Introduction to Model Theory

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

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1. Saturation
   - The Great and the Good
   - Big Models Exist
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Subsection 1

The Great and the Good

Intuition Behind Monster Models

- In arguments which involve several structures and maps between them, things usually go smoother when the maps are inclusions.
- There are at least two good mathematical reasons for this:
  - First, if the maps are inclusions, then diagrams automatically commute.
  - Second, if A is a substructure of B, then we can specify A by giving B and $\text{dom}(A)$; there is no need to describe the relations of A as well as those of B.
- Thoughts of this kind have led to the use of *big models*, sometimes known as *monster models*.
- Informally, a big model is a structure M such that every commutative diagram of structures and maps that we want to consider is isomorphic to a diagram of inclusions between substructures of M.
- Of course a structure M with this property cannot exist.
  - It would have to contain isomorphic copies of all structures.
  - So its domain would be a proper class and not a set.

Splendid Models

- We demand something less by calling a model M **splendid** if the following holds:

  Suppose L^+ is a first-order language got by adding a new relation symbol R to L. If N is an L^+-structure such that $M \equiv N|_L$, then we can interpret R by a relation S on the domain of M so that $(M, S) \equiv N$.

- Informally this says that M is compatible with any extra structural features which are consistent with $\text{Th}(M)$.

  Example (**Equivalence Relations**):

  Let M be a structure consisting of an equivalence relation with two equivalence classes, whose cardinalities are ω and ω_1.

  Then M is not splendid.

  - Take an elementary extension N where the two equivalence classes have the same size;
  - Add a bijection between these classes.

Big and Monster Models

- For any cardinal λ, we shall say that M is λ-**big** if $(M,\overline{a})$ is splendid whenever $\overline{a}$ is a sequence of fewer than λ elements of M.
- Thus, splendid is the same as 0-big.
- One can define a **big model** (or **monster model**) to be a model which is λ-big for some cardinal λ (which is taken "large enough to cover everything interesting").
- This is vague, but in practice there is no need to make it more precise.
  - In stability theory one is interested in the models of some complete first-order theory T; the usual habit is to choose a big model of T without specifying how large λ is.
- It will emerge that every structure has λ-big elementary extensions for any λ.

Types Revisited

- Let A be an L-structure and X a set of elements of A.
- Write $L(X)$ for the first-order language formed from L by adding constants for the elements of X.
- If $n < \omega$, then a **complete n-type over X with respect to A** is a set of the form

$$\{\phi(x_0, \ldots, x_{n-1}) : \phi \text{ is in } L(X) \text{ and } B \models \phi(\overline{b})\},$$

  where B is an elementary extension of A and $\overline{b}$ is an n-tuple of elements of B.
- We write this n-type as $\mathrm{tp}_B(\overline{b}/X)$.
- We say that $\overline{b}$ **realizes** this n-type in B.
- We write $S_n(X; A)$ for the set of all complete n-types over X with respect to A.
- A **type** is an n-type for some $n < \omega$.

Saturation, Homogeneity and Universality

- Let λ be a cardinal.
- We say that A is λ-**saturated** if, for every set X of elements of A, if $|X| < \lambda$, then all complete 1-types over X with respect to A are realized by elements in A.
- We say that A is **saturated** if A is $|A|$-saturated.
- We say that A is λ-**homogeneous** if, for every pair of sequences $\overline{a}$, $\overline{b}$ of length less than λ, if $(A, \overline{a}) \equiv (A, \overline{b})$ and d is any element of A, then there is an element c such that $(A, \overline{a}, c) \equiv (A, \overline{b}, d)$.
- We say that A is **homogeneous** if A is $|A|$-homogeneous.
- We say that A is λ-**universal** when, if B is any L-structure of cardinality $< \lambda$ and $B \equiv A$, then B is elementarily embeddable in A.

From Large to Smaller Cardinalities

- The following is straightforward from the definitions.

Lemma

Let A be a structure and suppose that $\kappa < \lambda$.

- If A is λ-big, then it is κ-big;
- If A is λ-saturated, then it is κ-saturated;
- If A is λ-homogeneous, then it is κ-homogeneous;
- If A is λ-universal, then it is κ-universal.

Relations Between the Concepts

- The simplest links between these concepts run as follows:

$$\begin{array}{ccccc} & & & & \lambda\text{-Homogeneous} \\ & & & \nearrow & \\ \lambda\text{-Big} & \Longrightarrow & \lambda\text{-Saturated} & & \\ & & & \searrow & \\ & & & & \lambda\text{-Universal} \end{array}$$

Big and Saturated

Theorem

Suppose A is λ-big. Then A is λ-saturated.

- Suppose A is a λ-big L-structure.

  Let $\overline{a}$ be a sequence of fewer than λ elements of A.

  Let B be an elementary extension of A and b an element of A.

  We must show that $\text{tp}_B(b/\overline{a})$ is realized in A.

  Let L^+ be obtained by L by adding a unary relation symbol R.

  Make B into an L^+-structure B^+ by interpreting R as $\{b\}$.

  By λ-bigness, there is a relation S on $\text{dom}A$, with $(A, S, \overline{a}) \equiv (B^+, \overline{a})$.

  Now $B^+ \models$ "Exactly one element satisfies $R(x)$".

  So S is a singleton $\{c\}$.

  Clearly c realizes $\text{tp}_B(b/\overline{a})$.

Saturation and Homogeneity

Lemma

Let A be an L-structure and λ a cardinal. The following are equivalent:

(a) A is λ-saturated.

(b) For every L-structure B and every pair of sequences $\overline{a}$, $\overline{b}$ of elements of A, B respectively, if $\overline{a}$ and $\overline{b}$ have the same length $< \lambda$ and $(A, \overline{a}) \equiv (B, \overline{b})$, and d is any element of B, then there is an element c of A such that $(A, \overline{a}, c) \equiv (B, \overline{b}, d)$.

(a)⇒(b): Assume (a). Suppose $\overline{a}, \overline{b}$ are as in the hypothesis of (b). By Elementary Amalgamation, there are an elementary extension D of A and an elementary embedding $f : B \to D$, such that $f\overline{b} = \overline{a}$. Since f is elementary, if d is in B, then $(D, \overline{a}, f(d)) \equiv (B, \overline{b}, d)$. But $\overline{a}$ contains fewer than λ elements and A is λ-saturated. So A contains an element c, such that $\text{tp}_A(c/\overline{a}) = \text{tp}_D(f(d)/\overline{a})$. Then $(A, \overline{a}, c) \equiv (D, \overline{a}, f(d)) = (B, \overline{b}, d)$, as required.

Saturation and Homogeneity (Cont'd)

- The implication (b)⇒(a) is immediate from the definitions.

  Let $\mathrm{tp}_B(b/\overline{a})$ be a complete 1-type with respect to A.

  By (b), there exists c in A, such that

$$(A, \overline{a}, c) \equiv (B, \overline{a}, b).$$

  This clearly implies that $\mathrm{tp}_B(b/\overline{a}) = \mathrm{tp}_A(c/\overline{a})$.

  Thus, every complete 1-type with respect to A is realized by an element of A. So A is λ-saturated.

Theorem

If A is λ-saturated, then A is λ-homogeneous.

- The definition of λ-homogeneity is the special case of Part (b) of the preceding lemma where $A = B$.

Using Saturation to Build Maps

- The preceding lemma can be applied over and over again, to build up maps between structures.

Lemma

Let L be a first-order language and A an L-structure.

(a) Suppose A is λ-saturated, B is an L-structure and $\overline{a}, \overline{b}$ are sequences of elements of A, B, respectively, such that $(A, \overline{a}) \equiv (B, \overline{b})$. Suppose $\overline{a}, \overline{b}$ have length $< \lambda$, and let d be a sequence of elements of B, of length $\leq \lambda$. Then there is a sequence $\overline{c}$ of elements of A, such that $(A, \overline{a}, c) \equiv (B, \overline{b}, d)$.

(b) The same holds if we replace λ-saturated by λ-homogeneous and add the assumption that $A = B$.

Using Saturation to Build Maps (Cont'd)

- We prove (a). The proof of (b) is similar.
  By induction, we define a sequence $\overline{c} = (c_i : i < \lambda)$ of elements of A so that, for each $i \leq \lambda$,
  $$(A, \overline{a}, \overline{c}|_i) \equiv (B, \overline{b}, \overline{d}|_i).$$
  For $i = 0$, $(B, \overline{b}) \equiv (A, \overline{a})$. This holds by hypothesis.
  There is nothing to do at limit ordinals, since any formula of L has only finitely many free variables.
  Suppose then that $\overline{c}|_i$ has just been chosen and $i < \lambda$.
  Now A is λ-saturated and $\overline{c}|_i$ has length $< \lambda$.
  By a previous lemma, there exists an element c_i in A such that
  $$(A, \overline{a}, \overline{c}|_i, c_i) \equiv (B, \overline{b}, \overline{d}|_i, d_i).$$

Saturation and Universality

Theorem

Let L be a first-order language and A a λ-saturated L-structure. Then A is λ^+-universal.

- We have to show that, if B is an L-structure of cardinality $\leq \lambda$ and $B \equiv A$, then there is an elementary embedding $e : B \to A$.

  List the elements of B as $\overline{d} = (d_i : i < \lambda)$, with repetitions allowed.

  By the lemma there is a sequence $\overline{c}$ in A, such that $(B, \overline{d}) \equiv (A, \overline{c})$.

  By the Elementary Diagram Lemma, there is an elementary embedding of B into A taking $\overline{d}$ to $\overline{c}$.
- We note that, actually, λ-saturation is exactly λ-homogeneity plus λ-universality.

