Introduction to Probability

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1. Combinatorial Analysis
   - Introduction
   - The Basic Principle of Counting
   - Permutations
   - Combinations
   - Multinomial Coefficients

Subsection 1

Introduction

A Combinatorial Probability Problem

- A communication system is to consist of n seemingly identical antennas that are to be lined up in a linear order.
- The resulting system will then be able to receive all incoming signals - and will be called **functional** - as long as no two consecutive antennas are defective.
- If it turns out that exactly m of the n antennas are defective, what is the probability that the resulting system will be functional?
- Consider the special case where $n = 4$ and $m = 2$.
  There are 6 possible system configurations, namely:

  $$0110 \quad 0101 \quad 1010 \quad 0011 \quad 1001 \quad 1100$$

  where 1 means that the antenna is working and 0 that it is defective.
  - The resulting system will be functional in the first 3 arrangements;
  - It will not be functional in the remaining 3.

  It seems reasonable to take $\frac{3}{6} = \frac{1}{2}$ as the desired probability.

Subsection 2

The Basic Principle of Counting

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The Basic Principle of Counting

Suppose that two experiments are to be performed.

- Experiment 1 can result in any one of m possible outcomes;
- For each outcome of Experiment 1, there are n possible outcomes of Experiment 2.

Then together there are mn possible outcomes of the two experiments.

- Enumerate all the possible outcomes of the two experiments:

$$\begin{array}{cccc} (1,1), & (1,2), & \ldots, & (1,n) \\ (2,1), & (2,2), & \ldots, & (2,n) \\ \vdots & & & \\ (m,1), & (m,2), & \ldots, & (m,n) \end{array}$$

  where we say that the outcome is (i,j) if Experiment 1 results in its ith outcome and Experiment 2 then results in its jth outcome. Hence, the set of possible outcomes consists of mn elements.

Example

- A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

  Regard:
  - The choice of the woman as the outcome of the first experiment;
  - The subsequent choice of one of her children as the outcome of the second experiment.

  We see from the Basic Principle that there are

  $$10 \times 3 = 30 \text{ possible choices.}$$

The Generalized Basic Principle of Counting

The Generalized Basic Principle of Counting

Suppose r experiments are to be performed in such a way that:

- The first one may result in any of n_1 possible outcomes;
- For each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment;
- For each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment;
- $\cdots$

Then there is a total of $n_1 \cdot n_2 \cdot \cdots \cdot n_r$ possible outcomes of the r experiments.

Example

- A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen.

  How many different subcommittees are possible?

  We may regard the choice of a subcommittee as the combined outcome of the four separate experiments of choosing a single representative from each of the classes.
  - The freshman can be chosen in 3 ways;
  - The sophomore in 4 ways;
  - The junior in 5 ways;
  - The senior in 2 ways.

  It then follows from the Generalized Basic Principle that there are

  $$3 \times 4 \times 5 \times 2 = 120 \text{ possible subcommittees.}$$

Example

- How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

  Perform 7 sequential experiments:
  - The first has 26 possible outcomes;
  - For each, the second has 26 possible outcomes;
  - For each pair, the third has 26 possible outcomes;
  - For each triple, the fourth has 10 possible outcomes;

    $\vdots$
  - For each six-tuple, the seventh has 10 possible outcomes.

  By the Generalized Basic Principle, the answer is

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175,760,000.$$

Example

- How many functions defined on n points are possible if each functional value is either 0 or 1?

  Let the points be $1, 2, ..., n$.

  View the assignment of a value $f(i)$ to input i as the outcome of the i-th experiment:

  - The first experiment has two possible outcomes;
  - For each, the second experiment has two possible outcomes;

    $\vdots$
  - For each tuple, the n-th experiment has two possible outcomes.

  It follows that there are

$$\underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{n \text{ factors}} = 2^n \text{ possible functions.}$$

Example

- How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers and repetitions among letters or numbers are prohibited?

  Perform 7 sequential experiments:
  - The first has 26 possible outcomes;
  - For each, the second has 25 possible outcomes;
  - For each pair, the third has 24 possible outcomes;
  - For each triple, the fourth has 10 possible outcomes;
  - For each quadruple, the fifth has 9 possible outcomes;
  - For each quintuple, the sixth has 8 possible outcomes;
  - For each six-tuple, the seventh has 7 possible outcomes.

  By the Generalized Basic Principle, the answer is

  $$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 78,624,000$$

  possible license plates.

Subsection 3

Permutations

Permutations

- How many different ordered arrangements of the letters a, b and c are possible?

