Introduction to Projective Geometry

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Subsection 1

The Idea of Analytic Geometry

Coordinates for Projective Geometry

- For Euclidean geometry we use the classical "non-homogeneous" coordinates, which may be illustrated by the description of a point in ordinary space (with reference to a chosen origin) as being at distances x_1 east, x_2 north, and x_3 up.
- For projective geometry it is more convenient to use "homogeneous coordinates", which may be illustrated by the description of a point in a plane (with reference to a triangle $A_1A_2A_3$) as being at the center of gravity of masses x_1 at A_1, x_2 at A_2, and x_3 at A_3.
- The important idea is to take an ordered set of numbers (x_1, x_2, x_3) and call it a point.
- The "numbers" that we use may be thought of as real numbers.
- However, they can be the elements of any commutative field in which $1+1 \neq 0$.

    In particular, they can form a finite field (thus, related to finite projective geometries).

Points and Lines in the Projective Plane

- In order to be able to interpret lines as well as points, we consider two types of ordered triads of numbers:
  - (x_1, x_2, x_3);
  - $[X_1, X_2, X_3]$.
- We exclude the "trivial" triads $(0,0,0)$, $[0,0,0]$;
- We regard two triads of the same type as being equivalent (that is, geometrically indistinguishable) if they are proportional, i.e., for all $\lambda \neq 0$,
  - (x_1, x_2, x_3) is equivalent to $(\lambda x_1, \lambda x_2, \lambda x_3)$;
  - $[X_1, X_2, X_3]$ is equivalent to $[\lambda X_1, \lambda X_2, \lambda X_3]$.
- With two triads of opposite types we associate a single number, their "**inner product**"

$$\{xX\} = \{Xx\} = X_1 x_1 + X_2 x_2 + X_3 x_3,$$

  which may be zero.

Subsection 2

Definitions

Points and Lines

- A **point** is the set of all triads equivalent to a given triad (x_1, x_2, x_3).

  In other words, a point is an ordered set of three numbers (x_1, x_2, x_3), not all zero, with the understanding that $(\lambda x_1, \lambda x_2, \lambda x_3)$ is the same point for any nonzero λ.

  Example: $(2,3,6)$ is a point, and $(-\frac{1}{3}, -\frac{1}{2}, -1)$ is another way of writing the same point.
- A **line** is the set of all triads equivalent to a given triad $[X_1, X_2, X_3]$.

  In other words, a line is defined in the same manner as a point, but with square brackets instead of ordinary parentheses, and with capital letters to represent the three numbers.

  Example: $[3, 2, -2]$ is a line, and $[-1, -\frac{2}{3}, \frac{2}{3}]$ is the same line.
- We will see that the point (x_1, x_2, x_3) and the line $[x_1, x_2, x_3]$ (with the same x's) are related by a polarity.

Incidence and Coordinates

- The point $(x) = (x_1, x_2, x_3)$ and line $[X] = [X_1, X_2, X_3]$ are said to be **incident** (the point lying on the line and the line passing through the point) if and only if
$$\{xX\} = 0.$$
Example: $(2,3,6)$ lies on $[3,2,-2]$.
- Any discussion can be dualized by interchanging small and capital letters, round and square brackets.
- The three numbers x_i are called the **coordinates** (or "**homogeneous coordinates**", or "**projective coordinates**") of the point (x).
- The three numbers X_i are called the **coordinates** (or "**line coordinates**", or "**envelope coordinates**", or "**tangential coordinates**") of the line $[X]$.

Equations of Lines and Points

- If (x) is a variable point on a fixed line $[X]$, we call $\{Xx\} = 0$ the **equation of the line** $[X]$, because it is a characteristic property of points on the line.

