Introduction to Projective Geometry

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Subsection 1

What is Projective Geometry

From Euclidean to Projective Geomery

- The plane geometry of the first six books of Euclid's Elements may be described as the geometry of lines and circles: Its tools are the straight-edge (unmarked ruler) and the compass.
- The Danish geometer Georg Mohr (1640-1697) and the Italian Lorenzo Mascheroni (1750-1800) proved independently that nothing is lost by discarding the straight-edge and using the compass alone.
- It is natural to ask how much remains if we discard the compass instead, and use the straight-edge alone.

  Is it possible to develop a geometry having no circles, no distances, no angles, no intermediacy (or "betweenness"), and no parallelism?

  The answer is Yes! What remains is projective geometry.
- The passage from axioms and "obvious" theorems to unexpected theorems resembles Euclid's work in spirit, but not in detail.

Features of Projective Geometry

- The geometry of the straight-edge seems at first to have very little connection with the familiar derivation of the name geometry as "earth measurement":
  - It deals with points, lines, and planes, but no attempt is ever made to measure the distance between two points or the angle between two lines.
  - It does not even admit the possibility that two lines in a plane might fail to meet by being "parallel".

The Building Objects

- We think of a **point** as "position without magnitude" or "an infinitesimal dot", represented in a diagram by a material dot only just big enough to be seen.
- By a **line** we mean a straight line of unlimited extent.

  Part of a line is reasonably well represented by a thin, tightly stretched thread, or a ray of light.
- A **plane** is a flat surface of unlimited extent, i.e., a surface that contains, for any two of its points, the whole of the line joining them.
- Any number of points that lie on a line are said to be **collinear**.
- Any number of lines that pass through a point are said to be **concurrent**.
- Any number of points or lines (or both) that lie in a plane are said to be **coplanar**.

Parallelism

- In Euclidean geometry two coplanar lines with a common perpendicular are parallel, in the sense that, however far we extend them, they will remain the same distance apart.
- By assuming that two coplanar lines always meet, we obtain a system of propositions which is just as logically consistent as Euclid's different system.
- From the point of view of logic, we are free to make any assumptions we please, so long as they are consistent and serve some useful purpose.

Subsection 2

Historical Remarks

Lines and Circles in Perspective

- In 1425 the Italian architect Brunelleschi began to discuss the geometrical theory of perspective.

  This was consolidated into a treatise by Alberti a few years later.
- Plane projective geometry may be described as the study of geometrical properties that are unchanged by "central projection":
  - When an artist draws a picture of a tiled floor on a vertical canvas, the square tiles cease to be square, as their sides and angles are distorted by foreshortening.
    Thus, projective geometry deals with triangles, quadrangles, and so on, but not with right-angled triangles, parallelograms, and so on.
  - When a lamp casts a shadow on a wall or on the floor, the circular rim of a lampshade usually casts a large circular or elliptic shadow on the floor and a hyperbolic shadow on the nearest wall.
    Thus projective geometry waives the customary distinction between a circle, an ellipse, a parabola, and a hyperbola; these curves are simply conics, all alike.

A Historical Timeline

- Conics were studied by Menaechmus, Euclid, Archimedes and Apollonius, in the fourth and third centuries B.C.
- The earliest truly projective theorems were discovered by Pappus of Alexandria in the third century A.D.
- J. V. Poncelet (1788-1867) first proved such theorems by purely projective reasoning.
- The important concept of a point at infinity occurred independently to the German astronomer Johann Kepler (1571-1630) and the French architect Girard Desargues (1591-1661).
- Incorporation of points at infinity on a par status with ordinary points was carried out by von Staudt (1798-1867).
- In 1871, Felix Klein provided an algebraic foundation for projective geometry in terms of "homogeneous coordinates", which had been discovered independently by Feuerbach and Möbius in 1827.

Duality Principle

- The determination of a point by two lines nicely balances the determination of a line by two points.
- We will find that every statement about points and lines (in a plane) can be replaced by a dual statement about lines and points.
- The possibility of making such a replacement is known as the "principle of duality".

  Poncelet claimed this principle as his own discovery;

  Its nature was more clearly understood by another Frenchman, J. D. Gergonne (1771-1859).
- Duality gives projective geometry a peculiar charm, making it more symmetrical than ordinary (Euclidean) geometry.

Euclidean and Non-Euclidean Geometry

- Projective geometry is useful as supplying a fresh approach to Euclidean geometry.
- In the theory of conics, a single projective theorem may yield several Euclidean theorems by different choices of the line at infinity.