  By direct enumeration we see that there are 6, namely,

  $$abc, acb, bac, bca, cab, cba.$$

- Each arrangement is known as a **permutation**.
- There are 6 possible permutations of a set of 3 objects.
- This can be obtained from the Basic Principle:
  - The first object in the permutation can be any of the 3;
  - The second object can then be chosen from any of the remaining 2;
  - The third object in the permutation is then the remaining 1.

  Thus, there are $3 \cdot 2 \cdot 1 = 6$ possible permutations.
- If we have n objects a similar reasoning shows that there are

  $$n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = n!$$

  different permutations of the n objects.

Example

- How many different batting orders are possible for a baseball team consisting of 9 players?

  The different batting orders are the possible permutations of the set of 9 players.

  Thus, there are

  $$9! = 362,880 \text{ possible batting orders.}$$

Example

- A class in probability theory consists of 6 men and 4 women.

  An examination is given, and the students are ranked according to their performance.

  Assume that no two students obtain the same score.

  (a) How many different rankings are possible?
  (b) If the men are ranked just among themselves and the women just among themselves, how many different rankings are possible?

  (a) Each ranking corresponds to a particular ordered arrangement of the 10 people. So there are $10! = 3,628,800$ possible rankings.
  (b) View the ranking of the men as the first experiment and that of the women as the second experiment:
    - There are $6!$ possible rankings of the men among themselves;
    - For each, there are $4!$ possible rankings of the women among themselves.

    By the Basic Principle, there are $(6!)(4!) = (720)(24) = 17,280$ possible rankings.

Example

- Ms. Jones has 10 books that she is going to put on her bookshelf.

  Of these, 4 are math books, 3 are chemistry, 2 are history, and 1 is a language book.

  Ms. Jones wants to arrange her books so that all the books dealing with the same subject are together on the shelf.

  How many different arrangements are possible?

  We view the arrangement as a series of five experiments.

  - In the first, the math books are arranged: 4! possible outcomes;
  - In the second, the chemistry books are arranged: 3! possible outcomes;
  - In the third, the history books are arranged: 2! possible outcomes;
  - In the fourth, the language book(s) are arranged: 1! possible outcome;
  - In the fifth, the groups are arranged on the shelf: 4! possible outcomes.

  By the Generalized Basic Principle, there are $4!4!3!2!1! = 6,912$ possible arrangements.

Permutations With Some Indistinguishable Objects

- How many different letter arrangements can be formed from the letters PEPPER?

  Consider the letters P_1 E_1 P_2 P_3 E_2 R, when the 3 P's and the 2 E's are distinguished from each other. There are 6! permutations of these.

  However, consider any one of these permutations - for instance, $P_1P_2E_1P_3E_2R$. If we permute the P's among themselves and the E's among themselves, then the resultant arrangement would still be of the form PPEPER. The number of this identified permutations, all of the form PPEPER, is 3!2!.

  Hence, there are

  $$\frac{6!}{3!2!} = 60$$

  possible letter arrangements of the letters PEPPER.

Example

- The same reasoning shows that there are

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike,..., n_r are alike.

Example: A chess tournament has 10 competitors, of which 4 are Russian, 3 are from the United States, 2 are from Great Britain, and 1 is from Brazil.

If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?

There are

$$\frac{10!}{4!3!2!1!} = 12,600$$

possible outcomes.

Example

- How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags and 2 blue flags if all flags of the same color are identical?

  This is the number of permutations of 9 objects, split in three groups, such that:
  - Group 1 consists of 4 identical objets;
  - Group 2 consists of 3 identical objects;
  - Group 3 consists of 2 identical objects.

  Thus, there are
  $$\frac{9!}{4!3!2!} = 1,260$$
  different signals.

Subsection 4

Combinations

Example of Combinations

- How many different groups of 3 could be selected from the 5 items A, B, C, D and E?
  There are:
  - 5 ways to select the initial item;
  - 4 ways to then select the next item;
  - 3 ways to select the final item.

  So, there are $5 \cdot 4 \cdot 3$ ways of selecting the group of 3 when the order in which the items are selected is relevant.
  But every group of 3 - say, the group consisting of items A, B and C - will be counted 6 times (all of the permutations ABC, ACB, BAC, BCA, CAB and CBA will be counted when the order of selection is relevant).
  It follows that the total number of groups that can be formed is

$$\frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10.$$

Combinations

- In general, the number of different ways that a group of r items could be selected from n items when the order of selection is relevant is $n(n-1)\cdots(n-r+1)$.
- Each group of r items will be counted $r!$ times.
- It follows that the number of different groups of r items that could be formed from a set of n items is $\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!}$.