λ-Saturation and Multi-Variate Types

Theorem

Let L be a first-order language, A an L-structure, λ an infinite cardinal and $\overline{y}$ any tuple of variables. Suppose A is λ-saturated. Let $\overline{a}$ be a sequence of fewer than λ elements of A, and $\Phi(\overline{x},\overline{y})$ a set of formulas of L, such that for each finite subset Ψ of Φ, $A \models \exists \overline{y} \bigwedge \Psi(\overline{a},\overline{y})$. Then there is a tuple $\overline{b}$ of elements of A, such that $A \models \bigwedge \Phi(\overline{a},\overline{b})$.

- By the Compactness Theorem, there is an elementary extension B of A containing a tuple $\overline{d} = (d_0,\ldots,d_{m-1})$, such that $B \models \bigwedge \Phi(\overline{a},\overline{d})$.
  Now $(A,\overline{a}) \equiv (B,\overline{a})$. Since λ is infinite, $\overline{d}$ has fewer than λ elements.
  By the previous lemma, there is $\overline{c}$ in A, such that $(A,\overline{a},\overline{c}) \equiv (B,\overline{a},\overline{d})$.
  Hence, $A \models \bigwedge \Phi(\overline{a},\overline{c})$.

Saturation, Homogeneity and Isomorphism

Theorem

Let A and B be elementarily equivalent L-structures of the same cardinality λ.

(a) If A and B are both saturated then $A \cong B$.

(b) If A and B are both homogeneous and realize the same n-types over $\emptyset$, for all $n < \omega$, then $A \cong B$.

(a) First assume that λ is infinite.

List the elements of A as $(a_i : i < \lambda)$ and those of B as $(b_i : i < \lambda)$.

Claim: There are sequences $\overline{c}, \overline{d}$ of elements of A and B, respectively, both of length λ, such that, for each $i < \lambda$, $(A, \overline{a}|_i, \overline{c}|_i) \equiv (B, \overline{d}|_i, \overline{b}|_i)$.

The proof is by induction on i.

Again the case $i = 0$ is given in the theorem hypothesis.

Moreover, there is nothing to do at limit ordinals.

Saturation, Homogeneity and Isomorphism (Part (a) Cont'd)

- Suppose the condition has been established for some $i < \lambda$.

  Then fewer than λ parameters have been chosen (since λ is infinite).
  - By saturation of B, we find d_i, such that $(A, \overline{a}\,|_i, a_i, \overline{c}\,|_i) \equiv (B, \overline{d}\,|_i, d_i, \overline{b}\,|_i)$;
  - By saturation of A we find c_i, such that $(A, \overline{a}\,|_i, a_i, \overline{c}\,|_i, c_i) \equiv (B, \overline{d}\,|_i, d_i, \overline{b}\,|_i, b_i)$.

  At the end of the construction, the Diagram Lemma gives us an embedding $f : A \to B$, such that $f(\overline{a}) = \overline{d}$ and $f(\overline{c}) = \overline{b}$.

  The embedding is onto B since $\overline{b}$ includes all the elements of B.

  Now assume that λ is finite.

  A previous theorem gives an elementary embedding $f : A \to B$.

  But A and B both have cardinality λ.

  So f must be an isomorphism.

Saturation, Homogeneity and Isomorphism (Part (b))

(b) We can assume that λ is infinite.
Claim: If $i < \lambda$ and $\overline{b}$ is a sequence in B of length i, then there is a sequence $\overline{a}$ of elements of A such that $(A,\overline{a}) \equiv (B,\overline{b})$. (And the same with A and B transposed.)
The proof is by induction on i. Since the hypotheses are symmetrical in A and B, we only prove one way round.
If i is finite, the Claim is given by the theorem hypothesis.
If i is infinite we distinguish two cases:

- Suppose that i is a cardinal.
  Then we build up $\overline{a}$ so that for each $j < i$, $(A,\overline{a}|_j) \equiv (B,\overline{b}|_j)$.
  The theorem hypothesis gives the case $j = 0$.
  If j is a limit ordinal, there is nothing to do.
  Suppose $(A,\overline{a}|_j) \equiv (B,\overline{b}|_j)$ for some j. By the induction hypothesis, since $|j+1| < i$, there is a sequence $\overline{c} = (c_k : k \leq j)$ in A such that $(A,\overline{c}) \equiv (B,\overline{b}|_{(j+1)})$. Then $(A,\overline{a}|_j) \equiv (A,\overline{c}|_j)$. So by the homogeneity of A, there is a_j, such that $(A,\overline{a}|_j,a_j) \equiv (A,\overline{c}) \equiv (B,\overline{b}|_{(j+1)})$.

Saturation, Homogeneity and Isomorphism (Part (b) Cont'd)

- Suppose i is not a cardinal. We reduce to the case where it is a cardinal by rearranging the elements of $\overline{b}$ into a sequence of order-type $|i|$.

To prove the theorem, we go back and forth as in (a).

For example, to find d_i:

- First, use the Claim to find $\overline{e}$ in D, such that $(A, \overline{a}|_i, a_i, \overline{c}|_i) \equiv (B, \overline{e})$;
- Then, by the homogeneity of B, find d_i so that $(B, \overline{e}) \equiv (B, \overline{d}|_i, d_i, \overline{b}|_i)$.

Example: Finite Structures

- Suppose the structure A is finite.

  Then any structure elementarily equivalent to A is isomorphic to A.

  It follows that A is λ-big for all cardinals λ.

  In particular A is saturated and homogeneous.
- This is an exceptional case, but it explains why the word "infinite" keeps appearing.

More Examples Without Details

- Let K be a field.

  Let λ an infinite cardinal $\geq |K|$.
  - If A is a vector space of dimension λ over K, then A is λ-big.
  - If A is infinite but has dimension less than λ, then A is no longer λ-saturated.
- Every algebraically closed field A of infinite transcendence degree over the prime field is $|A|$-big and hence saturated.
- Every countable ω-categorical structure is saturated.

  So every countable dense linear ordering without endpoints is ω-saturated.

Subsection 2

Big Models Exist

Existence of λ-Big Models

- The cardinal $\mu^{<\lambda}$ is the sum of all cardinals μ^κ with $\kappa < \lambda$.

  Example: If $\lambda = \kappa^+$, then $\mu^{<\lambda}$ is just μ^κ.

Theorem

Let L be a first-order language, A an L-structure and λ a regular cardinal $> |L|$. Then A has a λ-big elementary extension B, such that $|B| \leq |A|^{<\lambda}$.

- If A is finite, then A is already λ-big for any cardinal λ.

  So we can assume henceforth that A is infinite.

  Let C and D be structures. We call D is an **expanded elementary extension** of C if D is an expansion of some elementary extension of C.

  An **expanded elementary chain** is a chain $(C_i : i < \kappa)$ of structures, such that whenever $i < j < \kappa$, C_j is an expanded elementary extension of C_i.

Existence of λ-Big Models II

- Using the Tarski-Vaught Theorem on elementary chains it is not hard to see that each expanded elementary chain has a union D which is an expanded elementary extension of every structure in the chain.

  Let $\bigcup_{i<\kappa} C_i$ be the union of the expanded elementary chain $(C_i : i < \kappa)$.

  Put $\mu = (|A| + |L|^+)^{<\lambda}$. Then $\mu = \mu^{<\lambda} \geq \lambda$. The ordinal $\mu^2 \cdot \lambda$ consists of $\mu \cdot \lambda$ copies of μ laid end to end. The object will be to construct B (or rather, an expansion of B) as the union of an expanded elementary chain $(A_i : 0 < i < \mu \cdot \lambda)$, where for each $i < \mu \cdot \lambda$, the domain of A_i is the ordinal $\mu \cdot i$. Then B will have cardinality $|\mu^2 \cdot \lambda| = \mu$ as required.

  The ordinals $< \mu^2 \cdot \lambda$ will be called **witnesses**. We can regard them either as elements of the structure to be built, or as new constants which will be used as names of themselves.

Existence of λ-Big Models III

- Since A is infinite, we can suppose without loss that A has cardinality μ by the Upward Löwenheim-Skolem Theorem.

  Identify dom(A) with the ordinal $\mu \cdot 1 = \mu$ and put $A_1 = A$.

  At limit ordinals $\delta < \mu$ we put $A_\delta = \bigcup_{0<i<\delta} A_i$.

  It remains to define A_{i+1} when A_i has been defined.

  Suppose L_0 is a first-order language and L', L'' are first-order languages got by adding new relation symbols R', R'', respectively to L_0.

  We say that theories T', T'' in L', L'', respectively, are **conjugate** if T'' comes from T' by replacing R' by R'' throughout.

  List as $((X_i, T_i) : 0 < i < \mu \cdot \lambda)$ the set of "all" pairs (X_i, T_i), where:
  - X_i is a set of fewer than λ witnesses;
  - T_i is a complete theory in the first-order language L_i formed by adding to L the witnesses in X_i and one new relation symbol R_i.

  Here "all" means that for each such pair (X, T), there is a pair (X_i, T_i), with $X = X_i$ and T conjugate to T_i.

Existence of λ-Big Models IV

- Checking the arithmetic, note first that for each cardinal $\nu < \lambda$:
  - The number of sets X consisting of ν witnesses is $\mu^\nu = \mu$;
  - The number of complete theories T (up to conjugacy) in the language got by adding X and a relation symbol R to L is at most $2^{|L|+\nu} \leq \mu^{<\lambda} = \mu$.

  So the total number of pairs that we need is at most $\mu \cdot \lambda = \mu$.

  The listing can be done so that:
  1. The relation symbols R_i are all distinct;
  2. Up to conjugacy, each possible pair (X, T) appears as (X_i, T_i) cofinally often in the listing.

  In fact $\mu \cdot \lambda$ consists of λ blocks of length μ.

  We can make sure that each (X, T) appears at least once - up to conjugacy - in each of these blocks.

