{n-2},\overline{z}_{n-1},\overline{x})$, with θ_{n-1} quantifier-free.

ϕ is equivalent to $\exists\overline{x}_0\forall\overline{x}_1\cdots\forall\overline{x}_{n-2}\overline{z}_{n-1}\theta_{n-1}(\overline{x}_0,\overline{x}_1,\ldots,\overline{x}_{n-2},\overline{z}_{n-1},\overline{x})$.

By (d), taking negations, $\forall\overline{x}_{n-2}\overline{z}_{n-1}\theta_{n-1}(\overline{x}_0,\overline{x}_1,\ldots,\overline{x}_{n-2},\overline{z}_{n-1},\overline{x})$ is equivalent modulo T to a formula $\exists\overline{z}_{n-2}\theta_{n-2}(\overline{x}_0,\overline{x}_1,\ldots,\overline{z}_{n-2},\overline{x})$, with θ_{n-2} quantifier-free.

So ϕ is equivalent to $\exists\overline{x}_0\forall\overline{x}_1\cdots\exists\overline{x}_{n-3}\overline{z}_{n-2}\theta_{n-2}(\overline{x}_0,\overline{x}_1,\ldots,\overline{z}_{n-2},\overline{x})$.

After n steps in this style, all the quantifiers will have been gathered up into a universal quantifier $\forall\overline{z}_0$.

(e)⇒(a): This follows from the fact that $\forall_1$ formulas are preserved in substructures.

Robinson's Test

Corollary (Robinson's Test)

For the first-order theory T to be model-complete, it is necessary and sufficient that if A and B are any two models of T, with $A \subseteq B$, then $A \preccurlyeq_1 B$.

- This is a restatement of (b) in the theorem.

Theorem

Let T be a model-complete theory in a first-order language L. Then T is equivalent to an $\forall_2$ theory in L.

- Every chain of models of T is elementary.

  So its union is a model of T, by the Tarski-Vaught Theorem.

  Hence, T is equivalent to an $\forall_2$ theory by the Chang-Łoś-Suszko Theorem.

Quantifier Elimination and Model-Completeness

- We saw one method for showing that a theory is model-complete.
- Let L be a first-order language and T a theory in L.
- We say that T **has quantifier elimination** if for every formula $\phi(\overline{x})$ of L, there is a quantifier-free formula $\phi^*(\overline{x})$ of L which is equivalent to ϕ modulo T.
- Every quantifier-free formula is an $\forall_1$ formula.
- So, by condition (e) in the theorem, every theory with quantifier elimination is model-complete.

Model-Completeness: Examples

- This observation gives us plenty of examples, thanks to the quantifier elimination technique seen previously.
- The theory of infinite vector spaces over a fixed field is model-complete.
- The theory of real-closed fields in the language of ordered fields is model-complete.
- The theory of dense linear orderings without endpoints is model-complete.

Lindström Test

- Recall that a theory is λ-**categorical** if it has models of cardinality λ and all its models of cardinality λ are isomorphic.

Theorem (Lindström's Test)

Let L be a first-order language and T an $\forall_2$ theory in L which has no finite models. If T is λ-categorical for some cardinal $\lambda \geq |L|$, then T is model complete.

- We use Robinson's test. Suppose $\lambda \geq |L|$ and T is λ-categorical.
  For contradiction, assume that T has models A and B such that $A \subseteq B$, but there are an $\exists_1$ formula $\phi(\overline{x})$ of L and a tuple $\overline{a}$ in A, such that $A \models \neg\phi(\overline{a})$ and $B \models \phi(\overline{a})$.
  Extend L to a language L^+ by adding a new unary relation symbol P.
  Expand B to an L^+-structure B^+ by interpreting P as $\mathrm{dom}(A)$.
  Let T^+ be $\mathrm{Th}(B^+)$. By the Relativization Theorem, T^+ contains the sentence $\exists\overline{x}(P(x_0) \wedge \cdots \wedge P(x_{n-1}) \wedge \phi(\overline{x}) \wedge \neg\phi^P(\overline{x}))$.

Lindström Test (Cont'd)

- Since $|L^+| \leq \lambda$ and T has no finite models, an easy argument with the Compactness Theorem shows that T^+ has a model D^+ of cardinality λ, such that $|P^{D^+}| = \lambda$. Let D be $D^+|_L$.

  Now T is λ-categorical. So $D^+ \equiv B^+$.

  Hence, there is an L-structure C with domain P^{D^+}, such that $C \subseteq D$.

  Using the Relativization Theorem, C is a model of T of cardinality λ.

  By a previous corollary, there is an e.c. model of T of cardinality λ.