Notation and Terminology

We define $\binom{n}{r}$, for $r \leq n$, by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

and say that $\binom{n}{r}$ represents the number of possible **combinations** of n objects taken r at a time.

Example

- A committee of 3 is to be formed from a group of 20 people.

  How many different committees are possible?

  This is the number of combinations of 3 out of a group of 20 objects.

  So there are

$$\binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140$$

  possible committees.

Example

- Consider a group of 5 women and 7 men.
  (a) How many different committees consisting of 2 women and 3 men can be formed?
  (b) What if 2 of the men are feuding and refuse to serve on the committee together?

(a) View the formation of a committee as a series of two experiments.
- In the first a group of 2 women is chosen: $\binom{5}{2}$ possible outcomes;
- In the second a group of 3 men is chosen: $\binom{7}{3}$ possible outcomes.

By the Basic principle there are

$$\binom{5}{2}\binom{7}{3} = (\frac{5 \cdot 4}{2 \cdot 1})\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 350$$

possible committees consisting of 2 women and 3 men.

Example

(b) Now suppose that 2 of the men refuse to serve together.

How many committees of 2 women and 3 men are there which contain both feuding men?

View the formation as a series of two experiments:

- In the first, a group of 2 women is chosen: $\binom{5}{2}$ possible outcomes;
- In the second, the remaining male member is chosen: $\binom{5}{1}$ possible outcomes.

By the Basic Principle, there are $\binom{5}{2}\binom{5}{1} = 10 \cdot 5 = 50$ possible committees in which both feuding men are selected.

Since the total number of committees is 350, it follows that there are

$$350 - 50 = 300$$

committees that do not contain both of the feuding men.

Example

- Consider a set of n antennas of which m are defective and $n - m$ are functional. Assume that all of the defectives and all of the functionals are considered indistinguishable.

  How many linear orderings are there in which no two defectives are consecutive?

  Imagine that the $n - m$ functional antennas are lined up.

  If no two defectives are to be consecutive, then the spaces between the functionals must each contain at most one defective.

  That is, in the $n - m + 1$ possible positions - represented in $\_1\_1\_1 \ldots \_1\_1\_$ by spaces between the $n - m$ functional antennas, we must select m in which to put the defective antennas.

  Hence, there are $\binom{n-m+1}{m}$ possible orderings in which there is at least one functional antenna between any two defective ones.

A Combinatorial Identity

- A useful combinatorial identity is

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \quad 1 \leq r \leq n.$$

  Consider a group of n objects. Fix attention on some particular one of these objects - call it object 1.
  - The number of groups of size r containing object 1 is $\binom{n-1}{r-1}$ (each such group is formed by selecting $r-1$ from the remaining $n-1$ objects).
  - The number of groups of size r not containing object 1 is $\binom{n-1}{r}$.

  There are in total $\binom{n}{r}$ groups of size r.

  It follows that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.
- The values $\binom{n}{r}$ are often referred to as **binomial coefficients** because of their prominence in the binomial theorem.

The Binomial Theorem

The Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof by Induction: When $n = 1$, the equation reduces to

$$x + y = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 = y + x.$$

Assume the equation for $n-1$.

We obtain:

$$\begin{array}{rcl}
(x+y)^n & = & (x+y)(x+y)^{n-1} \\
 & = & (x+y)\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k} \\
 & = & \sum_{k=0}^{n-1}\binom{n-1}{k}x^{k+1}y^{n-1-k} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-k}.
\end{array}$$

The Binomial Theorem (Induction Cont'd)

- We got $(x+y)^n = \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k}$.
  Lett $i = k+1$ in the first sum and $i = k$ in the second sum.

$$\begin{array}{rcl} (x+y)^n & = & \sum_{i=1}^{n} \binom{n-1}{i-1} x^i y^{n-i} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\ & = & \sum_{i=1}^{n-1} \binom{n-1}{i-1} x^i y^{n-i} + x^n + y^n + \sum_{i=1}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\ & = & x^n + \sum_{i=1}^{n-1} [\binom{n-1}{i-1} + \binom{n-1}{i}] x^i y^{n-i} + y^n \\ & = & x^n + \sum_{i=1}^{n-i} \binom{n}{i} x^i y^{n-i} + y^n \\ & = & \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}, \end{array}$$

  where the next-to-last equality follows by the previous identity.
  By induction, the theorem is now proved.