Existence of λ-Big Models V

- Assume A_i has been defined with domain $\mu \cdot i$.
  To define A_{i+1}, consider the pair (X_i, T_i).
  - Suppose some witness $\geq \mu \cdot i$ appears in X_i.
    Then we take A_{i+1} to be an arbitrary elementary extension of A_i with domain $\mu \cdot (i+1)$ (possible using compactness).
  - Suppose every witness in X_i is an element of A_i.
    - Suppose T_i is inconsistent with the elementary diagram of A_i.
      Then again we take A_{i+1} to be an arbitrary elementary extension of A_i with domain $\mu \cdot (i+1)$.
    - Suppose T_i is consistent with the elementary diagram of A_i.
      Then some expanded elementary extension D of A_i is a model of T_i.
      By the Downward Löwenheim-Skolem, assume D has cardinality μ.
      So again (after adding at most μ elements if necessary) we can identify the elements of D with the ordinals $< \mu \cdot (i+1)$.
      This done, we take A_{i+1} to be D.

Existence of λ-Big Models VI

- We have defined the chain $(A_i : 0 < i < \mu \cdot \lambda)$.

  We put $B^+ = \bigcup_{0<i<\mu\cdot\lambda} A_i$.

  Let $B = B^+ |_L$.

  The structure B^+ is an expanded elementary extension of A.

  So B is an elementary extension of A.

  B is the union of a chain of length μ in which every structure has cardinality μ.

  So B has cardinality μ.

Existence of λ-Big Models VII

- We show that B is λ-big.
  Suppose $\overline{a}$ is a sequence of fewer than λ elements of B.
  Let C be a structure with a new relation symbol R, such that $(C|_L, \overline{c}) \equiv (B, \overline{a})$, for some sequence $\overline{c}$ in C.
  Adjusting C, we can suppose without loss that $\overline{c}$ is $\overline{a}$.
  Now $\mu \cdot \lambda$ is an ordinal of cofinality λ and λ is regular.
  So there is some $j < \mu \cdot \lambda$ such that all the witnesses in $\overline{a}$ are less than j.
  Thus, $(C|_L, \overline{a}) \equiv (A_j, \overline{a})$.
  By Condition 2, for some $i \geq j$, T_i is conjugate to $\mathrm{Th}(C, \overline{a})$.
  Then $\mathrm{Th}(A_i|_L, \overline{a}) \cup T_i$ is consistent.
  So by Condition 1 and a previous theorem, T_i is consistent with the elementary diagram of A_i.
  So, by construction, A_{i+1} is a model of T_i.
  Hence, B^+ is also a model of T_i.
  Thus, B expands to a model of T_i, as required.

Consequences

Corollary

Let A be an L-structure and λ a cardinal $\geq |L|$. Then A has a λ^+-big (and, hence, λ^+-saturated) elementary extension of cardinality $\leq |A|^\lambda$.

- Direct from the theorem.

Corollary

Let λ be any cardinal. Then every structure is elementarily equivalent to a λ-big structure.

- Thus, if we want to classify the models of a first-order theory T up to elementary equivalence, it is enough to choose a cardinal λ and classify the λ-big models up to elementary equivalence.
- Since the λ-big models of T may be a much better behaved collection than the models of T in general, this is real progress.

Existence of λ-Homogeneous Models

- Since every λ-big structure is λ-homogeneous, the preceding theorem creates λ-homogeneous elementary extensions too.
- If all we want is λ-homogeneity, we can get it with a smaller structure.

Theorem

Let L be a first-order language, A an L-structure and λ a regular cardinal. Then A has a λ-homogeneous elementary extension C such that $|C| \leq (|A| + |L|)^{<\lambda}$.

- By the preceding theorem, we have a λ-big elementary extension B of A; never mind its cardinality. Write ν for $(|A| + |L|)^{<\lambda}$.
  Note that $\nu \geq \lambda$. Otherwise $\nu = \nu^{<\lambda} = (\nu^{<\lambda})^{\nu} \geq 2^{\nu} > \nu$.
  If D is any elementary substructure of B with cardinality at most ν, we can find a structure D^* with $D \preccurlyeq D^* \preccurlyeq B$, such that:

  If $\overline{a}$ and $\overline{b}$ are two sequences of elements of D, both of length $< \lambda$, and $(D, \overline{a}) \equiv (D, \overline{b})$, then, for every element c of D there is an element d of D^* such that $(D, \overline{a}, c) \equiv (D^*, \overline{b}, d)$.

Existence of λ-Homogeneous Models (Cont'd)

- We can find D^* as the union of a chain of elementary substructures of B, taking one such substructure for each triple $(\overline{a}, \overline{b}, c)$, such that $(D, \overline{a}) \equiv (D, \overline{b})$ and c is in D. Such a chain is automatically elementary. As we move one step up the chain, we choose the next structure so that it contains some d with $(D, \overline{a}, c) = (B, \overline{b}, d)$. This is possible since B is λ-homogeneous.
  - The number of triples $(\overline{a}, \overline{b}, c)$ is at most $\nu^{<\lambda} = \nu$;
  - Each structure in the chain can be chosen of cardinality at most ν.

  So the union D^* can be found with cardinality at most ν.
  Now we build a chain $(A_i : i < \lambda)$ of elementary substructures of B, so that for each $i < \lambda$, A_{i+1} is A_i^*. At limit ordinals we take unions.
  Let C be $\bigcup_{i<\lambda} A_i$. Then C has cardinality at most $\nu \cdot \lambda = \nu$.
  Let $(C, \overline{a}) = (C, \overline{b})$, where $\overline{a}$ and $\overline{b}$ are sequences of length $< \lambda$ in C.
  Let c is an element of C. λ being regular, $\overline{a}$, $\overline{b}$, c must lie in some A_i.
  So A_{i+1} contains d with $(C, \overline{a}, c) \equiv (A_{i+1}, \overline{a}, c) \equiv (A_{i+1}, \overline{b}, d) \equiv (C, \overline{b}, d)$.
  Thus C is λ-homogeneous.

Consequences

- A structure A is called **strongly ω-homogeneous** if, whenever $\overline{a}, \overline{b}$ are in A, such that $(A, \overline{a}) \equiv (A, \overline{b})$, there exists an automorphism of A taking $\overline{a}$ to $\overline{b}$.

Corollary

Let A be an infinite L-structure and μ a cardinal $\geq |A| + |L|$.

(a) A has an ω-homogeneous elementary extension of cardinality μ. In particular every complete and countable first-order theory with infinite models has a countable homogeneous model.

(b) A has a strongly ω-homogeneous elementary extension B of cardinality μ.

Subsection 3

Syntactic Characterizations

Embedding a Structure in a λ-Saturated Structure

- Recall that if A and B are L-structures, then "$A \Rrightarrow_1 B$" means that, for every $\exists_1$ first-order sentence ϕ of L, if $A \models \phi$ then $B \models \phi$.

Theorem

Let L be a first-order language. Let A and B be L-structures, and suppose B is $|A|$-saturated and $A \Rrightarrow_1 B$. Then A is embeddable in B.

- List the elements of A as $\overline{a} = (a_i : i < \lambda)$, where $\lambda = |A|$.

  Claim: There is a sequence $\overline{b} = (b_i : i < \lambda)$ of elements of B such that for each $i \leq \lambda$, $(A, \overline{a}|_i) \Rrightarrow_1 (B, \overline{b}|_i)$.

  The proof is by induction on i.

  When $i = 0$, $A \Rrightarrow_1 B$ by assumption.

  When i is a limit ordinal, the condition holds at i provided it holds at all smaller ordinals.

  This leaves the case where i is a successor ordinal $j + 1$.

Embedding a Structure in a λ-Saturated Structure (Cont'd)

- Let $\overline{x}$ be the sequence of variables $(x_\alpha : \alpha < i)$.

  Let $\Phi(\overline{x}, y)$ be the set of all $\exists_1$ formulas $\phi(\overline{x}, y)$, such that $A \models \phi(\overline{a}\,|_i, a_i)$.

  For each finite set $\phi_0, \ldots, \phi_{n-1}$ from Φ, $A \models \exists y \bigwedge_{k<n} \phi_k(\overline{a}\,|_j, y)$.

  But $\exists y \bigwedge_{k<n} \phi_k$ is equivalent to an $\exists_1$ formula.

  So, by the induction hypothesis, $B \models \exists y \phi(\overline{b}\,|_j, y)$.

  By a previous theorem, $\Phi(\overline{b}\,|_j, y)$ is a type with respect to B.

  Since $j < \lambda$ and B is λ-saturated, this type is realized in B, say by b_j.

  Then $(A, \overline{a}\,|_i) \Rrightarrow_1 (B, \overline{b}\,|_i)$ as required. This proves the claim.

  Hence $(A, \overline{a}) \Rrightarrow_1 (B, \overline{b})$.

  By the Diagram Lemma, there is an embedding $f : A \to B$, such that $f(\overline{a}) = \overline{b}$.

A Corollary

Corollary

Let L be a first-order language, T a theory in L and $\Phi(\overline{x})$ a set of formulas of L (where the sequence $\overline{x}$ may be infinite). Suppose that whenever A and B are models of T with $A \subseteq B$, and $\overline{a}$ is a sequence of elements of A such that $A \models \bigwedge\Phi(\overline{a})$, we have $B \models \bigwedge\Phi(\overline{a})$. Then Φ is equivalent modulo T to a set $\Psi(\overline{x})$ of $\exists_1$ formulas of L.

- Putting new constants for the variables $\overline{x}$, we can suppose that the formulas in Φ are sentences. Let Ψ be the set of all $\exists_1$ sentences ψ of L such that $T \cup \Phi \vdash \psi$. It suffices to show that $T \cup \Psi \vdash \bigwedge\Phi$.

  If $T \cup \Psi$ has no models then this holds trivially.

  If $T \cup \Psi$ has models, let B' be one.

A Corollary (Cont'd)

- We set $\Psi = \{\psi\ \exists_1\text{-sentence} : T \cup \Phi \vdash \psi\}$. We must show $T \cup \Psi \vdash \bigwedge \Phi$.

