# Introduction to Projective Geometry

**George Voutsadakis**[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 400

## Subsection 1

# The Idea of a Finite Geometry

# The Geometry PG($n, q$)

- Abandoning the "intuitive" idea that the number of points is infinite, we find that all our theorems remain valid (although the figures are somewhat misleading).
- In 1892, Fano described an $n$-dimensional geometry in which the number of points on each line is $p+1$, for a fixed prime $p$.
- In 1906, Veblen and Bussey gave this finite Projective Geometry the name PG($n, p$) and extended it to PG($n, q$), where $q = p^k$, $p$ is prime, and $k$ is any positive integer.

# The Number $q$

- Any range or pencil can be related to any other by a sequence of elementary correspondences:
  - The number of points on a line must be the same for all lines;
  - The number of points on a line must be the same as the number of lines in a pencil;
  - The number of points on a line must be the same as the number of planes through a line.

  We call this number $q+1$.

# von Staudt's Formula for Number of Points in PG$(n,q)$

- In a plane, any point is joined to the remaining points by a pencil consisting of $q+1$ lines, each containing the one point and $q$ others.

  Hence, the plane contains $q(q+1)+1=q^2+q+1$ points and (dually) the same number of lines.
- In space, any line $\ell$ is joined to the points outside $\ell$ by $q+1$ planes, each containing the $q+1$ points on $\ell$ and $q^2$ others.

  Hence the whole space contains $(q+1)(q^2+1)=q^3+q^2+q+1$ points and (dually) the same number of planes.
- The general formula for the number of points in PG$(n,q)$ is

$$q^n+q^{n-1}+\cdots+q+1=\frac{q^{n+1}-1}{q-1}.$$

Subsection 2

## A Combinatorial Scheme for PG(2,5)

# The Projective Geometry PG(2,5)

- The finite projective plane PG(2,5) has:
  - 6 points on each line;
  - 6 lines through each point;
  - a total of $5^2+5+1=\frac{5^3-1}{5-1}=31$ points;
  - 31 lines.
- The appropriate scheme uses symbols $P_0,P_1,\ldots,P_{30}$ for the 31 points, and $\ell_0,\ell_1,\ldots,\ell_{30}$ for the 31 lines, with a table telling us which are the 6 points on each line and which are the 6 lines through each point:

**Table of possible values of $s$, given $r$, such that $P_r$ and $l_s$ (or $l_r$ and $P_s$) are incident**

| *r* | 30 | 29 | 28 | 27 | 26 | 25 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 |
| *s* | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 |
| | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

# A Polarity $P_r \leftrightarrow \ell_r$

- For good measure, this table gives every relation of incidence twice:

**Table of possible values of $s$, given $r$, such that $P_r$ and $l_s$ (or $l_r$ and $P_s$) are incident**

| $r$ | 30 | 29 | 28 | 27 | 26 | 25 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 |
| $s$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 |
| | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

Each column tells us which points lie on a line and also which lines pass through a point.

Example: The last column says that the line $\ell_0$ contains the six points $P_0, P_1, P_3, P_8, P_{12}, P_{18}$ and that the point $P_0$ belongs to the six lines $\ell_0, \ell_1, \ell_3, \ell_8, \ell_{12}, \ell_{18}$.

Thus the notation exhibits a polarity $P_r \leftrightarrow \ell_r$.

# Using Congruences to Express Incidence

- By regarding the subscripts as residues modulo 31, so that $r+31$ has the same significance as $r$ itself, we can condense the whole table into the simple statement that the point $P_r$, and line $\ell_s$, are incident if and only if

$$r+s \equiv 0,\ 1,\ 3,\ 8,\ 12,\ \text{or } 18 \pmod{31}.$$

- The **congruence** $a \equiv b \pmod{n}$ is a convenient abbreviation for the statement that $a$ and $b$ leave the same remainder (or "residue") when divided by $n$.
- The residues $0,1,3,8,12,18 \pmod{31}$ are said to form a **perfect difference set** because every possible residue except zero (namely, $1,2,3,\ldots,30$) is uniquely expressible as the difference between two of these special residues:

$$1 \equiv 1-0,\ 2 \equiv 3-1,\ 3 \equiv 3-0,\ 4 \equiv 12-8,\ldots,$$
$$13 \equiv 0-18,\ldots,\ 30 \equiv 0-1.$$

Subsection 3

## Verifying the Axioms

# Axioms of Two-Dimensional Projective Geometry

- The following five axioms suffice for the development of two-dimensional projective geometry:

  **Axiom 3** Any two distinct points are incident with just one line.

  **New Axiom 1** Any two lines are incident with at least one point.

  **New Axiom 2** There exist four points of which no three are collinear.

  **Axiom 7** The three diagonal points of a quadrangle are never collinear.

  **Axiom 8** If a projectivity leaves invariant each of three distinct points on a line, it leaves invariant every point on the line.