The Binomial Theorem: Combinatorial Proof

- Consider the product

$$(x+y)^n = (x+y)(x+y)\cdots(x+y).$$

Let k be fixed, with $0 \leq k \leq n$.
In how many ways can we form a product

$$x^k y^{n-k}?$$

Each such term corresponds to a choice of a group of k from the n parentheses from which x will be chosen (and y from the remaining $n-k$).
Thus, there are $\binom{n}{k}$ such terms.
Since k can vary from 1 to n, we get

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Example

- Expand $(x+y)^3$.

$$\begin{aligned}(x+y)^3 &= \sum_{k=0}^{3}\binom{3}{k}x^k y^{3-k}\\ &= \binom{3}{0}x^0y^3+\binom{3}{1}x^1y^2+\binom{3}{2}x^2y+\binom{3}{3}x^3y^0\\ &= y^3+3xy^2+3x^2y+x^3.\end{aligned}$$

Example: Subsets of Set of n Elements

- How many subsets are there of a set consisting of n elements?

  The number of subsets of size k are $\binom{n}{k}$.

  Since k can vary from 0 to n, we get total number of subsets

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n.$$

- This result could also have been obtained by assigning either the number 0 or the number 1 to each element in the set.

  The number of subsets equals the number of assignments:

  > Each subset is paired to the assignment assigning 1 to its elements and 0 to all other elements.

  There are 2^n possible assignments.

  It follows that there are 2^n possible subsets.

Subsection 5

Multinomial Coefficients

Dividing Items Into Distinct Groups

- A set of n distinct items is to be divided into r distinct groups of respective sizes n_1, n_2, $\ldots$, n_r, where $\sum_{i=1}^r n_i = n$.
  How many different divisions are possible?
  Note that there are:
  - $\binom{n}{n_1}$ possible choices for the first group;
  - For each, there are $\binom{n-n_1}{n_2}$ possible choices for the second group;
  - For each, there are $\binom{n-n_1-n_2}{n_3}$ possible choices for the third group;

  $\vdots$

  From the Generalized Basic Principle, there are

$$\begin{aligned} & \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r} \\ = & \frac{n!}{(n-n_1)!n_1!}\frac{(n-n_1)!}{(n-n_1-n_2)!n_2!}\cdots\frac{(n-n_1-n_2-\cdots-n_{r-1})!}{0!n_r!} \\ = & \frac{n!}{n_1!n_2!\cdots n_r!} \end{aligned}$$

  possible divisions.

Alternative Proof

- Another way to see this result is to consider the n values $1, 1, \ldots, 1$, $2, \ldots, 2$, ..., $r, \ldots, r$, where i appears n_i times, for $i = 1, \ldots, r$. Every permutation of these values corresponds to a division of the n items into the r groups in the following manner:

  The permutation $i_1, i_2, \ldots, i_n$ correspond to assigning item 1 to group i_1, item 2 to group i_2, and so on.

  E.g., if $n = 8$ and $n_1 = 4$, $n_2 = 3$, $n_3 = 1$, then the permutation $1, 1, 2, 3, 2, 1, 2, 1$ corresponds to assigning: items $1, 2, 6, 8$ to the first group; items $3, 5, 7$ to the second; item 4 to the third group.
  Note that:
  - Every permutation yields a division of the items;
  - Every possible division results from some permutation.

  Hence, the number of divisions of n items into r distinct groups of sizes $n_1, n_2, \ldots, n_r$ is the same as the number of permutations of n items of which n_1 are alike, and n_2 are alike, ..., and n_r are alike. The latter was shown in a previous section to be $\frac{n!}{n_1! n_2! \cdots n_r!}$.

Notation

- If $n_1 + n_2 + \cdots + n_r = n$, we define $\binom{n}{n_1, n_2, \ldots, n_r}$ by

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}.$$

Thus,

$$\binom{n}{n_1, n_2, \ldots, n_r}$$

represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

Example

- A police department in a small city consists of 10 officers.

  If the department policy is to have:
  - 5 of the officers patrolling the streets;
  - 2 of the officers working full time at the station;
  - 3 of the officers on reserve at the station;

  how many different divisions of the 10 officers into the 3 groups are possible?

  This is tantamount to dividing a set of 10 distinct objects to three groups, each having 5, 2 and 3 members, respectively.

  Hence there are

$$\frac{10!}{5!2!3!} = 2520 \text{ possible divisions.}$$

Example

- Ten children are to be divided into an A team and a B team of 5 each.

  The A team will play in one league and the B team in another.

  How many different divisions are possible?

  We want to divide a group of 10 distinct objects into 2 distinct groups of 5 objects each.