  Let B' be a model of $T \cup \Psi$. By a previous corollary, B' is elementarily equivalent to a λ-saturated structure B, where $\lambda \geq |L|$.

  Write U for the set of all $\forall_1$ sentences of L which are true in B.

  Claim: $T \cup \Phi \cup U$ has a model.

  Suppose not. By the Compactness Theorem there is a finite subset $\{\theta_0, \ldots, \theta_{m-1}\}$ of U, such that $T \cup \Phi \vdash \neg\theta_0 \vee \cdots \vee \neg\theta_{m-1}$. Hence, $\neg\theta_0 \vee \cdots \vee \neg\theta_{m-1}$ is equivalent to a sentence in Ψ. Thus, it is true in B' and B, a contradiction.

  Let A be a model of $T \cup \Phi \cup U$ of cardinality $\leq |L|$.

  By the choice of U, $A \Rrightarrow_1 B$.

  So A is embeddable in B, by the theorem.

  Thus, since Φ is a set of $\exists_1$ sentences, $B \models \bigwedge \Phi$.

  Thus, $B' \models \bigwedge \Phi$.

Relation Symbols Fixed By Homomorphisms

- Let L be a first-order language and R a relation symbol of L.
- Let $f : A \to B$ be a homomorphism of L-structures.
- We say that f **fixes** R if, for every tuple $\overline{a}$ of elements of A,

$$A \models R(\overline{a}) \quad \text{if and only if} \quad B \models R(f(\overline{a})).$$

- In this definition we allow R to be the equality symbol $=$.
- The following properties hold.
  - f fixes $=$ if and only if f is injective.
  - f fixes all relation symbols if and only if f is an embedding.

Relation Symbols Positive In Formulas

- Let L be a first-order language and R a relation symbol of L.
- Let ϕ be an L-formula.
- ϕ is said to be **negation normal** if in ϕ the symbol $\neg$ never occurs except immediately in front of an atomic formula.
- We say that the relation symbol R is **positive in** ϕ if ϕ can be brought to negation normal form in such a way that there are no subformulas of the form $\neg R(\overline{t})$.
- Recall that a formula is **positive** if $\neg$ never occurs in it.
- Up to logical equivalence, a formula ϕ is positive if and only if every relation symbol, including $=$, is positive in ϕ.

Lyndon's Theorem

Theorem

Let L be a first-order language, Σ a set of relation symbols of L (possibly including $=$) and $\phi(\overline{x})$ a formula of L in which every relation symbol in Σ is positive.

(a) If $f : A \to B$ is a surjective homomorphism of L-structures, and f fixes all relation symbols (including possibly $=$) which are not in Σ, then f preserves ϕ.

(b) Suppose that every surjective homomorphism between models of T which fixes all relation symbols not in Σ preserves ϕ. Then ϕ is equivalent modulo T to a formula $\psi(\overline{x})$ of L in which every relation symbol in Σ is positive.

(a) This is a variant of a previous theorem concerning positive formulas.

(b) We start along the same track as the proof of the preceding corollary.

Lyndon's Theorem (Cont'd)

- Replacing the variables $\overline{x}$ by distinct new constants, we can assume that ϕ is a sentence. Let Θ be the set of all formulas of L in which every relation symbol in Σ is positive.

  We use Θ in the same way as we used $\exists_1$ in the preceding theorem.

  For L-structures C and D, write $(C,\overline{c}) \Rrightarrow_\Theta (D,\overline{d})$ to mean that if $\theta(\overline{x})$ is any formula in Θ such that $C \models \theta(\overline{c})$, then $D \models \theta(\overline{d})$.

  So $C \Rrightarrow_\Theta D$ means that every sentence in Θ true in C is also true in D.

  In place of the previous theorem, we shall show the following.

Lemma

Let L, Σ and Θ be as in the theorem. Let λ be a cardinal $\geq |L|$, and suppose A and B are λ-saturated structures such that $A \Rrightarrow_\Theta B$. Then there are elementary substructures A', B' of A, B, respectively, and a surjective homomorphism $f : A' \to B'$ which fixes all relation symbols not in Σ.

Lyndon's Theorem (Proof of the Lemma)

- We build up sequences $\overline{a},\overline{b}$ of elements of A,B, respectively, both of length λ, in such a way that:
  1. For every $i \leq \lambda$, $(A,\overline{a}|_i) \Rrightarrow_\Theta (B,\overline{b}|_i)$;
  2. $\overline{a}$ is the domain of an elementary substructure of A;
  3. $\overline{b}$ is the domain of an elementary substructure of B.

  The construction is by induction on i, as in the previous theorem.

  In that proof, each a_j was given and we found an element b_j to match.

  Here we sometimes choose the b_j first and, then, an answering a_j.

  One can think of the process as a back-and-forth game of length λ between A and B:
  - Player $\forall$ chooses an element a_j (or b_j);
  - Player $\exists$ has to find a corresponding element b_j (or a_j).

  Player $\exists$ wins iff Condition 1 holds after λ steps.

Lyndon's Theorem (Proof of the Lemma Cont'd)

Claim: Player $\exists$ can always win this game.

At the beginning of the game, $A \Rrightarrow_\Theta B$ by assumption.

Suppose i is a limit ordinal and $(A, \overline{a}|_j) \Rrightarrow_\Theta (B, \overline{b}|_j)$, for all $j < i$.

Since all formulas are finite, $(A, \overline{a}|_i) \Rrightarrow_\Theta (B, \overline{b}|_i)$.

Suppose i is a successor ordinal $j+1$. There are two cases, according as player $\forall$ chooses from A or from B.

- Suppose first that player $\forall$ has just chosen a_j from A.
  Let $\Phi(\overline{x}, y)$ be the set of all $\phi(\overline{x}, y)$ in Θ such that $A \vDash \phi(\overline{a}|_j, a_j)$.
  Φ is closed under conjunctions and existential quantification.
  Exactly the same argument as in the proof of the previous theorem shows that $\Phi(\overline{b}|_j, y)$ is a type over $\overline{b}|_j$ with respect to B.
  So there exists b in B, such that $(A, \overline{a}|_j, a_j) \Rrightarrow_\Theta (B, \overline{b}|_i, b)$.
  Let player $\exists$ choose b_j to be this element b.
- Suppose player $\forall$ chose b_j from B. So player $\exists$ must find a suitable a_j.
  The argument is just the same but from right to left, using the set $\{\neg\theta : \theta \in \Theta\}$ in place of Θ.

Lyndon's Theorem (Proof of the Lemma Conclusion)

- To enforce Conditions 2 and 3, we issue some instructions to player $\forall$.

  As the play proceeds, he must keep a note of all the formulas of the form $\phi(\overline{a}|_i, y)$, with ϕ in L, such that $A \models \exists y \phi(\overline{a}|_i, y)$.

  For each such formula he must make sure that at some stage j later than i, he chooses a_j so that $A \models \phi(\overline{a}|_i, a_j)$.

  He must do the same with B.

  At the end of the play, Conditions 2 and 3 will hold by the Tarski-Vaught Criterion.

Lyndon's Theorem (Conclusion)

- Finally suppose the game is played.

  Assume $\overline{a}, \overline{b}$ satisfying Conditions 1 and 2 have been found.

  Let A' be the substructure of A, with domain listed by $\overline{a}$.

  Let B' be the substructure of B, with domain listed by $\overline{b}$.

  All atomic formulas of L are in Θ.

  By the Diagram Lemma, we get a homomorphism $f: A' \to B'$ such that $f(\overline{a}) = \overline{b}$.

  Clearly f is surjective.

  If R is a relation symbol not in Σ, then $\neg R(\overline{z})$ is in Θ.

  So Condition 1 implies that f fixes R.

  The rest of the argument is much as in the proof of the preceding corollary.

Lyndon's Preservation Theorem

Corollary (Lyndon's Preservation Theorem)

Let T be a theory in a first-order language L and $\phi(\overline{x})$ a formula of L which is preserved by all surjective homomorphism between models of T. Then ϕ is equivalent modulo T to a positive formula $\psi(\overline{x})$ of L.

- Let Σ in the theorem be the set of all relation symbols of L, including the symbol $=$.
- Using the same argument, we can replace ϕ and ψ in this corollary by sets Φ, Ψ of formulas.

Keisler Games

- Let L be a first-order language and λ an infinite cardinal.
- A **Keisler sentence** of length λ in L is an infinitary expression of the form
$$\underbrace{Q_0x_0Q_1x_1\cdots Q_ix_i\cdots}_{i<\lambda}\bigwedge\Phi,$$
where:
  - Each Q_i is either $\forall$ or $\exists$;
  - Φ is a set of formulas $\phi(x_0,x_1,\ldots)$ of L.
- If χ is the Keisler sentence above and A is an L-structure, then the **Keisler game** $G(\chi,A)$ involves λ steps and is played as follows:
  - At the i-th step, one of the players chooses an element a_i of A.
    - Player $\forall$ makes the choice if Q_i is $\forall$;
    - Player $\exists$ makes the choice otherwise.
  - At the end of the play, player $\exists$ wins if $A\vDash\bigwedge\Phi(a_0,a_1,\ldots)$.
- $A\vDash\chi$ means that player $\exists$ has a winning strategy for $G(\chi,A)$.

Finite Approximations to Keisler Senences

- A **finite approximation** to the Keisler sentence

$$\underbrace{Q_0x_0Q_1x_1\cdots Q_ix_i\cdots}_{i<\lambda}\bigwedge\Phi$$

  is a sentence $\overline{Q}\bigwedge\Psi$, where:
  - Ψ is a finite subset of Φ;
  - $\overline{Q}$ is a finite subsequence of the quantifier prefix, containing quantifiers to bind all the free variables of Ψ.
- We denote by $\text{app}(\chi)$ the set of all finite approximations to the Keisler sentence χ.