  E.g., if the line at infinity is a tangent or a secant, the conic is a parabola or a hyperbola, respectively.
- Arthur Cayley (1821-1895) and Felix Klein (1849-1925) noticed that projective geometry is equally powerful in its application to non-Euclidean geometries.

  Cayley said:

  > Metrical geometry is a part of descriptive geometry, and descriptive geometry is all geometry.
  > (Cayley used "descriptive" where today we would say "projective".)

Subsection 3

Definitions

Incidence

- We regard:
  - a line as a certain set of points;
  - a plane as a certain set of points and lines.
- A point and a line, or a point and a plane, or a line and a plane, are said to be **incident** if the former belongs to the latter.
- We also say that the former **lies on** (or **in**) the latter, and that the latter **passes through** the former.
- We shall consistently use:
  - capital letters for points;
  - small (lower case) letters for lines;
  - Greek letters for planes.

Join and Meet

- If a line ℓ passes through two points P and Q, we say that it **joins** them and write $\ell = PQ$.
- If a plane α passes through two lines ℓ and m, or through ℓ and a nonincident point P, we say that α **joins** the two lines, or the line and point, and write $\alpha = \ell m = m\ell = \ell P = P\ell$.
- If P lies on both ℓ and m, we say that these lines **meet** in P, or that P is their **common point** (or "**intersection**"): $P = \ell \cdot m$.

  Note ℓm is a plane, but $\ell \cdot m$ is a point.
- Similarly:
  - A line and a plane may have a **common point** $\ell \cdot \alpha$;
  - Two planes may have a **common line** $\alpha \cdot \beta$.

Primitive Concepts and Axioms

- Any definition of a word must inevitably involve other words, which require further definitions.
- The only way to avoid a vicious circle is to regard certain **primitive concepts** as being so simple and obvious as to be left undefined.
- Similarly, the proof of any statement uses other statements.
- Since we must begin somewhere, we agree to leave a few simple statements, called **axioms**, unproved.

Subsection 4

The Simplest Geometric Objects

Primitive Concepts, Planes and Triangles

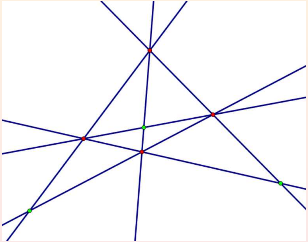
- We use three primitive concepts: **point**, **line**, and **incidence**.
- In terms of these we can easily define "lie on", "pass through", "join", "meet", "collinear", "concurrent", and so on.
- If a point P and a line ℓ are not incident, the **plane** $P\ell$ may be taken to consist of:
  - all the points that lie on lines joining P to points on ℓ;
  - all the lines that join pairs of distinct points so constructed.
- A **triangle** PQR consists of:
  - three noncollinear points P, Q, R, called its **vertices**;
  - the three joining lines QR, RP, PQ, called its **sides**.

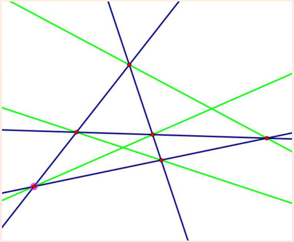
  Thus, if 3 points are joined in pairs by 3 lines, they form a triangle, which is equally well formed by 3 lines meeting by pairs in 3 points.

Complete Quadrangles and Complete Quadrilaterals

If 4 points in a plane are joined in pairs by 6 distinct lines, they are called the **vertices** of a **complete quadrangle** and the lines are its 6 sides. Two sides are said to be **opposite** if their common point is not a vertex. The common point of two opposite sides is called a **diagonal point**. There are 3 diagonal points.

If 4 lines in a plane meet by pairs in 6 distinct points, they are called the **sides** of a **complete quadrilateral**, and the points are its 6 vertices. Two vertices are said to be **opposite** if their join is not a side. The join of two opposite vertices is called a **diagonal line**. There are 3 diagonal lines.

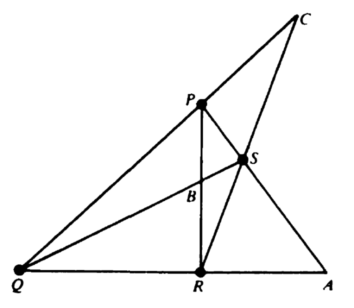


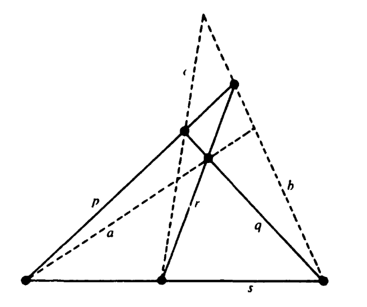


Diagrams of Quadrangles and Quadrilaterals

In the figure, the quadrangle is $PQRS$, its sides are PS, QS, RS, QR, RP, PQ, and its diagonal points are A, B, C.