  So C is an e.c. model of T, since T is λ-categorical.

  It follows that for every tuple $\overline{c}$ in C, if $D \models \phi(\overline{c})$, then $C \models \phi(\overline{c})$.

  This contradicts D^+ being a model of the sentence exhibited above.

  Example: Consider the theory of dense linear orderings without endpoints.

  - It is $\forall_2$ by inspection;
  - It is ω-categorical by a previous example.

  So, by Lindström's Test, it is model-complete.

Model Completeness and Completeness

- Let L be a first-order language and T a theory in L.
- A model A of T is called an **algebraically prime model** if A is embeddable in every model of T.

Theorem

Let L be a first-order language and T a theory in L. If T is model complete and has an algebraically prime model, then T is complete.

- Let A and B be any two models of T.

  Let C be an algebraically prime model of T.

  Then C is embeddable in A and in B.

  By hypothesis, T is model complete.
  - Since A and C are both models of T and $C \subseteq A$, $C \preccurlyeq A$.
  - Since B and C are both models of T and $C \subseteq B$, $C \preccurlyeq B$.

  It follows that $A \equiv C \equiv B$.

Model Companions

- Let T be a theory in a first-order language L.
- A theory U in L is called a **model companion** of T if:
  1. U is model-complete;
  2. Every model of T has an extension which is a model of U;
  3. Every model of U has an extension which is a model of T.
- By a previous corollary, Conditions 2 and 3 together are equivalent to the equation
$$T_\forall = U_\forall.$$
- A theory T might not have a model companion.
- T is called **companionable** if it has a model companion.

Model Companions and e.c. Models

Theorem

Let T be an $\forall_2$ theory in a first-order language L.

(a) T is companionable if and only if the class of e.c. models of T is axiomatizable by a theory in L.

(b) If T is companionable, then up to equivalence of theories, its model companion is unique and is the theory of the class of e.c. models of T.

- Suppose first that T is companionable, with a model companion T'. We show that the e.c. models of T are precisely the models of T'. First assume A is a model of T'. By a previous theorem, some extension B of A is an e.c. model of T. But T' is a model companion of T. So some extension C of B is a model of T'. Also, $A \preccurlyeq C$. Let $\phi(\overline{x})$ be an $\exists_1$ formula of L and $\overline{a}$ in A, such that $B \models \phi(\overline{a})$. Since $B \subseteq C$, $C \models \phi(\overline{a})$. Hence, since $A \preccurlyeq C$, $A \models \phi(\overline{a})$. So $A \preccurlyeq_1 B$. It follows that A is an e.c. model of T.

Model Companions and e.c. Models (Cont'd)

- Conversely, suppose A is an e.c. model of T.
  Then some extension B of A is a model of T'.
  But T' is equivalent to an $\forall_2$ theory. So A is a model of T'.
  This proves (b) and the left-to-right in (a).
  Now we prove the right-to-left in (a).
  Suppose the class of e.c. models of T is axiomatized by a theory U.
  Then Conditions 2 and 3 hold. So $T_\forall = U_\forall$.
  The class of e.c. models of T is closed under unions of chains.
  By the Chang-Łoś-Suszko theorem, we can take U to be an $\forall_2$ theory.
  Every model A of U is an e.c. model of T.
  By a previous theorem, A is an e.c. model of $T_\forall = U_\forall$.
  By the same theorem, A is now an e.c. model of U.
  By a previous theorem, U is model-complete.
  So U is a model companion of T.

Model Companions and $\exists_1$ Formulas

Corollary

Let L be a first-order language and T a $\forall_2$ theory in L with a model companion T^*. Then for every $\exists_1$ formula $\phi(\overline{x})$ of L, $\mathrm{Res}_\phi(\overline{x})$ is equivalent modulo T^* to a single $\forall_1$ formula $\psi(\overline{x})$ of L.

- By the preceding theorem and a previous one, $\phi(\overline{x}) \dashv\vdash_{T^*} \bigwedge \mathrm{Res}_\phi(\overline{x})$.

  As T^* is model-complete, by the first theorem of the section, ϕ is equivalent modulo T^* to an $\forall_1$ formula $\psi(\overline{x})$ of L.
- To show that a theory T is not companionable, we may:
  - Show that some e.c. model has an elementary extension which is not e.c..
  - Conclude that the class of e.c. models of T is not axiomatizable.

Example: Fields

- Let T be the theory of fields.

  Its model companion is the theory of algebraically closed fields.
- In fact it is the *model completion* of the theory of fields.
- This is a stronger notion to be seen in the next section.