Keisler Formulas

- These definitions adapt in an obvious way to give:
  - **Keisler formulas** $\chi(\overline{w})$;
  - **Keisler games** $G(\chi(\overline{w}), A, \overline{c})$.
- $A \models \chi(\overline{c})$ holds if player $\exists$ has a winning strategy for $G(\chi(\overline{w}), A, \overline{c})$.
- In particular, let χ be the Keisler sentence

$$\underbrace{Q_0 x_0 Q_1 x_1 \cdots Q_i x_i \cdots}_{i<\lambda} \bigwedge \Phi$$

  and α be an ordinal $< \lambda$.
- We write $\chi^\alpha(x_i : i < \alpha)$ for the Keisler formula got from χ by removing the quantifiers $Q_i x_i$, $i < \alpha$.

Detaching the Leftmost Quantifier

- The following lemma tells us that we can detach the leftmost quantifier Q_0x_0 of a Keisler sentence and treat it exactly like an ordinary quantifier.
- The lemma generalizes to cover also Keisler formulas $\chi(\overline{w})$.

Lemma

With the notation above, we have $A \models \chi$ iff $A \models Q_0x_0\chi^1(x_0)$.

- Suppose first that Q_0 is $\forall$. If $A \models \chi$, then the initial position in $G(\chi, A)$ is winning for player $\exists$. So every choice a of player $\forall$ puts player $\exists$ into winning position in $G(\chi^1, A, a)$. Hence $A \models \chi^1(a)$. So $A \models \forall x_0\chi^1(x_0)$.

  The converse, and the corresponding arguments for the case $Q_0 = \exists$, are similar.

Keisler Games and Saturation

Theorem

Let A be a non-empty L-structure, λ an infinite cardinal and χ a Keisler sentence of L of length λ.

(a) If $A \models \chi$, then $A \models \bigwedge \text{app}(\chi)$.
(b) If $A \models \bigwedge \text{app}(\chi)$ and A is λ-saturated then $A \models \chi$.

(a) Let $\chi = \underbrace{Q_0 x_0 Q_1 x_1 \cdots Q_i x_i \cdots}_{i<\lambda} \bigwedge \Phi$.

Suppose $\alpha < \lambda$, $\theta(x_i : i < \alpha)$ is a finite approximation to χ^α, and $\overline{a}$ is a sequence of elements of A, such that $A \models \chi^\alpha(\overline{a})$. We show $A \models \theta(\overline{a})$.

Use induction on the number n of quantifiers in the prefix of θ.

If $n = 0$, then θ is a conjunction of formulas $\phi(x_i : i < \alpha)$ from Φ.

If $A \models \chi^\alpha(\overline{a})$, then player $\exists$ has a winning strategy for $G(\chi^\alpha, A, \overline{a})$.

Therefore, $A \models \theta(\overline{a})$.

Keisler Games and Saturation (Part (a) Cont'd)

- Suppose $n > 0$.

  Let the quantifier prefix of θ begin with a universal quantifier $\forall x_\beta$.

  Then $\beta \geq \alpha$ and we can write θ as $\forall x_\beta \theta'(x_i : i \leq \beta)$ (note that none of the variables x_i, with $i > \alpha$ are free in θ).

  If $A \models \chi^\alpha(\overline{a})$, then player $\exists$ has a winning strategy for $G(\chi^\alpha, A, \overline{a})$.

  Let players play this game through the steps $Q_i x_i$, $\alpha \leq i < \beta$, with player $\exists$ using her winning strategy. Let $\overline{b}$ be the sequence of elements chosen (possible because A is not empty).

  Then $A \models \chi^\beta(\overline{a}, \overline{b})$.

  By the preceding lemma, $A \models \forall x_\beta \chi^{\beta+1}(\overline{a}, \overline{b}, x_\beta)$.

  So, for every element c of A, $A \models \chi^{\beta+1}(\overline{a}, \overline{b}, c)$.

  By the induction hypothesis, $A \models \theta'(\overline{a}, \overline{b}, c)$. Thus, $A \models \theta(\overline{a})$.

  The argument when θ begins with an existential quantifier is similar.

  Finally putting $\alpha = 0$, we get Part (a).

Keisler Games and Saturation (Part (b))

(b) Assume A is λ-saturated and $A \models \bigwedge \text{app}(x)$.

Player $\exists$ should adopt the following rule for playing $G(\chi, A)$:

> Always choose so that for each $\alpha < \lambda$, if $\overline{b}$ is the sequence of elements chosen before the α-th step, then $A \models \bigwedge \text{app}(\chi^\alpha)(\overline{b})$.

Suppose she succeeds in following this rule until the end of the game.

Then a sequence $\overline{a}$ of length λ has been chosen, with $A \models \bigwedge \Phi(\overline{a})$.

So, in that case, she wins $G(\chi, A)$.

Claim: $\exists$ can follow this rule.

Suppose she has followed this rule up to the choice of $\overline{b} = (b_i : i < \alpha)$.

So, it holds that $A \models \bigwedge \text{app}(\chi^\alpha)(\overline{b})$.

Keisler Games and Saturation (Part (b) Cont'd)

- First suppose that Q_α is $\exists$. Without loss write any finite approximation θ to χ^α as $\exists x_\alpha \theta'(x_i : i \leq \alpha)$. To maintain the rule, player $\exists$ has to choose an element b_α so that $A \models \theta'(\overline{b}, b_\alpha)$, for each $\theta \in \mathrm{app}(\chi^\alpha)$. Now A is λ-saturated and $\overline{b}$ has length less than λ. Hence we only need show that if $\{\theta_0, \ldots, \theta_{n-1}\}$ is a finite set of formulas in $\mathrm{app}(\chi^\alpha)$, then $A \models \exists x_\alpha(\theta'_0, \ldots, \theta'_{n-1})(\overline{b}, x_\alpha)$. But clearly there is some finite approximation θ to χ^α which begins with $\exists x_\alpha$ and is such that θ' implies $\theta'_0, \ldots, \theta'_{n-1}$. By assumption $A \models \theta(\overline{b})$, in other words $A \models \exists x_\alpha \theta'(\overline{b}, x_\alpha)$. This completes the argument when Q_α is $\exists$.
- Next suppose that Q_α is $\forall$, and let $\theta'(x_i : i \leq \alpha)$ be a finite approximation to $\chi^{\alpha+1}$. Then $\forall x_\alpha \theta'$ is a finite approximation to χ^α. So, by assumption, $A \models \forall x_\alpha \theta'(\overline{b}, x_\alpha)$. Hence $A \models \theta'(\overline{b}, b_\alpha)$ regardless of the choice of b_α. So player $\forall$ can never break player $\exists$'s rule.

  Limit ordinals are no threat to player $\exists$'s rule.

  So she can follow the rule and win. Therefore, $A \models \chi$.

Generalization to Keisler Formulas

- The preceding theorem generalizes to Keisler formulas $\chi(\overline{w})$ with fewer than λ free variables $\overline{w}$.
- Part (a) of the theorem reads

$$\text{If } A \models \chi(\overline{b}), \text{ then } A \models (\bigwedge \text{app}(\chi))(\overline{b}).$$

- Part (b) takes the form

$$\text{if } A \models \bigwedge \text{app}(\chi)(\overline{b}) \text{ and } A \text{ is } \lambda\text{-saturated, then } A \models \chi(\overline{b}).$$

Relating the First and Last Theorems of the Section

- Let L be a first-order language. Let A and B be L-structures, and suppose B is $|A|$-saturated and $A \Rrightarrow_1 B$.

  We list the elements of A as $\overline{a} = (a_i : i < \lambda)$.

  Let $\overline{x}$ be the sequence of variables $(x_i : i < \lambda)$.

  Write Θ for the set of $\exists_1$ formulas $\theta(\overline{x})$ of L, such that $A \models \theta(\overline{a})$.

  Let χ be the sentence $\exists_0 x_0 \exists_1 x_1 \cdots \bigwedge \Theta$.

  Then $\bigwedge \mathrm{app}(\chi)$ is a conjunction of $\exists_1$ sentences true in A.

  Hence, since $A \Rrightarrow_1 B$, $B \models \bigwedge \mathrm{app}(\chi)$.

  By Part (b) of the preceding theorem, it follows that $B \models \chi$.

  Therefore, A is embeddable in B.

Subsection 4

Ultraproducts and Ultrapowers

Direct Products

- Let L be a signature and I a non-empty set.
- Suppose that for each $i \in I$ a non-empty L-structure A_i is given.
- The **direct product** (or **Cartesian product** or, simply, **product**) $\prod_{i \in I} A_i$ (or $\prod_i A_i$ for short) is the L-structure B defined as follows:
  - Write X for the set of all maps $a : I \to \bigcup_{i \in I} \mathrm{dom}(A_i)$, such that for each $i \in I$, $a(i) \in \mathrm{dom}(A_i)$. We put $\mathrm{dom}(B) = X$.
  - For each constant c of L we take c^B to be the element a of X, such that $a(i) = c^{A_i}$, for each $i \in I$.
  - For each n-ary function symbol F of L and n-tuple $\overline{a} = (a_0, \ldots, a_{n-1})$ from X, we define $F^B(\overline{a})$ to be the element b of X such that for each $i \in I$, $b(i) = F^{A_i}(a_0(i), \ldots, a_{n-1}(i))$.
  - For each n-ary relation symbol R of L and n-tuple $\overline{a}$ from X, we put $\overline{a}$ in R^B iff for every $i \in I$, $(a_0(i), \ldots, a_{n-1}(i)) \in R^{A_i}$.
- The structure A_i is called the ***i*-th factor** of the product.
- If $I = \{0, \ldots, n-1\}$, we write $A_0 \times \cdots \times A_{n-1}$ for $\prod_i A_i$.

