Introduction to Projective Geometry

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LSSU Math 400

1. Parallelism
   - Is the Circle a Conic?
   - Affine Space: Basic Notions
   - How Coplanar Planes Determine Flat Pencil and Bundle
   - How Two Planes Determine Axial Pencil
   - The Language of Pencils and Bundles
   - The Plane at Infinity
   - Euclidean Space

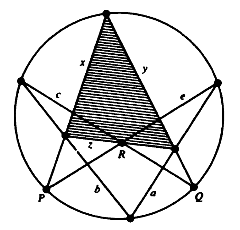
Subsection 1

Is the Circle a Conic?

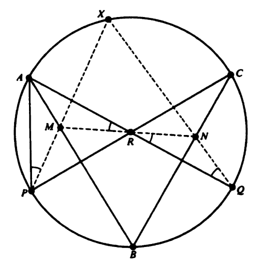
Euclidean Proof of the Braikenridge-MacLaurin Theorem

The Braikenridge-MacLaurin Theorem

If the sides of a variable triangle pass through three fixed noncollinear points P, Q, R, while two vertices lie on fixed lines a and b, not concurrent with PQ, then the third vertex describes a conic.

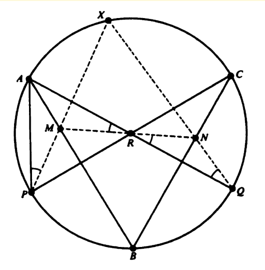


- Select five points on a circle and prove that the unique conic that can be drawn through these points coincides with the circle.
  For convenience we shall take the five points A, P, B, Q, C to be five of the six vertices of a regular hexagon inscribed in the circle.



Braikenridge-MacLaurin Theorem (Cont'd)

- Let AQ and CP meet in R (which is, of course, the center of the circle), let a variable diameter meet AB in M, BC in N, and let PM meet QN in X. Since AB, being a median of the equilateral triangle APR, is the perpendicular bisector of PR, and similarly BC is the perpendicular bisector of QR, we have



$$\angle XPA = \angle MPA = \angle ARM = \angle QRN = \angle NQR = \angle XQA.$$

By Euclid's results, the locus of X is the circle APQ. By the Braikenridge-MacLaurin construction, the locus of X is the conic $APBQC$. Hence the conic coincides with the circle.

Subsection 2

Affine Space: Basic Notions

Bundles and Pencils

- A **bundle** is the set of all lines and planes through a point.
- An **axial pencil** is the set of all planes through a line.
- When there is any possibility of confusion, the other kind of pencil (the set of all lines that lie in a plane and pass through a point) is called a **flat pencil**.
- An axial pencil, like a flat pencil, is a "one-dimensional form".

  But a bundle is a combination of two two-dimensional forms:
  - the set of lines through a point (which is the space-dual of the set of lines in a plane);
  - the set of planes through the same point (which is the space-dual of the set of points in a plane).
- These forms admit a description that is precisely the same in affine space as in projective space.

Determining Bundles and Pencils

- A range is determined by any two of its points, and these may be any two distinct points;
- A flat pencil is determined by any two of its lines, and these may be any two lines that meet;
- An axial pencil is determined by any two of its planes, and these may be any two planes that meet;
- A bundle (like a flat pencil) is determined by any two of its lines.

Subsection 3

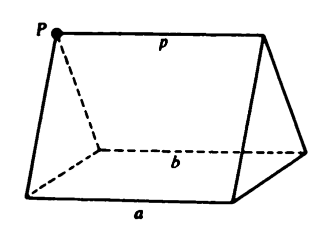
How Coplanar Planes Determine Flat Pencil and Bundle

Pencil Determined by Two Lines

- Suppose we are given in ordinary space a point P and two coplanar lines a and b whose point of intersection O is inconveniently far away.
- We construct the line through P of the pencil or bundle determined by a and b.
- If O were available we could simply draw OP; We can still locate this line without using O.