In the figure the quadrilateral is $pqrs$, its vertices are $p \cdot s$, $q \cdot s$, $r \cdot s$, $q \cdot r$, $r \cdot p$, $p \cdot q$, and its diagonal lines are a, b, c.





Often, we speak simply of **quadrangles** and **quadrilaterals**, omitting the word "complete".

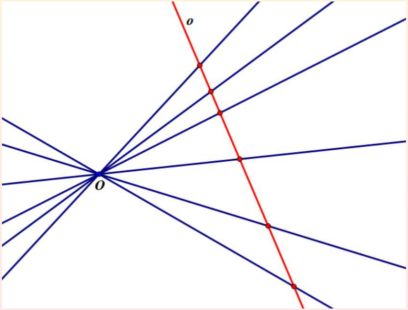
Subsection 5

Projectivities

Ranges and Pencils

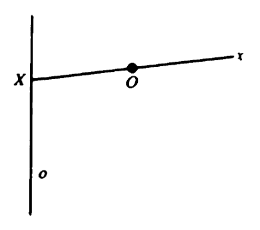
- The set of all points on a line is called a **range**.
- The set of all lines that lie in a plane and pass through a point is called a **pencil**.
- Ranges and pencils are instances of **one-dimensional forms**.
- Suppose that the line o on which the points of a range lie is not incident with the point O through which the lines of a pencil pass.

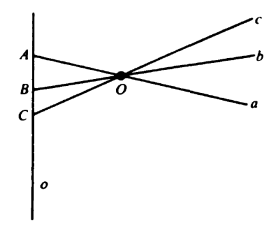
  Then:
  - The range is a **section** of the pencil (the section by the line o);
  - The pencil **projects** the range (from the point O).



Correspondence Between Ranges and Pencils

- This **elementary correspondence** is written either $X \,\overline{\wedge}\, x$, where X is a variable point of the range and x is the corresponding line of the pencil, or $ABC\cdots \,\overline{\wedge}\, abc\cdots$, where $A, B, C, \ldots$ are particular positions of X and $a, b, c, \ldots$ are the corresponding positions of x.





  - The order in which the symbols for the points or lines are written does not necessarily agree with the order in which they occur in the range or pencil.
  - Corresponding symbols are placed in corresponding positions, but the statement $ABC\cdots \,\overline{\wedge}\, abc\cdots$ has the same meaning as $BAC\cdots \,\overline{\wedge}\, bac\cdots$.

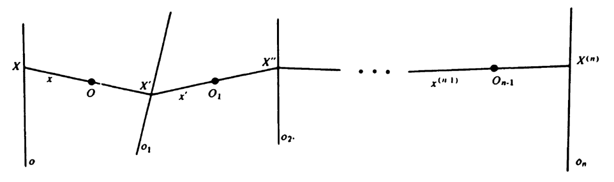
Inversion of Correspondences

- Since the statement $X \overline{\wedge} x$ means that X and x are incident, we can also write $x \overline{\wedge} X$.
- It is convenient to make we make a distinction:
  - The correspondence $X \overline{\wedge} x$ is directed "from X to x": it **transforms** X into x;
  - The **inverse** correspondence $x \overline{\wedge} X$ transforms x into X.

Projectivities

- We may use a sequence of lines and points occurring alternately:

$$o, O, o_1, O_1, o_2, \ldots, O_{n-1}, o_n, O_n.$$



We allow the sequence to begin with a point or to end with a line, but we insist that:
  - Adjacent members shall be nonincident;
  - Alternate members shall be distinct.
- This arrangement establishes a transformation relating the range of points X on o (or the pencil of lines x through O) to the pencil of lines $x^{(n)}$ through O_n (or the range of points $X^{(n)}$ on o_n).
- We call such a transformation a **projectivity**.