Filters, Ultrafilters and Principal Ultrafilters

- By a **filter** over a non-empty set I we mean a non-empty set $\mathscr{F}$ of subsets of I such that:
  1. $\emptyset \notin \mathscr{F}$;
  2. $X \in \mathscr{F}$ and $X \subseteq Y \subseteq I$ imply $Y \in \mathscr{F}$;
  3. $X, Y \in \mathscr{F}$ implies $X \cap Y \in \mathscr{F}$.
- In particular $I \in \mathscr{F}$ by the second condition and the fact that $\mathscr{F} \neq \emptyset$.
- A filter $\mathscr{F}$ is called an **ultrafilter** if it has the further property:

  For every set $X \subseteq I$, exactly one of X, $I \backslash X$ is in $\mathscr{F}$.
- Given an element $i \in I$, the set $\mathscr{U}$ of all subsets X of I, such that $i \in X$ is an ultrafilter over I.
- Ultrafilters of this form are called **principal**.

The Boolean Value

- Let L be a first-order language and I a non-empty set.
- Let $(A_i : i \in I)$ a family of non-empty L-structures.
- Let $\phi(\overline{x})$ be a formula of L.
- Let $\overline{a}$ be a tuple of elements of the product $\prod_i A_i$.
- We define the **boolean value** of $\phi(\overline{a})$, in symbols $\|\phi(\overline{a})\|$, to be the set

$$\|\phi(\overline{a})\| := \{i \in I : A_i \models \phi(\overline{a}(i))\}.$$

- Note the following properties:
  - $\|\phi \wedge \psi\| = \|\phi\| \cap \|\psi\|$;
  - $\|\phi \vee \psi\| = \|\phi\| \cup \|\psi\|$;
  - $\|\neg\phi\| = I \setminus \|\phi\|$.

Boolean Value and Quantification

- The property for the existential quantifier should say that $\|\exists x\phi(x)\|$ is the union of the sets $\|\phi(a)\|$, with a in $\prod_i A_i$.
- Something stronger is true, both for $\prod_i A_i$ and for some of its substructures C.
- We say that C **respects** $\exists$ if, for every formula $\phi(x)$ of L with parameters from C,

$$\|\exists x\phi(x)\| = \|\phi(a)\|, \text{ for some element } a \text{ of } C.$$

Claim: $\prod_i A_i$ respects $\exists$.

For each $i \in \|\exists x\phi(x)\|$, choose an element a_i, such that $A_i \models \phi(a_i)$.

Consider the element a of $\prod_i A_i$, such that $a(i) = a_i$, $i \in \|\exists x\phi(x)\|$ (invoking the axiom of choice).

The Equivalence Relation $\sim$

- Let L be a first-order language and I a non-empty set.
- Let $(A_i : i \in I)$ a family of non-empty L-structures.
- Let $\mathscr{F}$ a filter over I.
- Form the product $\prod_I A_i$ and, using $\mathscr{F}$, define a relation $\sim$ on $\operatorname{dom} \prod_I A_i$ by
$$a \sim b \quad \text{iff} \quad \|a = b\| \in \mathscr{F}.$$
Claim: $\sim$ is an equivalence relation.
  - Reflexive: For each element a of $\prod_I A_i$, $\|a = a\| = I \in \mathscr{F}$.
  - Symmetric: This is clear.
  - Transitive: $\|a = b\| \cap \|b = c\| \subseteq \|a = c\|$.
    So if $\|a = b\|, \|b = c\| \in \mathscr{F}$, then $\|a = c\| \in \mathscr{F}$.

  Thus, $\sim$ is an equivalence relation.
- We write $a/\mathscr{F}$ for the equivalence class of the element a.

The L-structure D

- We define an L-structure D as follows:
  - The domain $\mathrm{dom}(D)$ is the set of equivalence classes $a/\mathscr{F}$, with $a \in \mathrm{dom}\prod_I A_i$.
  - For each constant symbol c of L we put $c^D = a/\mathscr{F}$, where $a(i) = c^{A_i}$, for each $i \in I$.
  - Let F be an n-ary function symbol of L, and $a_0, \ldots, a_{n-1}$ elements of $\prod_I A_i$. We define $F^D(a_0/\mathscr{F}, \ldots, a_{n-1}/\mathscr{F}) = b/\mathscr{F}$, where $b(i) = F^{A_i}(a_0(i), \ldots, a_{n-1}(i))$, for each $i \in I$.
  - Finally if R is an n-ary relation symbol of L and $a_0, \ldots, a_{n-1}$ are elements of $\prod_I A_i$, then we put $(a_0/\mathscr{F}, \ldots, a_{n-1}/\mathscr{F}) \in R^D$ iff $\|R(a_0, \ldots, a_{n-1})\| \in \mathscr{F}$

Soundness of the Definition

- Claim: The definition of D is sound.
  - Let F be an n-ary function symbol.
    Suppose that $a_i \sim a'_i$, $i < n$.
    We must show $F(a_0,\ldots,a_{n-1}) \sim F(a'_0,\ldots,a'_{n-1})$.
    Since $a_i \sim a'_i$, $\|a_i = a'_i\| \in \mathscr{F}$.
    Since $\mathscr{F}$ is a filter, $\bigcap_{i<n} \|a_i = a'_i\| \in \mathscr{F}$.
    But $\bigcap_{i<n} \|a_i = a'_i\| \subseteq \|F(a_0,\ldots,a_{n-1}) = F(a'_0,\ldots,a'_{n-1})\|$.
    So, again by the filter property, $\|F(a_0,\ldots,a_{n-1}) = F(a'_0,\ldots,a'_{n-1})\| \in \mathscr{F}$.
    This proves that $F(a_0,\ldots,a_{n-1}) \sim F(a'_0,\ldots,a'_{n-1})$.
  - Let R be an n-ary relation symbol.
    Suppose that $\|R(a_0,\ldots,a_{n-1})\| \in \mathscr{F}$ and $a_i \sim a'_i$, $i < n$.
    We must show $\|R(a'_0,\ldots,a'_{n-1})\| \in \mathscr{F}$.
    Since $a_i \sim a'_i$, $\|a_i = a'_i\| \in \mathscr{F}$.
    Since $\mathscr{F}$ is a filter, $\bigcap_{i<n} \|a_i = a'_i\| \cap \|R(a_0,\ldots,a_{n-1})\| \in \mathscr{F}$.
    But $\bigcap_{i<n} \|a_i = a'_i\| \cap \|R(a_0,\ldots,a_{n-1})\| \subseteq \|R(a'_0,\ldots,a'_{n-1})\|$.
    So, again by the filter property, $\|R(a_0,\ldots,a_{n-1})\| \in \mathscr{F}$.

Reduced Products

- The L-structure D is called the **reduced product** of $(A_i : i \in I)$ over $\mathscr{F}$, in symbols $\prod_I A_i/\mathscr{F}$.
- When $\mathscr{F}$ is an ultrafilter, the structure is called the **ultraproduct** of $(A_i : i \in I)$ over $\mathscr{F}$.
- For every unnested atomic formula $\phi(\overline{x})$ of L and every tuple $\overline{a}$ of elements of $\prod_I A_i$,

$$\prod_I A_i/\mathscr{F} \vDash \phi(\overline{a}/\mathscr{F}) \quad \text{iff} \quad \|\phi(\overline{a})\| \in \mathscr{F}.$$

- Note that $\prod_I A_i$ itself is just the reduced product $\prod_I A_i/\{I\}$.
- So every direct product is a reduced product.

Reduced Products and Relativized Reducts

Theorem

Let L and L^+ be signatures and P a 1-ary relation symbol of L^+. Let $(A_i : i \in I)$ be a non-empty family of non-empty L^+-structures such that $(A_i)_P$ is defined and $\mathscr{F}$ a filter over I. Then $(\prod_I A_i/\mathscr{F})_P = \prod_I((A_i)_P)/\mathscr{F}$.

- Define $f : \prod_I((A_i)_P)/\mathscr{F} \to \prod_I A_i/\mathscr{F}$ by

$$\prod_I((A_i)_P)/\mathscr{F} \ni a/\mathscr{F} \mapsto a/\mathscr{F} \in \prod_I A_i/\mathscr{F}.$$

One can check from the definition of reduced products that this definition is sound.

Moreover f is an embedding with image $(\prod_I A_i/\mathscr{F})_P$.

Reduced Powers

- When all the structures A_i are equal to a fixed structure A, we call $\prod_I A/\mathscr{F}$ the **reduced power** $A^I/\mathscr{F}$.
- If $\mathscr{F}$ is an ultrafilter, we call the structure the **ultrapower** of A over $\mathscr{F}$.
- There is an embedding (as will follow from the next lemma) $e : A \to A^I/\mathscr{F}$ defined by
$$e(b) = a/\mathscr{F},$$
where $a(i) = b$, for all $i \in I$.
- We call e the **diagonal embedding**.

Reduced Products and Positive Primitive Formulas

- Recall that a **positive primitive (p.p.) formula** is a first-order formula of the form $\exists \overline{y} \bigwedge \Phi$, where Φ is a set of atomic formulas.

Lemma

Let L be a signature and $\phi(\overline{x})$ a p.p. formula of L. Let $(A_i : i \in I)$ be a non-empty family of non-empty L-structures and $\overline{a}$ a tuple of elements of $\prod_I A_i$. Let $\mathscr{F}$ be a filter over I. Then $\prod_I A_i/\mathscr{F} \vDash \phi(\overline{a}/\mathscr{F})$ iff $\|\phi(\overline{a})\| \in \mathscr{F}$.

- By induction on the complexity of ϕ. If $\psi \equiv \chi$, $\|\psi\| = \|\chi\|$.

  By a previous corollary, we may assume that ϕ is unnested.

  For atomic formulas the result has bee asserted in a preceding slide.

  - Suppose the conclusion holds for $\phi(\overline{x})$, $\psi(\overline{x})$. Then it holds for their conjunction.