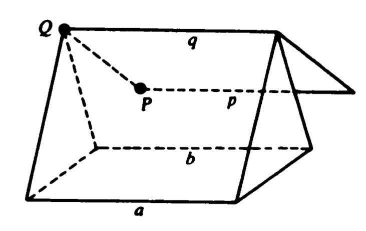
If P is not in the plane ab, we merely have to draw the planes Pa and Pb; these meet in a line p through P, which is the desired line of the bundle.
In a single symbol, the member through P (outside the plane ab) is the line

$$p = Pa \cdot Pb.$$



Pencil Determined by Two Lines (Cont'd)

- If P is in the plane ab,
  - we can use an auxiliary point Q outside the plane;
  - locate the member q through Q (which is the line OQ);
  - consider the line of intersection of the planes ab and Pq.



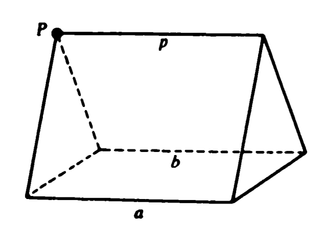
  In other words, the line through P (of the bundle or pencil) is now

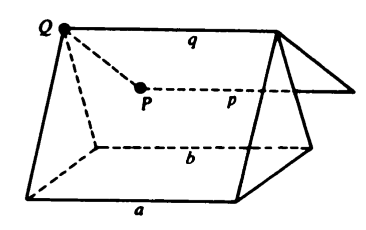
$$p = ab \cdot Pq, \quad \text{where} \quad q = Qa \cdot Qb.$$

- This construction owes its importance to the fact that it remains valid when a and b are parallel, so that O does not exist!

  The lines a, b, q, p, which originally passed through O, are now all parallel.

Bundle and (Flat) Pencil of Parallels



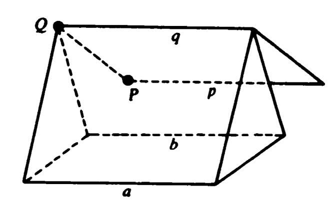


- Since we can derive p without inquiring whether a and b are parallel or not, we are justified in extending the meaning of the words **pencil** and **bundle** so as to allow the determining lines a and b to be any two coplanar lines.
- If a and b happen to be parallel:
  - The bundle consists of all the lines and planes parallel to them;
  - The pencil consists of all the lines parallel to them in their own plane.

  Accordingly, we speak of a **bundle of parallels** and a (**flat**) **pencil of parallels**.

Some Remarks

- It must be remembered that two planes may be parallel to a line without being parallel to each other (e.g., Qa and Qb are both parallel to the line p).
  Thus, a bundle of parallels contains:



  - A lot of lines, all parallel to one another;
  - A lot of planes, not all parallel to one another but each containing two (and therefore infinitely many) of the lines.
- A familiar instance of a bundle of parallels is the set of all "vertical" lines and planes.

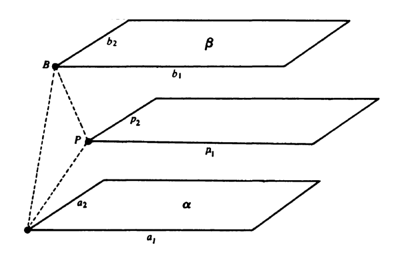
  If we take a cosmic standpoint and insist that two vertical lines are not strictly parallel but meet in the center of the earth, then we have an ordinary bundle instead of a bundle of parallels.

Subsection 4

How Two Planes Determine Axial Pencil

Axial Pencil Determined by Two Planes

- If we are given a point P, and two planes α and β whose line of intersection is far away, we construct the member through P of the axial pencil determined by α and β:
  - Take any two intersecting lines a_1 and a_2 in α;
  - Take a point B in β (but not in either of the planes Pa_1, Pa_2);
  - Draw the lines $b_1 = Ba_1 \cdot \beta$, $b_2 = Ba_2 \cdot \beta$, $p_1 = Pa_1 \cdot Pb_1$, $p_2 = Pa_2 \cdot Pb_2$;
  - The desired plane through P is $p_1 p_2$.