Subsection 4

Quantifier Elimination Revisited

Quantifier Elimination Through Model Theory

- Let T be a theory in a first-order language L.
- T is said to have **quantifier elimination** if every formula $\phi(\overline{x})$ of L is equivalent modulo T to a quantifier-free formula $\psi(\overline{x})$ of L.
- To show that certain theories T have quantifier elimination we find, given a formula ϕ, an equivalent formula ψ modulo T, which has a certain form.
- Sometimes, to prove quantifier elimination, we can use model theory rather than syntax, when the model theory fits into some known algebraic structure theory.
- To do this we must know some model theoretic criteria for a theory to have quantifier elimination.

Amalgamation Property

- If A and B are L-structures, we write $A \equiv_0 B$ to mean that exactly the same quantifier-free sentences of L are true in A as in B.
- As before, we write $A \Rrightarrow_1 B$ to mean that for every $\exists_1$ sentence ϕ of L, if $A \models \phi$ then $B \models \phi$.
- Recall that $T_\forall$ is the set of $\forall_1$ first-order consequences of T.
- We say that a first-order theory T has the **amalgamation property** (**AP**) if the class $\mathbf{K}$ of all models of T has the AP.

  In other words, if the following holds:

  If A, B, C are models of T and $e : A \to B$, $f : A \to C$ are embeddings, then there are D in $\mathbf{K}$ and embeddings $g : B \to D$ and $h : C \to D$ such that $ge = hf$.

$$\begin{tikzcd}
 & D & \\
B \arrow[ur, hook, "g"] & & C \arrow[ul, hook', "h"'] \\
 & A \arrow[ul, hook', "e"] \arrow[ur, hook, "f"'] &
\end{tikzcd}$$

Characterization of Theories with Quantifier Elimination

Theorem

Let L be a first-order language and T a theory in L. The following are equivalent:

(a) T has quantifier elimination.

(b) If A and B are models of T, and $\overline{a}, \overline{b}$ are tuples from A, B respectively. such that $(A, \overline{a}) \equiv_0 (B, \overline{b})$, then $(A, \overline{a}) \Rrightarrow_1 (B, \overline{b})$.

(c) If A and B are models of T, $\overline{a}$ a sequence from A and $e : \langle \overline{a} \rangle_A \to B$ is an embedding, then there are an elementary extension D of B and an embedding $f : A \to D$ which extends e.

$$\begin{tikzcd}
 & D & \\
A \arrow[ur, hook, "f"] & & B \arrow[ul, hook', "\preccurlyeq"'] \\
 & \langle \overline{a} \rangle_A \arrow[ul, hook] \arrow[ur, hook', "e"'] &
\end{tikzcd}$$

(d) T is model-complete and $T_\forall$ has the amalgamation property.

(e) For every quantifier-free formula $\phi(\overline{x}, y)$ of L, $\exists y \phi$ is equivalent modulo T to a quantifier-free formula $\psi(\overline{x})$.

Theories with Quantifier Elimination (Cont'd)

(a)⇒(b) This is immediate.

(b)⇒(c) Assume (b). For every tuple $\overline{a}'$ inside $\overline{a}$, the hypothesis of (c) implies that $(A,\overline{a}') \equiv_0 (B,e(\overline{a}'))$. Hence, $(A,\overline{a}') \Rrightarrow_1 (B,e(\overline{a}'))$. Every sentence of $L(\overline{a})$ (L with parameters $\overline{a}$ added) mentions just finitely many elements. So $(A,\overline{a}) \Rrightarrow_1 (B,e(\overline{a}))$. Hence, the conclusion of (c) holds by the Existential Amalgamation Theorem.

(c)⇒(d) Assume (c). Let $e : B \to A$ be any embedding between models of T. Put $\langle\overline{a}\rangle_A = B$ in the diagram. By (c) of the Characterization Theorem for model-completion, T is model-complete.

We prove, next, that $T_\forall$ has the amalgamation property.

Let $g : C \to A'$ and $e : C \to B'$ be embeddings between models of $T_\forall$.

By a previous corollary, A' and B' can be extended to models A, B of T respectively. Now apply (c) with the embedding $C \to A$ in place of the inclusion $\langle\overline{a}\rangle_A \subseteq A$.

Theories with Quantifier Elimination (Conclusion)

(d)⇒(e) Assume (d) holds. Let **K** to be the class of all models of T.

- **K** is closed under isomorphisms.
- By hypothesis, the class of all substructures of structures in **K** (i.e., the class of all models of $T_\forall$) has the amalgamation property.
- By model completion, all models of T are e.c. models.

By a previous theorem, for every quantifier-free formula $\phi(\overline{x}, y)$ of L, $\exists y \phi$ is equivalent modulo T to a quantifier-free formula $\psi(\overline{x})$.