Extension of the $\overline{\wedge}$-Notation

- Instead of
$$X \,\overline{\wedge}\, x \,\overline{\wedge}\, X' \,\overline{\wedge}\, x' \,\overline{\wedge}\, X'' \,\overline{\wedge} \cdots \overline{\wedge}\, X^{(n)} \,\overline{\wedge}\, x^{(n)},$$
we write simply $X \,\overline{\wedge}\, x^{(n)}$ or $x \,\overline{\wedge}\, x^{(n)}$ or $x \,\overline{\wedge}\, X^{(n)}$ or $X \,\overline{\wedge}\, X^{(n)}$.
- In other words, we extend the meaning of the sign $\overline{\wedge}$ from an elementary correspondence to the **product** (or "**resultant**") of any number of elementary correspondences.

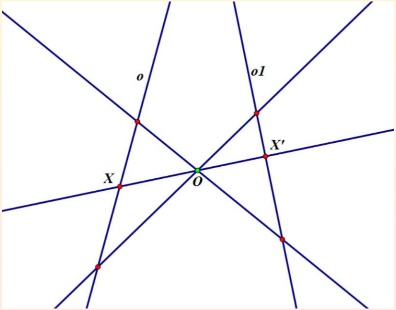
Subsection 6

Perspectivities

Perspectivities of Ranges

- The product of two elementary correspondences is called a **perspectivity** and is indicated by the sign $\overline{\wedge}$:

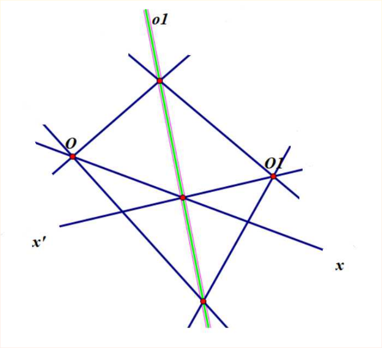
Two ranges are related by a perspectivity with **center** O if they are sections of one pencil (consisting of all the lines through O) by two distinct lines o and o_1; that is, if the join XX' of corresponding points continually passes through the point O.



In symbols: $X \overline{\wedge} X'$ or $X \overset{O}{\overline{\wedge}} X'$.

Perspectivities of Pencils

- The product of two elementary correspondences is called a **perspectivity** and is indicated by the sign $\overline{\overline{\wedge}}$:

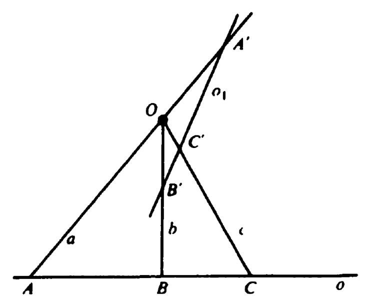


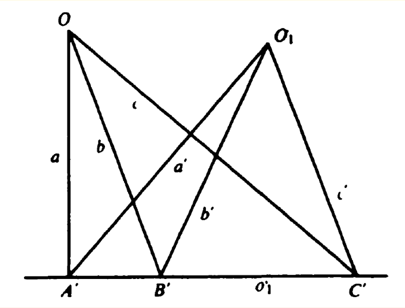
Two pencils are related by a perspectivity with **axis** o_1 if they project one range (consisting of all the points on o_1) from two distinct points O and O_1; that is, if the intersection $x \cdot x'$ of corresponding lines continually lies on the line o_1.

In symbols: $x \overline{\overline{\wedge}} x'$ or $x \overset{o_1}{\overline{\overline{\wedge}}} x'$.

Example

- In the figures (where A, B, C are particular instances of the variable point X, and a, b, c of the variable line x),

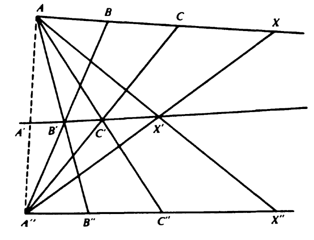




we have the perspectivities $ABC \overset{O}{\doublebarwedge} A'B'C'$ and $abc \overset{o_1}{\doublebarwedge} a'b'c'$.
These can be analyzed in terms of elementary correspondences:
  - $ABC \barwedge abc \barwedge A'B'C'$;
  - $abc \barwedge A'B'C' \barwedge a'b'c'$.

Projectivity Determined by Three Points

- Given three distinct points A, B, C on a line, and three distinct points A'', B'', C'' on another line, we can set up two perspectivities whose product has the effect $ABC \barwedge A''B''C''$: The **axis** (or "**intermediary line**") of the projectivity joins the points $B' = AB'' \cdot BA''$, $C' = AC'' \cdot CA''$,



so that, if $A' = AA'' \cdot B'C'$, $ABC \overset{A''}{\barwedge} A'B'C' \overset{A}{\barwedge} A''B''C''$.