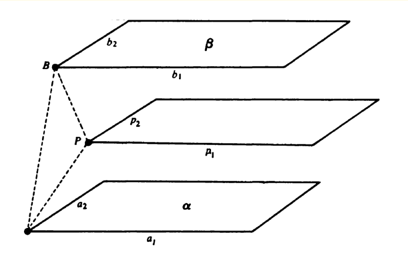


Axial Pencil Determined by Two Planes (Cont'd)

- If α and β meet in a line o, we may assume a_1 and a_2 to be chosen so as to meet o in two distinct points O_1 and O_2.
  - Since $O_1 = o \cdot a_1$ lies in both the planes Pa_1 and β, it lies on their common line b_1;
  - Since O_1 lies in both the planes Pa_1 and Pb_1, it lies on their common line p_1.

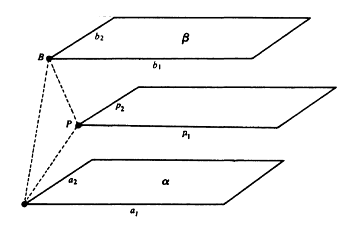
  Similarly, O_2 lies on p_2.

  Thus, the join $o = O_1 O_2$ lies in the plane $p_1 p_2$.



Pencil of Parallel Lines

- If, on the other hand, α and β are parallel planes, the construction makes the lines b_1 and p_1 parallel to a_1, and the lines b_2 and P_2 parallel to a_2; therefore, the plane $p_1 p_2$ is parallel to α and β.



  Allowing P to take various positions, we thus obtain a **pencil of parallel planes**, consisting of all the planes parallel to a given plane.

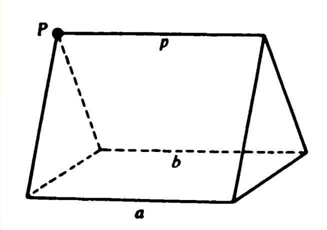
- A familiar instance is the set of all "horizontal" planes.

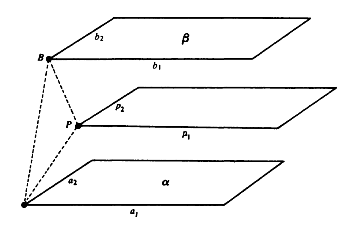
  If we insist that two horizontal planes are not strictly parallel, but intersect in a line called the "horizon", then we have an ordinary axial pencil instead of a pencil of parallel planes.

Subsection 5

The Language of Pencils and Bundles

Statements about Bundles and Axial Pencils





- Since an ordinary bundle consists of all the lines and planes through a point, and an ordinary axial pencil consists of all the planes through a line, any simple statement about points and lines can be "translated" into a corresponding statement about bundles and axial pencils.

  Example: The statement

  Any two distinct points lie on a line

  becomes:

  The common planes of any two distinct bundles form an axial pencil.

The Language of Bundles of Parallels

- The statement

  The common planes of any two distinct bundles form an axial pencil.

  remains true when:
  - One of the bundles is replaced by a bundle of parallels;
  - Both are replaced by bundles of parallels.
- In fact:
  - The common planes of an ordinary bundle and a bundle of parallels form the ordinary axial pencil whose axis is the common line of the two bundles.
  - The common planes of two bundles of parallels form a pencil of parallel planes.

Subsection 6

The Plane at Infinity

Ideal Points and Ideal Lines

- An ordinary bundle consists of all the lines and planes through an ordinary point.
- We regard a bundle of parallels as consisting of all the lines and planes through an **ideal point**.
- Similarly, we regard a pencil of parallel planes as consisting of all the planes through an **ideal line**.
- We say that an ideal point **lies on** an ideal line if the bundle contains the pencil.
- We can still assert that any two distinct points lie on a line:
  - If one of the points is ordinary, so is the line;
  - If both are ideal, the line is ideal.