(e)⇒(a) This is by a previous lemma.

Quantifier Eliminable and Ultrahomogeneous Structures

- An L-structure A is said to be **quantifier-eliminable** (**q.e.** for short), or to **have quantifier elimination**, if A is a model of a theory in L which has quantifier elimination.
- This is equivalent to saying that $\mathrm{Th}(A)$ has quantifier elimination.
- Recall that a structure A is called **ultrahomogeneous** if every isomorphism between finitely generated substructures of A extends to an automorphism of A.

Quantifier Eliminable and Ultrahomogeneous Structures

Corollary

Let A be a finite L-structure. Then A is quantifier eliminable if and only if A is ultrahomogeneous.

- By hypothesis, A is finite.

  So $\mathrm{Th}(A)$ says how many elements A has.

  Thus, in the diagram D must be A.

  Hence, f must be an isomorphism.

$$\begin{tikzcd} & D & \\ A \arrow[ur, hook, "f"] & & A \arrow[ul, hook', "\cong"'] \\ & \langle \overline{a} \rangle_A \arrow[ul, hook] \arrow[ur, hook, "e"'] & \end{tikzcd}$$

Example: Vector Spaces

- Let R be a field.

  Let T be the theory of infinite (left) vector spaces over R.

  The axioms of T consist of the axioms for vector spaces together with the sentences $\exists_{\geq n} x x = x$, for each positive integer n.

  Inspection shows that these axioms are $\forall_2$ first-order sentences.

  Certainly T is λ-categorical for any infinite cardinal $\lambda > |R|$.

  By definition T has no finite models.

  So by Lindström's Test, T is model-complete.

  Elementary algebra shows that $T_\forall$, which is the theory of vector spaces over R, has the amalgamation property.

  So by (d) of the theorem, T has quantifier elimination.

  Now by the corollary, every finite vector space is quantifier eliminable.

  So we have shown that every vector space is quantifier eliminable.

Example: Algebraically Closed Fields

- By a theorem of Steinitz any two algebraically closed fields of the same characteristic and the same uncountable cardinality are isomorphic.

  Also algebraically closed fields are infinite.

  So by Lindström's Test, the theory of algebraically closed fields of a fixed characteristic is model-complete.

  These are precisely the completions of the theory of algebraically closed fields, and they are obtained (up to equivalence) by adding sets of $\exists_1$ sentences.

  It follows that the theory of algebraically closed fields is model-complete.

  We saw already in a previous corollary how this implies that every algebraically closed field is quantifier eliminable.

Sufficient Conditions for Quantifier Elimination

Corollary

Let L be a first-order language and T a theory in L. Suppose that T satisfies the following conditions:

(a) For any two models A and B of T, if $A \subseteq B$, $\phi(\overline{x}, y)$ is a quantifier free formula of L and $\overline{a}$ is a tuple of elements of A, such that $B \models \exists y \phi(\overline{a}, y)$, then $A \models \exists y \phi(\overline{a}, y)$. ("$T$ is 1-model-complete".)

(b) For every model A of T and every substructure C of A, there is a model A' of T, such that:

  (i) $C \subseteq A' \subseteq A$;

  (ii) If B is another model of T with $C \subseteq B$, then there is an embedding of A' into B over C.

Then T has quantifier elimination.

Proof

- Assuming (a) and (b), we prove Part (b) of the previous theorem.

  Suppose A and B are models of T, $\overline{a}$ and $\overline{b}$ are tuples of elements of A and B, respectively, and $(A,\overline{a}) \equiv_0 (B,\overline{b})$.

  Let $\phi(\overline{x},\overline{y})$ be a quantifier-free formula of L, such that $A \models \exists y \phi(\overline{a},\overline{y})$.

  We show that $B \models \exists \overline{y} \phi(\overline{b},\overline{y})$. Without loss we can suppose that $\overline{b}$ is $\overline{a}$.

  Suppose $A \models \phi(\overline{a},\overline{c})$, where $\overline{c}$ is $(c_0,\ldots,c_{k-1})$.

  Claim: There is an element d_0 in some elementary extension B_0 of B, such that $(A,\overline{a},c_0) \equiv_0 (B_0,\overline{a},d_0)$. (Proven in the next slide.)

  Now we repeat to find an elementary extension B_1 of B_0 with an element d_1, such that $(A,\overline{a},c_0,c_1) \equiv_0 (B_1,\overline{a},d_0,d_1)$, and so on.