    Suppose $\prod_I A_i/\mathscr{F} \vDash (\phi \wedge \psi)(\overline{a}/\mathscr{F})$. By definition, $\prod_I A_i/\mathscr{F} \vDash \phi(\overline{a}/\mathscr{F})$ and $\prod_I A_i/\mathscr{F} \vDash \psi(\overline{a}/\mathscr{F})$. By assumption, $\|\phi(\overline{a})\|$ and $\|\psi(\overline{a})\|$ are both in $\mathscr{F}$. So $\|(\phi \wedge \psi)(\overline{a})\| \in \mathscr{F}$.

Reduced Products and Positive Primitive Formulas (Cont'd)

- We continue the Induction:
  - We finish conjunction by looking at the converse.
    Suppose $\|(\phi\wedge\psi)(\overline{a})\|\in\mathscr{F}$. But $\|(\phi\wedge\psi)(\overline{a})\|\subseteq\|\phi(\overline{a})\|$ and $\|(\phi\wedge\psi)(\overline{a})\|\subseteq\|\psi(\overline{a})\|$. So $\|\phi(\overline{a})|\in\mathscr{F}$ and $\|\psi(\overline{a})|\in\mathscr{F}$. By the induction hypothesis, $\prod_I A_i/\mathscr{F}\models\phi(\overline{a}/\mathscr{F})$ and $\prod_I A_i/\mathscr{F}\models\psi(\overline{a}/\mathscr{F})$. So, by definition, $\prod_I A_i/\mathscr{F}\models(\phi\wedge\psi)(\overline{a}/\mathscr{F})$.
  - If the result holds for $\psi(\overline{x},\overline{y})$ then it holds for $\exists\overline{y}\psi(\overline{x},\overline{y})$.
    From left to right, suppose $\prod_I A_i/\mathscr{F}\models\exists y\psi(\overline{a}/\mathscr{F},\overline{y})$. Then there are elements $\overline{b}$ of $\prod_I A_i$ such that $\prod_I A_i/\mathscr{F}\models\psi(\overline{a}/\mathscr{F},\overline{b}/\mathscr{F})$. So $\|\psi(\overline{a},\overline{b})\|\in\mathscr{F}$ by assumption. But $\|\psi(\overline{a},\overline{b})\|\subseteq\|\exists\overline{y}\psi(\overline{a},\overline{y})\|$. So $\|\exists\overline{y}\psi(\overline{a},\overline{y})\|\in\mathscr{F}$.
    Conversely, suppose $\|\exists\overline{y}\psi(\overline{a},\overline{y})\|\in\mathscr{F}$. Now $\prod_I A_i$ respects $\exists$. So there are elements $\overline{b}$ of $\prod_I A_i$, such that $\|\psi(\overline{a},\overline{b})\|=\|\exists\overline{y}\psi(\overline{a},\overline{y})\|$. Thus, $\prod_I A_i/\mathscr{F}\models\psi(\overline{a}/\mathscr{F},\overline{b}/\mathscr{F})$, by assumption. Hence $\prod_I A_i\models\exists\overline{y}\psi(\overline{a}/\mathscr{F},\overline{y})$.

Łoś' Theorem

Theorem (Łoś' Theorem)

Let L be a first-order language, $(A_i : i \in I)$ a non-empty family of non-empty L-structures and $\mathscr{U}$ an ultrafilter over I. Then, for any formula $\phi(\overline{x})$ of L and tuple $\overline{a}$ of elements of $\prod_I A_i$,

$$\prod_I A_i/\mathscr{U} \vDash \phi(\overline{a}/\mathscr{U}) \quad \text{iff} \quad \|\phi(\overline{a})\| \in \mathscr{U}.$$

- By induction on the complexity of ϕ. Comparing with the proof of the preceding lemma, only one more thing is needed. Assuming the conclusion holds for ϕ, we have to deduce it for $\neg\phi$ also. We have

$$\begin{array}{lll} \prod_I A_i/\mathscr{U} \vDash \neg\phi(\overline{a}/\mathscr{U}) & \text{iff} & \prod_I A_i/\mathscr{U} \nvDash \phi(\overline{a}/\mathscr{U}) \\ & \text{iff} & \|\phi(\overline{a})\| \notin \mathscr{U} \\ & \text{iff} & I \setminus \|\phi(\overline{a})\| \in \mathscr{U} \\ & \text{iff} & \|\neg\phi(\overline{a})\| \in \mathscr{U}. \end{array}$$

Constructing Elementary Extensions

Corollary

If $A^I/\mathscr{U}$ is an ultrapower of A, then the diagonal map $e: A \to A^I/\mathscr{U}$ is an elementary embedding.

- By he corollary, we may regard A as an elementary substructure of $A^I/\mathscr{U}$.
- So ultrapowers give elementary extensions.
- But this is useful only when ultrafilters are non-principal.

The Finite Intersection Property and Ultrafilters

- Let I be a nonempty set and W a set of subsets of I.
- We say that W has the **finite intersection property** if for every finite set $X_0, \ldots, X_{m-1}$ of elements of W, $X_0 \cap \cdots \cap X_{m-1}$ is not empty.
- Every filter over I has the finite intersection property.

Lemma

Let I be a nonempty set and W a set of subsets of I with the finite intersection property. Then, there is an ultrafilter $\mathscr{U}$ over I, with $W \subseteq \mathscr{U}$.

- Let L be the first-order language with the following signature:
  - Each subset of I is a constant;
  - There is one unary relation symbol P.

  Let T be the theory

$$\{P(a) \to P(b) : a \subseteq b\} \cup \{P(a) \wedge P(b) \to P(c) : a \cap b = c\} \\ \cup \{P(a) \leftrightarrow \neg P(b) : b = I \setminus a\} \cup \{P(a) : a \in W\}.$$

The Finite Intersection Property and Ultrafilters (Claim)

Claim: T has a model.

Suppose T does not have a model. By the Compactness Theorem, some finite subset U of T does not have a model.

Let $X_0, \ldots, X_{m-1}$ be the elements a of W, such that "$P(a)$"$\in U$.

By hypothesis, W has the finite intersection property.

So there is some $i \in I$, such that $i \in X_0 \cap \cdots \cap X_{m-1}$.

Let $\mathcal{V}$ be the principal ultrafilter consisting of all the subsets of I that contain i. Then we form a model of U by:

- Interpreting each subset of I as a name of itself;
- Reading "$P(c)$" as "$c \in \mathcal{V}$".

This proves the claim.

Let B be a model of T.

Define a set $\mathcal{U}$ of subsets of I by $b \in \mathcal{U}$ if and only if $B \models P(b)$.

By reading T, $\mathcal{U}$ is an ultrafilter containing all of W.

Regular and Incomplete Filters

- Let I be an infinite set.
- Let $\mathscr{F}$ be a filter over I.
- $\mathscr{F}$ is **regular** if there exists a countable $\mathscr{G} \subseteq \mathscr{F}$, such that, for all $i \in I$,

$$|\{X \in \mathscr{G} : i \in X\}| < \omega.$$

- $\mathscr{F}$ is **incomplete** if there exists countable $\mathscr{G} \subseteq \mathscr{F}$, such that $\bigcap \mathscr{G} \notin \mathscr{F}$.

Characterization of Regularity

Proposition

Let I be an infinite set and $\mathscr{F}$ be a filter over I. $\mathscr{F}$ is regular if and only if, there exists a countable decreasing chain

$$I = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$$

of elements in $\mathscr{F}$, such that $\bigcap_n I_n = \emptyset$.

- Suppose $\mathscr{F}$ is regular. Let $\mathscr{G} = \{G_0, G_1, \ldots\} \subseteq \mathscr{F}$ be countable such that, each $i \in I$ is in finitely many elements of $\mathscr{G}$. Take $I_i = \bigcap_{j<i} G_i$, $i \geq 1$. For every $i \in I$, there exists k, such that $i \notin G_k$. Hence, $i \notin \bigcap_n I_n$. This shows that $\bigcap_n I_n = \emptyset$.

  Suppose the given condition holds. Let $\mathscr{G} = \{I_0, I_1, \ldots\}$. Since $i \notin \bigcap \mathscr{G}$, there exists k, such that $i \notin I_k$. Hence, since the I_i's form a decreasing sequence, $|\{I_j : i \in I_j\}| < \omega$. Thus, $\mathscr{F}$ is regular.

Regularity and Incompleteness

Proposition

Let I be an infinite set and $\mathscr{F}$ be an ultrafilter over I. $\mathscr{F}$ is regular if and only if it is incomplete.

- Suppose $\mathscr{F}$ is regular. Let $\mathscr{G} \subseteq \mathscr{F}$ be countable, such that, for each $i \in I$, i is in finitely many $G \in \mathscr{G}$. Then $\bigcap \mathscr{G} = \emptyset$. Hence, $\bigcap \mathscr{G} \notin \mathscr{F}$. So $\mathscr{F}$ is incomplete.

  Suppose $\mathscr{F}$ is incomplete. Let $\mathscr{G} = \{G_0, G_1, \ldots\} \subseteq \mathscr{F}$ be such that $\bigcap \mathscr{G} \notin \mathscr{F}$. Define

$$G_0' = G_0 \setminus \bigcap \mathscr{G}, \quad G_{i+1}' = G_i' \cap G_{i+1}, \quad i \geq 0.$$

  Then $\mathscr{G}' = \{G_0', G_1', \ldots\} \subseteq \mathscr{F}$ and $\bigcap \mathscr{G}' = \emptyset$. Hence, $\mathscr{F}$ is regular.

Regular Ultrafilters

Lemma

Let I be an infinite set. Then there is a regular ultrafilter $\mathscr{F}$ over I.