  There are
  $$\frac{10!}{5!5!} = 252 \text{ possible divisions.}$$

Example

- In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each.

  How many different divisions are possible?

  This example is different from the preceding one because now the order of the two teams is irrelevant.

  That is, there is no A and B team, but just a division consisting of 2 groups of 5 each.

  Hence, the desired answer is

$$\frac{\frac{10!}{5!5!}}{2!} = \frac{10!}{5!5!2!} = 126.$$

The Multinomial Theorem

The Multinomial Theorem

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1,\ldots,n_r): \\ n_1+\cdots+n_r=n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}.$$

That is, the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$.

- The numbers $\binom{n}{n_1, n_2, \ldots, n_r}$ are known as **multinomial coefficients**.

Example

- In the first round of a knockout tournament involving $n = 2^m$ players, the n players are divided into $\frac{n}{2}$ pairs, with each of these pairs then playing a game.

  The losers of the games are eliminated while the winners go on to the next round, where the process is repeated until only a single player remains.

  Suppose we have a knockout tournament of 8 players.

  (a) How many possible outcomes are there for the initial round? (For instance, one outcome is that 1 beats 2, 3 beats 4, 5 beats 6, and 7 beats 8.)

  (b) How many outcomes of the tournament are possible, where an outcome gives complete information for all rounds?

(a) One way to determine the number of possible outcomes for the initial round is to first determine the number of possible pairings for that round.

Example (Cont'd)

- First, determine the number of possible pairings:
  - The number of ways to divide the 8 players into a first pair, a second pair, a third pair, and a fourth pair is $\binom{8}{2,2,2,2} = \frac{8!}{2^4}$;
  - The number of possible pairings when there is no ordering of the 4 pairs is $\frac{8!}{2^4 4!}$.
- For each such pairing, there are 2 possible choices from each pair as to the winner of that game.

  Thus, there are $\frac{8! 2^4}{2^4 4!} = \frac{8!}{4!}$ possible results of round 1.
- Another way to see this is to note that:
  - The number of possible choices of the 4 winners is $\binom{8}{4}$;
  - For each such choice, the number of ways to pair the 4 winners with the 4 losers is 4!.

  Hence, there are $4!\binom{8}{4} = \frac{8!}{4!}$ possible results for the first round.

Example (Cont'd)

(b) Similarly, for each result of round 1, there are:

- $\frac{4!}{2!}$ possible outcomes of round 2;
- For each of the outcomes of the first two rounds, there are $\frac{2!}{1!}$ possible outcomes of round 3.

Consequently, by the Generalized Basic Principle, there are

$$\frac{8!}{4!} \cdot \frac{4!}{2!} \cdot \frac{2!}{1!} = 8!$$

possible outcomes of the tournament.

- Indeed, the same argument can be used to show that a knockout tournament of $n = 2^m$ players has $n!$ possible outcomes.

Example (Tournament Outcomes & Permutations)

- There is a one-to-one correspondence between the set of possible tournament results and the set of permutations of $1, \ldots, n$.
- To obtain such a correspondence, rank the players as follows:
  - Give the tournament winner rank 1 and the final round loser rank 2.
  - For the two players who lost in the next-to-last round, give rank 3 to the one who lost to the player ranked 1 and give rank 4 to the one who lost to the player ranked 2.
  - For the four players who lost in the second-to-last round, give rank 5 to the one who lost to player ranked 1, rank 6 to the one who lost to the player ranked 2, rank 7 to the one who lost to the player ranked 3, and rank 8 to the one who lost to the player ranked 4.
  - Continuing on in this manner gives a rank to each player.
- In this manner, the result of the tournament can be represented by a permutation $i_1, i_2, \ldots, i_n$, where i_j is the player who was given rank j.
- This correspondence is a bijection.
- So the number of possible tournament results equals that of permutations of $1, \ldots, n$.

A Multinomial Expansion

- For $(x_1 + x_2 + x_3)^2$ we have:

$$\begin{aligned}(x_1 + x_2 + x_3)^2 &= \binom{2}{2,0,0} x_1^2 x_2^0 x_3^0 + \binom{2}{0,2,0} x_1^0 x_2^2 x_3^0 \\ &\quad + \binom{2}{0,0,2} x_1^0 x_2^0 x_3^2 \\ &\quad + \binom{2}{1,1,0} x_1^1 x_2^1 x_3^0 + \binom{2}{1,0,1} x_1^1 x_2^0 x_3^1 \\ &\quad + \binom{2}{0,1,1} x_1^0 x_2^1 x_3^1 \\ &= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3.\end{aligned}$$