# Logical Consistency of the Axioms

Claim: This is a logically consistent geometry.

We verify that all the axioms are satisfied in PG(2,5).

To verify Axiom 3 and New Axiom 1, we observe that:

Table of possible values of $s$, given $r$, such that $P_r$ and $l_s$ (or $l_r$ and $P_s$) are incident

| *r* | 30 | 29 | 28 | 27 | 26 | 25 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 |
| *s* | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 |
| | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

- Any two residues are found together in just one column of the table.
- Any two columns contain just one common number.

For New Axiom 2, we can cite $P_0P_1P_2P_5$.

# Logical Consistency of the Axioms (Cont'd)

- To check Axiom 7, for every complete quadrangle (or rather, for every one having $P_0$ for a vertex) is possible but tedious.

Table of possible values of $s$, given $r$, such that $P_r$ and $l_s$ (or $l_r$ and $P_s$) are incident

| $r$ | 30 | 29 | 28 | 27 | 26 | 25 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 |
| $s$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 |
| | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

We illustrate with a single instance: Take $P_0P_1P_2P_9$.

Its diagonal points are $\ell_0 \cdot \ell_{29} = P_3$, $\ell_1 \cdot \ell_7 = P_{11}$, $\ell_3 \cdot \ell_{30} = P_9$.

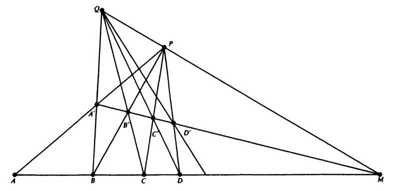
# Logical Consistency of the Axioms (Cont'd)

- Axiom 8 is superseded by

  If a projectivity leaves invariant each of three distinct points $A, B, C$ on a line, it leaves invariant every point of the harmonic net $R(ABC)$.

  because a harmonic net fills the whole line.

  In fact, the harmonic net $R(P_0P_1P_{18})$ contains the harmonic sequence $P_0P_1P_3P_{12}P_8\cdots$. To verify this, we use the procedure in the figure taking $A, B, M, P, Q$ to be $P_0$, $P_1$, $P_{18}$, $P_5$, $P_{30}$, so that $C = P_3$, $D = P_{12}$, $E = P_8$, $F = P_0 = A$. Since there are only six points on the line, the sequence is inevitably periodic:



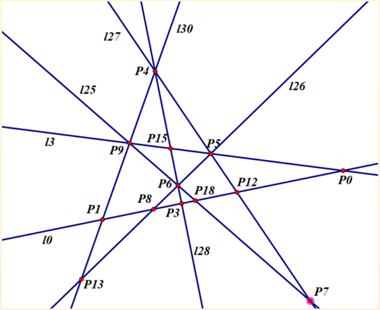

  The five points $P_0, P_1, P_3, P_{12}, P_8$ are repeated cyclically for ever. Instead of taking $P$ and $Q$ to be $P_5$ and $P_{30}$, we could just as well have taken them to be any other pair of points on $\ell_{13}$ or $\ell_{14}$ or $\ell_{18}$ or $\ell_{21}$ or $\ell_{25}$ (these being, with $\ell_0$, the lines through $P_{18}$). We would still have obtained the same harmonic sequence.

Subsection 4

# Involutions

# Projectivities

- The sections of the quadrangles $P_4P_5P_6P_9$, $P_{14}P_{15}P_{16}P_{19}$, $P_9P_{10}P_{11}P_{14}$ by the line $\ell_0$ yield the quadrangular and harmonic relations $(P_1P_8)(P_0P_3)(P_{18}P_{12})$, $(P_{12}P_{18})(P_8P_0)(P_3P_1)$, $\mathrm{H}(P_{12}P_{18}, P_3P_8)$.




The fundamental theorem shows that every projectivity on $\ell_0$ is expressible in the form $P_0P_1P_3 \barwedge P_iP_jP_k$, where $i, j, k$ are any three distinct numbers selected from 0, 1, 3, 8, 12, 18.

Hence there are just $6 \cdot 5 \cdot 4 = 120$ projectivities (including the identity).

# Classification of the Projectivities

- Of the 120 projectivities, 25 are involutions; 15 hyperbolic and 10 elliptic:

  Suppose $i$ and $j$ are any two of the six numbers.
  - Then, since an involution is determined by any two of its pairs, there is a hyperbolic involution $(P_iP_i)(P_jP_j)$ which interchanges the remaining four numbers in pairs in a definite way.
  - Since involutions are either hyperbolic or elliptic, the other two possible ways of pairing the remaining four numbers must each determine an elliptic involution which interchanges $P_i$ and $P_j$.