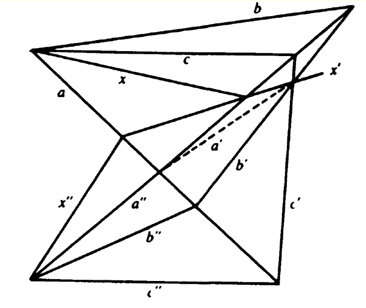
For each point X on AB, we can construct a corresponding point X'' on $A''B''$ by joining A to the point $X' = A''X \cdot B'C'$, so that

$$ABCX \overset{A''}{\barwedge} A'B'C'X' \overset{A}{\barwedge} A''B''C''X''.$$

- We will see that this projectivity $ABC \barwedge A''B''C''$ is unique, in the sense that any sequence of perspectivities relating ABC to $A''B''C''$ will have the same effect on X.

Projectivity Determined by Three Lines

- Interchanging points and lines, we obtain an analogous construction for the projectivity $abc \barwedge a''b''c''$, where a, b, c are three distinct lines through a point and a'', b'', c'' are three distinct lines through another point.



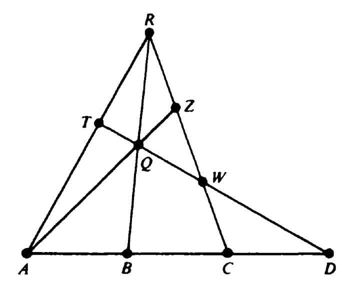
Set: $b' = (a \cdot b'')(a'' \cdot b)$; $c' = (a \cdot c'')(a'' \cdot c)$; $a' = (a \cdot a'')(b' \cdot c')$.

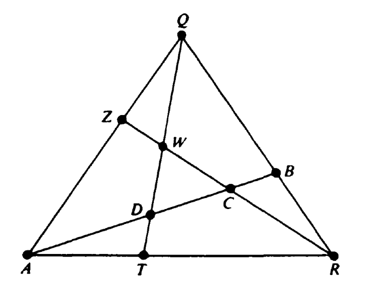
Then, we get

$$abc \overset{a''}{\barwedge} a'b'c' \overset{a}{\barwedge} a''b''c''.$$

Interchanging Points in Pairs by a Projectivity

- Suppose A, B, C, D are any four collinear points, R is a point outside their line, T, Q, W are the sections of RA, RB, RC by an arbitrary line through D, and Z is the point $AQ \cdot RC$.



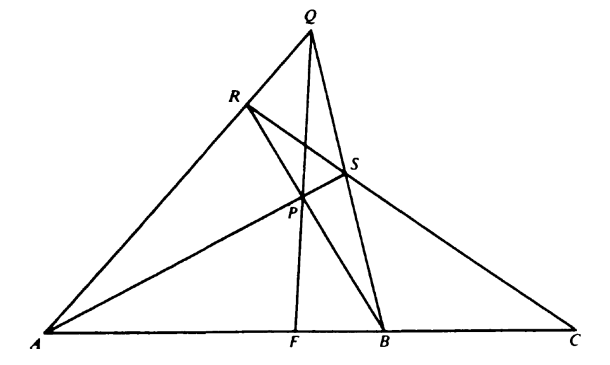


Then $ABCD \overset{Q}{\overline{\overline{\wedge}}} ZRCW \overset{A}{\overline{\overline{\wedge}}} QTDW \overset{R}{\overline{\overline{\wedge}}} BADC$. Hence, $ABCD \,\overline{\wedge}\, BADC$.

Theorem

Any four collinear points can be interchanged in pairs by a projectivity.

Invariant Points



$$ABC \overset{R}{\overline{\overline{\wedge}}} APS \overset{Q}{\overline{\overline{\wedge}}} AFB, \qquad ABC \overset{S}{\overline{\overline{\wedge}}} AQR \overset{P}{\overline{\overline{\wedge}}} AFB.$$

In this projectivity $ABC \,\overline{\wedge}\, AFB$, the point A corresponds to itself.

A point that corresponds to itself is said to be **invariant**.

Comments on the Notation

- The idea of a projectivity is due to Poncelet.
- Its analysis into elementary correspondences was suggested by Mathews.
- The sign $\overline{\wedge}$ was invented by von Staudt.
- For the special case of a perspectivity, the sign $\overset{=}{\wedge}$ was adopted by the great American geometer Oswald Veblen (1880-1960).