- It suffices to prove the lemma for a set J of the same cardinality as I.
  Let J be the set of all finite subsets of I.
  For $i \in I$, let $\widehat{i} = \{X \in J : i \in X\}$.
  Set $\mathscr{G} = \{\widehat{i} : i \in I\}$.
  $\mathscr{G}$ has the finite intersection property: This holds since
  $\{i_0, \ldots, i_{n-1}\} \subseteq \widehat{i}_0 \cap \cdots \cap \widehat{i}_{n-1}$.
  Hence $\mathscr{G}$ can be extended to an ultrafilter $\mathscr{F}$ over J.
  Clearly $\mathscr{G} \subseteq \mathscr{F}$, with $|\mathscr{G}| = \omega$.
  Moreover, if $X \in J$, X is finite and $X \in \widehat{i}$ means $i \in X$.
  So each $X \in J$ is in finitely many elements of $\mathscr{G}$.
  This proves that $\mathscr{F}$ is regular.

Cardinality and Realization Properties

Theorem

Let L be a first-order language, A an L-structure, I an infinite set and $\mathscr{U}$ a regular ultrafilter over I.

(a) If $\phi(x)$ is a formula of L such that $|\phi(A)|$ is infinite, then $|\phi(A^I/\mathscr{U})| = |\phi(A)|^{|I|}$.

(b) If $\Phi(\overline{x})$ is a type over $\mathrm{dom}(A)$ with respect to A, and $|\Phi| \leq |I|$, then some tuple $\overline{a}$ in $A^I/\mathscr{U}$ realizes Φ.

(a) We first prove $\leq$. By Łoś's theorem, each element of $\phi(A^I/\mathscr{U})$ is of the form $b/\mathscr{U}$, for some b such that $\|\phi(b)\| \in \mathscr{U}$. Since we can change b anywhere outside a set in $\mathscr{U}$ without affecting $b/\mathscr{U}$, we can choose b so that $\|\phi(b)\| = I$. This sets up an injection from $\phi(A^I/\mathscr{U})$ to the set $\phi(A)^I$ of all maps from I to $\phi(A)$.

Cardinality and Realization Properties ((a) Cont'd)

- Next we prove $|\phi(A^I/\mathscr{U})| \geq |\phi(A)|^{|I|}$.

  Since $\mathscr{U}$ is regular, there are sets X_i, $i \in I$, in $\mathscr{U}$ such that for each $j \in I$, the set $Z_j = \{i \in I : j \in X_i\}$ is finite.

  For each $j \in I$, let μ_j be a bijection taking the set $\phi(A)^{Z_j}$ (of all maps from Z_j to $\phi(A)$) to $\phi(A)$. Such a μ_j exists since $\phi(A)$ is, by hypothesis, infinite.

  For each function $f : I \to \phi(A)$, define f^μ to be the map from I to $\phi(A)$ such that, for each $j \in I$, $f^\mu(j) = \mu_j(f|_{Z_j})$.

  Each function $f^\mu : I \to \phi(A)$ is an element of A^I.

  By Łoś's theorem $f^\mu/\mathscr{U} \in \phi(A^I/\mathscr{U})$.

  We must show that if $f, g : I \to \phi(A)$, $f \neq g$, then $f^\mu/\mathscr{U} \neq g^\mu/\mathscr{U}$.

  Suppose then that $f(i) \neq g(i)$, for some $i \in I$.

  Then $f|_{Z_j} \neq g|_{Z_j}$ whenever $i \in Z_j$, i.e., whenever $j \in X_i$.

  Hence $X_i \subseteq \| f^\mu \neq g^\mu \|$. But $X_i \in \mathscr{U}$. So $f^\mu/\mathscr{U} \neq g^\mu/\mathscr{U}$.

Cardinality and Realization Properties (Part (b))

(b) Since $\mathscr{U}$ is regular, there is a family $\{X_\phi : \phi \in \Phi\}$ of sets in $\mathscr{U}$, such that for each $i \in I$ the set $Z_i = \{\phi \in \Phi : i \in X_\phi\}$ is finite.

But Φ is a type over dom(A).

So, for each $i \in I$, there is a tuple $\overline{a}_i$ in A which satisfies Z_i.

Let $\overline{a}$ be the tuple in A^I, such that $\overline{a}(i) = \overline{a}_i$, for each i.

Then, for each formula ϕ in Φ, if $i \in X_\phi$, then $\phi \in Z_i$.

So $A \models \phi(\overline{a}_i)$. Thus, $X_\phi \subseteq \|\phi(\overline{a})\|$.

By Łoś's theorem, we deduce that $A^I/\mathscr{U} \models \phi(\overline{a})$.

Arbitrarily Large Elementary Extensions

Corollary

Let L be a first-order language, A an L-structure and κ an infinite cardinal. Then A has an elementary extension B, such that, for every formula $\phi(\overline{x})$ of L, $|\phi(B)|$ is either finite or equal to $|\phi(A)|^\kappa$.

- Consider an ultrafilter $\mathscr{U}$ over a set I of cardinality κ.

  Then the conclusion follows from Part (a) of the preceding theorem combined with a previous corollary.

Keisler-Shelah Theorem

- We present an important theorem characterizing elementary equivalence using ultrapowers without proof.

Theorem (Keisler-Shelah Theorem)

Let L be a signature and let A, B be L-structures. The following are equivalent:

(a) $A \equiv B$.

(b) There are a set I and an ultrafilter $\mathscr{U}$ over I, such that $A^I/\mathscr{U} \cong B^I/\mathscr{U}$.

- The proof uses some quite difficult combinatorics.

Robinson's Joint Consistency Lemma

Corollary (Robinson's Joint Consistency Lemma)

Let L_1 and L_2 be first-order languages and $L = L_1 \cap L_2$. Let T_1 and T_2 be consistent theories in L_1 and L_2, respectively, such that $T_1 \cap T_2$ is a complete theory in L. Then $T_1 \cup T_2$ is consistent.

- Let A_1, A_2 be models of T_1, T_2 respectively.

  Then since $T_1 \cap T_2$ is complete, $A_1 |_L \equiv A_2 |_L$.

  By the Keisler-Shelah Theorem, there is an ultra-filter $\mathscr{U}$ over a set I, such that $(A_1 |_L)^I / \mathscr{U} \cong (A_2 |_L)^I / \mathscr{U}$.

  By a previous corollary, $A_1^I / \mathscr{U} \models T_1$ and $A_2^I / \mathscr{U} \models T_2$.

  By a previous theorem:
  - $A_1^I / \mathscr{U}$ is an expansion of $(A_1 |_L)^I / \mathscr{U}$;
  - $A_2^I / \mathscr{U}$ is an expansion of an isomorphic copy of $(A_1 |_L)^I / \mathscr{U}$.

  So we can use $A_2^I / \mathscr{U}$ as a template to expand $A_1^I / \mathscr{U}$ to a model of T_2.

Limit Points of Theories of Structures

- Let L be a first-order language.
- Let S be the set of all theories in L which are of the form $\text{Th}(A)$, for some L-structure A.
- Let X be a subset of S.
- Let T a set of sentences of L.
- We call T a **limit point of** X if:
  1. For every sentence ϕ of L, exactly one of ϕ, $\neg\phi$ is in T;
  2. For every finite $T_0 \subseteq T$, there is $T' \in X$ with $T_0 \subseteq T'$.
- The following theorem is one way of showing that such a set T is in fact an element of S.

Characterization of Limit Points

Theorem

Let L be a first-order language, $\mathbf{K}$ a class of L-structures and T a limit point of $\{\text{Th}(A) : A \in \mathbf{K}\}$. Then T is $\text{Th}(B)$, for some ultraproduct B of structures in $\mathbf{K}$.

- Let $\mathscr{U}$ be a regular ultrafilter over the set T.
  Then there is a family $\{X_\phi : \phi \in T\}$ of sets in $\mathscr{U}$, such that for each $i \in T$, the set $Z_i = \{\phi \in T : i \in X_\phi\}$ is finite.
  Since T is a limit point of $\{\text{Th}(A) : A \in \mathbf{K}\}$, for each $i \in T$, there is a structure $A_i \in \mathbf{K}$, such that $A_i \vDash Z_i$.
  Define $B = \prod_I A_i / \mathscr{U}$.
  If $i \in X_\phi$ then $\phi \in Z_i$. So $A_i \vDash \phi$.
  Hence, $X_\phi \subseteq \|\phi\|$, for each sentence ϕ in T.
  By Łoś's Theorem, $B \vDash T$.
  So, T being a limit point, $T = \text{Th}(B)$.

Criterion for First-Order Axiomatizability

Corollary

Let L be a first-order language and $\mathbf{K}$ a class of L-structures. Then the following are equivalent:

(a) $\mathbf{K}$ is axiomatizable by a set of sentences of L.

(b) $\mathbf{K}$ is closed under ultraproducts and isomorphic copies, and if A is an L-structure such that some ultrapower of A lies in $\mathbf{K}$, then A is in $\mathbf{K}$.

(a)⇒(b) This follows from previous results.

(b)⇒(a) Suppose (b) holds. Let T be the set of all sentences of L which are true in every structure in $\mathbf{K}$. To prove (a) it suffices to show that any model A of T lies in $\mathbf{K}$.

Criterion for First-Order Axiomatizability (Cont'd)

Claim: $\mathrm{Th}(A)$ is a limit point of $\{\mathrm{Th}(C) : C \in \mathbf{K}\}$.

For this, let U be a finite set of sentences of L which are true in A.

Then $\bigwedge U$ is a sentence ϕ which is true in A.

Since A is a model of T, $\neg\phi \notin T$.

By the definition of T, some structure in $\mathbf{K}$ is a model of ϕ.

Thus, $\mathrm{Th}(A)$ is a limit point of $\{\mathrm{Th}(C) : C \in \mathbf{K}\}$.

By the preceding theorem, A is elementarily equivalent to some ultraproduct of structures in $\mathbf{K}$.

Hence, by (b), it is elementarily equivalent to some structure B in $\mathbf{K}$.

By the Keisler-Shelah Theorem, some ultrapower of A is isomorphic to an ultrapower of B.

So by (b) again, A is in $\mathbf{K}$.