  Example: The hyperbolic involution $(P_{12}P_{12})(P_{18}P_{18})$, interchanging $P_3$ and $P_8$, must also interchange $P_0$ and $P_1$, and is expressible as $(P_0P_1)(P_3P_8)$;

  On the other hand, both the involutions $(P_1P_8)(P_0P_3)$, $(P_0P_8)(P_1P_3)$ interchange $P_{12}$ and $P_{18}$, and are therefore elliptic.

Subsection 5

# Collineations and Correlations

# Projective Collineations and Projective Correlations

- By previous results, every projective collineation or projective correlation is determined by its effect on a particular quadrangle, such as $P_0P_1P_2P_6$.
  - The collineation may transform $P_0$ into any one of the 31 points, and $P_1$ into any one of the remaining 30.
  - It may transform $P_2$ into any one of the $31-6=25$ points not collinear with the first two.
  - The number of points that lie on at least one side of a given triangle is evidently $3+(3\cdot 4)=15$; therefore the number not on any side is 16.

  Hence PG(2,5) admits altogether $31\cdot 30\cdot 25\cdot 16=372000$ projective collineations, and the same number of projective correlations.

# Examples

- Of the 372000 projective collineations, 775 are of period 2.

  By a previous result, the number of harmonic homologies is $31 \cdot 25 = 775$.
- Apart from the identity, the two most obvious collineations are:
  - $P_r \to P_{5r}$ (of period 3, since $5^3 \equiv 1 \pmod{31}$);
  - $P_r \to P_{r+1}$ (of period 31).
- The appropriate criterion (any collineation that transforms one range projectively is a projective collineation) assures us they are projective:
  - The corresponding ranges of the former on $P_0P_1$ and $P_0P_5$ are related by the perspectivity $P_0P_1P_3P_8P_{12}P_{18} \overset{P_{11}}{\overline{\overline{\wedge}}} P_0P_5P_{15}P_9P_{29}P_{28}$;
  - The corresponding ranges of the latter on $P_0P_1$, and $P_1P_2$ are related by a projectivity with axis $P_0P_2$:

$$P_0P_1P_3P_8P_{12}P_{18} \overset{P_9}{\overline{\overline{\wedge}}} P_0P_2P_{30}P_{11}P_{17}P_7 \overset{P_8}{\overline{\overline{\wedge}}} P_1P_2P_4P_9P_{13}P_{19}.$$

Subsection 6

# Conics

# From Polarities to Conics

- The most obvious correlation is, of course, $P_r \to \ell_r$.

  To verify that it is projective, we use a preceding result in the form

  $$P_1P_2P_4P_9P_{13}P_{19} \overset{P_8}{\overline{\overline{\wedge}}} P_0P_{29}P_{28}P_9P_5P_{15} \overline{\wedge} \ell_1\ell_2\ell_4\ell_9\ell_{13}\ell_{19}.$$

  Being of period 2, it is a polarity.

  Since $P_0$ lies on $\ell_0$, it is a hyperbolic polarity, and determines a conic.
- By Steiner's Construction, we see that the number of points on a conic (in any finite projective plane) is equal to the number of lines through a point, in the present case 6.

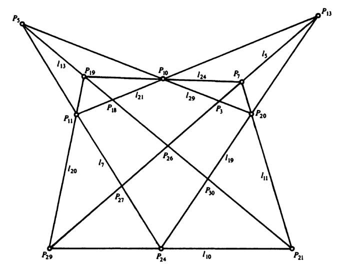
  By inspecting the incidence table, or by halving the residues 0, 1, 3, 8, 12, 18 (mod 31), we see that the conic determined by the polarity $P_r \leftrightarrow \ell_r$ consists of the 6 points and 6 lines $P_0P_4P_6P_9P_{19}P_{17}$, $\ell_0\ell_4\ell_6\ell_9\ell_{16}\ell_{17}$.

# Tangents, Secants and Nonsecants

- The 6 lines $\ell_0\ell_4\ell_6\ell_9\ell_{16}\ell_{17}$ are the tangents;
- By joining the 6 points $P_0P_4P_6P_9P_{19}P_{17}$ in pairs, we obtain the $\binom{6}{2} = 15$ secants

$$\begin{array}{lllll}
\ell_1 = P_0P_{17}, & \ell_2 = P_6P_{16}, & \ell_3 = P_0P_9, & \ell_8 = P_0P_4, & \ell_{12} = P_0P_6, \\
\ell_{14} = P_4P_{17}, & \ell_{15} = P_{16}P_{17}, & \ell_{18} = P_0P_{16}, & \ell_{22} = P_9P_{17}, & \ell_{23} = P_9P_{16}, \\
\ell_{25} = P_6P_9, & \ell_{26} = P_6P_{17}, & \ell_{27} = P_4P_{16}, & \ell_{28} = P_4P_6, & \ell_{30} = P_4P_9.
\end{array}$$

- It follows that the remaining 10 lines $\ell_5$, $\ell_7$, $\ell_{10}$, $\ell_{11}$, $\ell_{13}$, $\ell_{19}$, $\ell_{20}$, $\ell_{21}$, $\ell_{24}$, $\ell_{29}$ are nonsecants, each containing an elliptic involution of conjugate points.