Ordinary and Ideal Points and Lines

- Since an ordinary bundle contains no pair of parallel planes, an ordinary point cannot lie on an ideal line:

  All the "points" on an ideal line are ideal points.
- Since a bundle of parallels contains ordinary axial pencils as well as pencils of parallel planes, an ideal point lies on some ordinary lines as well as on some ideal lines.

Points and Lines at Infinity

- Since any ordinary line belongs to just one bundle of parallels (consisting of all the lines and planes parallel to it), it contains just one ideal point, which we call its **point at infinity**.
- Thus, we regard any two parallel lines as meeting in an ideal point: their common point at infinity.
- Since any plane belongs to just one pencil of parallel planes (consisting of all the planes parallel to it), it contains just one ideal line, which we call its **line at infinity**.
- Thus we regard any two parallel planes as meeting in an ideal line: their common line at infinity.
- In a given plane, each point on the line at infinity is the "center" of a pencil of parallel lines.

The Plane at Infinity

- Since any two pencils of parallel planes belong to a bundle of parallels,

  any two ideal lines meet in an ideal point.

- It follows that, if a and b are any two ideal lines, every other ideal line meets both a and b.
- This state of affairs resembles what happens in a plane:

  If a and b are two ordinary intersecting lines, every point in the plane ab lies on a line that meets both a and b.

- Thus, it is appropriate to regard the set of all ideal points and ideal lines as forming an ideal plane: the **plane at infinity**.
- This makes it possible to assert that any two intersecting (or parallel) lines determine a plane through both of them.
  - If one of the lines is ordinary this is an ordinary plane;
  - If both are ideal it is the plane at infinity.

Projective Plane and Bundles

- Since each point (or line) at infinity is joined to an ordinary point O by an ordinary line (or plane), the points and lines of the projective plane may simply be regarded as a "new language" for the lines and planes (respectively) through O:

  The projective plane is faithfully represented by a bundle.

Subsection 7

Euclidean Space

Affine vs. Euclidean Geometry

- Although elementary solid geometry operates in affine space, affine geometry is NOT merely another name for Euclidean geometry!
- Affine geometry is the part of Euclidean geometry in which distances are compared only on the same line or on parallel lines.
- Right angles lead to circles and spheres, and thus enable us to compare other more general distances.
- Thus,

  Affine geometry becomes Euclidean geometry as soon as we have said what we mean by perpendicular.

Correspondence Between Points and Lines at Infinity

- The set of all vertical lines is a familiar instance of a bundle of parallels.
- The set of all horizontal planes is a familiar instance of a pencil of parallel planes.
- More generally, every bundle of parallels in Euclidean space determines a unique axial pencil (of parallel planes), whose planes are perpendicular to the lines and planes of the bundle;

  Conversely, every pencil of parallel planes determines a perpendicular bundle (of parallels).
- In the language of the plane at infinity, we thus have a special one-to-one correspondence between points at infinity and lines at infinity.

The Absolute Polarity and Perpendicularity

- The preceding correspondence between points and lines of the plane at infinity, which is a projective plane, is a polarity, called the **absolute polarity**.
  - A line and a plane are perpendicular if the point at infinity on the line is the pole of the line at infinity in the plane.
  - Two lines (or two planes) are perpendicular if their sections by the plane at infinity are conjugate points (or lines).
- Since no line or plane is perpendicular to itself, the polarity is elliptic.
- We draw the following conclusions:
  - Affine space can be derived from projective space by singling out a plane ("at infinity") and using it to define parallelism.
  - Euclidean space can be derived from affine space by singling out an elliptic polarity in the plane at infinity and using it to define perpendicularity.