  Eventually we reach an elementary extension B_{n-1} of B and elements $\overline{d}$ such that $(A,\overline{a},\overline{c}) \equiv_0 (B_{n-1},\overline{a},\overline{d})$.

  In particular we have $B_{n-1} \models \phi(\overline{a},\overline{d})$. So $B_{n-1} \models \exists \overline{y} \phi(\overline{a},\overline{y})$.

  Thus, $B \models \exists y \phi(\overline{a},\overline{y})$ as required.

Sufficient Conditions for Quantifier Elimination (Claim)

- Claim: There is an element d_0 in some elementary extension B_0 of B, such that $(A,\overline{a},c_0) \equiv_0 (B_0,\overline{a},d_0)$.

  Write $\Psi_0(\overline{x},y)$ for the set of all quantifier-free formulas $\psi(\overline{x},y)$, such that $A \models \psi(\overline{a},c_0)$. Since $(A,\overline{a}) \equiv_0 (B,\overline{a})$, we can write $C = \langle\overline{a}\rangle_A = \langle\overline{a}\rangle_B$. By (b), there is a model A' of T, such that $C \subseteq A' \subseteq A$. Further, there is an embedding of A' into B over C. Without loss we can suppose that A' is a substructure of B. Now $\overline{a}$ is in A' and each formula $\psi(\overline{a},y)$ in $\Psi_0(\overline{a},y)$ has just one free variable. So, by (a), since $A \models \exists y\psi(\overline{a},y)$ $A' \models \exists y\psi(\overline{a},y)$. Since $A' \subseteq B$, we get $B \models \exists y\psi(\overline{a},y)$. Hence every finite subset of $\Psi_0(\overline{a},y)$ is satisfied by an element of B. By compactness, there exists an elementary extension B_0 of B with an element d_0 which realizes the type $\Psi_0(\overline{a},y)$. Thus, $(A,\overline{a},c_0) \equiv_0 (B_0,\overline{a},d_0)$, proving the claim.

The Theory of Real-Closed Fields

Theorem (Tarski)

The theory T of real-closed fields in the first-order language L of ordered fields has quantifier elimination.

- The proof borrows two facts from algebra:
  1. The intermediate value theorem holds in real-closed fields for all functions defined by polynomials $p(x)$, possibly with parameters. I.e. if $p(a) \cdot p(b) < 0$, then $p(c) = 0$, for some c strictly between a and b.
  2. If A is a real-closed field and C an ordered subfield of A, then:
     - There is a smallest real-closed field B such that $C \subseteq B \subseteq A$;
     - If A' is any real-closed field $\supseteq C$, then B is embeddable in A' over C.

     B is called the **real closure of** C **in** A.

  We prove Parts (a) and (b) of the corollary.

Proof of Tarski's Theorem

(a) Let A and B be real-closed fields with $A \subseteq B$.

Let $\phi(x)$ be quantifier-free in L, $\overline{a}$ in A, such that $B \models \exists x \phi(x, \overline{a})$.

We must show that $A \models \exists x \phi(x, \overline{a})$.

After bringing ϕ to disjunctive normal form and distributing the quantifier through it, we can assume that ϕ is a conjunction of literals.

Now we have:

- $y \neq z$ is equivalent to $y < z \vee z < y$;
- $\neg y < z$ is equivalent to $y = z \vee z < y$.

So we can suppose that ϕ has the form

$$p_0(x) = 0 \wedge \cdots \wedge p_{k-1}(x) = 0 \wedge q_0(x) > 0 \wedge \cdots \wedge q_{m-1}(x) > 0,$$

where $p_0, \ldots, p_{k-1}, q_0, \ldots, q_{m-1}$ are polynomials with coefficients in A.

- Suppose ϕ contains a non-trivial equation $p_i(x) = 0$.
  Then any element of B satisfying ϕ is algebraic over A.
  Hence, it is already in A.

Proof of Tarski's Theorem (Cont'd)

- Suppose ϕ contains no non-trivial equations, i.e., that $k = 0$.
  There are finitely many points $c_0 < \cdots < c_{n-1}$ in A which are zeros of one or more of the polynomials q_j $(j < m)$. By the intermediate value property, none of the q_j changes sign except at the points c_i, $i < n$.
  So we choose a point b of B, such that $B \models \phi(b)$. And then choose a point a in A which lies in the same interval of the c_is as b.

This proves (a) of the preceding corollary.

(b) Let A, B be real-closed fields and C a common substructure of A, B.

Then C is an ordered integral domain. We can show that the quotient field of C in A is isomorphic over C to the quotient field of C in B.

Identify these quotients and suppose that C is itself an ordered field.

By Fact 2, we can take A' to be the real closure of C in A.

This proves (b).