# Self-Polar Triangles

- Any two conjugate points on a secant or nonsecant determine a self-polar triangle.
- For instance, the secant $\ell_1$, containing the hyperbolic involution $(P_0P_0)(P_{17}P_{17})$ or $(P_2P_{30})(P_7P_{11})$, is a common side of the two self-polar triangles $P_1P_2P_{30}$, $P_1P_7P_{11}$.
  These two triangles are of different types:
  - Of the former, all three sides $\ell_1, \ell_2, \ell_{30}$ are secants;
  - The sides $\ell_7$ and $\ell_{11}$ of the latter are nonsecants.

  We speak of triangles of the **first type** and **second type**, respectively.
- Since each of the 15 secants belongs to one self-polar triangle of either type, there are altogether
  - 5 triangles of the first type;
  - 15 triangles of the second type.
- These properties of a conic are amusingly different from what happens in real geometry, where the sides of a self-polar triangle always consist of two secants and one nonsecant.

# Polarities Through a Triangle a Polar Pair

- There are, of course, many ways to express a given polarity by a symbol of the form $(ABC)(Pp)$;
- For example, the polarity $P_r \leftrightarrow \ell_r$ is $(P_1P_2P_{30})(P_3\ell_3)$ or $(P_1P_7P_{11})(P_3\ell_3)$ or $(P_1P_7P_{11})(P_4\ell_4)$.
- Such symbols will enable us to find the total number of polarities:
  - If $ABC$ is given, there are:
    - 16 possible choices for $P$ (not on any side);
    - 16 possible choices for $p$ (not through a vertex);

    So there are $16^2 = 256$ available symbols $(ABC)(Pp)$ for polarities in which ABC is self-polar.
- Of the 16 lines, each contains 3 of the 16 points.

  Thus, just 48 of the 256 symbols have $P$ lying on $p$, as in the case of $(P_1P_7P_{11})(P_4\ell_4)$.

# All Polarities in PG(2,5) are Hyperbolic

**Theorem**

There are no elliptic polarities in PG(2,5).

- Suppose that the self-polar triangle is of the first type (with every side a secant).

  Then, all the six points on the conic are on sides of the triangle. Hence, $P$ never lies on $p$. Each hyperbolic polarity (with $ABC$ of this type) is named 16 times by a symbol $(ABC)(Pp)$ with $P$ not on $p$.
- Suppose only one side is a secant.

  Then 2 of the 6 points are on this side and the remaining 4 are among the 16. Therefore each hyperbolic polarity (with $ABC$ of the second type) is named:
  - 4 times with $P$ on $p$;
  - 12 times with $P$ not on $p$.

# All Polarities in PG(2,5) are Hyperbolic (Cont'd)

- Conversely, if $P$ lies on $p$, $ABC$ can only be of the second type.

  Therefore the number of such hyperbolic polarities (each accounting for 4 of the 48 symbols) is 12.

  Since each hyperbolic polarity (or conic) has 5 self-polar triangles of the first type and 15 of the second, the number of hyperbolic polarities in which a given triangle $ABC$ is of the first type is one-third of 12, that is, 4.

  The total number of symbols $(ABC)(Pp)$ that denote hyperbolic polarities is thus

$$\underbrace{48}_{P \text{ on } p} + \underbrace{16}_{P \text{ not on } p} \cdot \underbrace{4}_{ABC \text{ of type I}} + \underbrace{12}_{P \text{ not on } p} \cdot \underbrace{12}_{ABC \text{ of type II}} = 256.$$

  Since we have accounted for all the available symbols, there are no elliptic polarities in PG(2,5).

# Number of Triangles and Conics

- The total number of triangles in PG(2,5) can be found as follows:
  - There are 31 choices for the first vertex;
  - There are 30 choices for the second vertex;
  - There are $31-6=25$ choices for the third vertex;

  The three vertices can be permuted in $3!=6$ ways.

  Hence, the number of triangles is $\frac{31\cdot 30\cdot 25}{6}=31\cdot 125=3875$.
- We now compute the number of conics:
  - Each conic has 5 self-polar triangles of the first type.
  - Each triangle plays this role for 4 conics.

  Therefore, the number of conics is $\frac{31\cdot 125\cdot 4}{5}=3100$.