  Claim: The line $[3, 2, -2]$ has the equation

  $$3x_1 + 2x_2 - 2x_3 = 0, \quad \text{or} \quad 3x_1 + 2x_2 = 2x_3.$$

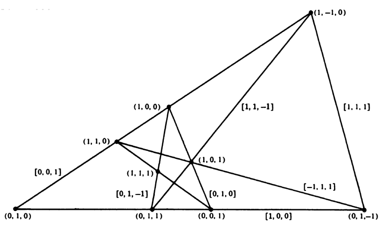
- Dually, if $[X]$ is a variable line through a fixed point (x), we call $\{xX\} = 0$ (which is the same as $\{Xx\} = 0$) the **equation of the point** (x), because it is a characteristic property of lines through the point.

  Example:
  - The point $(2, 3, 6)$ has the equation $2X_1 + 3X_2 + 6X_3 = 0$;
  - The point $(1, 0, 0)$ has the equation $X_1 = 0$.

- Thus, the coordinates of a line or point are the coefficients in its equation (with zero for any missing term).

Triangle of Reference

- The three points (1,0,0), (0,1,0), (0,0,1), or $X_i = 0$, $i = 1,2,3$, and the three lines [1,0,0], [0,1,0], [0,0,1], or $x_i = 0$, are evidently the vertices and sides of a triangle. We call this the **triangle of reference**.



- The point (1,1,1) and line [1,1,1] are called the **unit point** and **unit line**.
- The key features are:
  - The point does not lie on a side;
  - The line does not pass through a vertex;
  - The point is the trilinear pole of the line.
    Recall this means that the line passes through the points of intersection of corresponding sides of the original triangle and the triangle formed by the foots of the Cevians through the point.

Collinearity of Points

- By eliminating X_1, X_2, X_3 from the equations

$$\{xX\} = 0, \quad \{yX\} = 0, \quad \{zX\} = 0$$

  of three given points (x), (y), (z), we find the necessary and sufficient condition

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$$

  for the three points to be collinear.
- This condition is equivalent to the existence of numbers λ, μ, ν, not all zero, such that

$$\lambda x_i + \mu y_i + \nu z_i = 0, \quad i = 1, 2, 3.$$

Relative Positions of Collinear Points

- Suppose $(x),(y),(z)$ are collinear, i.e., $\lambda x_i + \mu yi + \nu z_i = 0$, $i = 1,2,3$.
  - If (y) and (z) are distinct points, $\lambda \neq 0$.
    Hence, the general point collinear with (y) and (z) is
    $$(\mu y_1 + \nu z_1, \mu y_2 + \nu z_2, \mu y_3 + \nu z_3) \quad \text{or} \quad (\mu y + \nu z),$$
    where μ and ν are not both zero.
    - When $\nu = 0$, this is the point (y) itself.
    - For any other position, since (νz) is the same point as (z), we can allow the coordinates of (z) to absorb the ν, and the point collinear with (y) and (z) is simply $(\mu y + z)$.
    - If we are concerned with only one such point, we may allow the μ to be absorbed too; thus three distinct collinear points may be expressed as (y), (z), $(y+z)$.
      The last simplification cannot be effected simultaneously on two lines if thereby one point would have to absorb two different parameters.
- The symbol $(\mu y + z)$ can be made to include every point on the line $(y)(z)$ if we adopt the convention that (y) is $(\mu y + z)$ with $\mu = \infty$.

Concurrent Lines

- Dually, the condition for three lines $[X]$, $[Y]$, $[Z]$ to be concurrent is

$$\begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{vmatrix} = 0.$$

- The general line concurrent with $[Y]$ and $[Z]$ is $[\mu Y + \nu Z]$;
- A particular line concurrent with $[Y]$ and $[Z]$, but distinct from them, may be expressed as $[Y + Z]$.

Subsection 3

Verifying the Axioms for the Projective Geometry

Axiom 3 and New Axiom 1

- To show that this analytic geometry provides a model for the synthetic geometry developed in earlier chapters, we must verify that Axiom 3, New Axioms 1, 2, and Axioms 7, 8 are all satisfied.

  The first two can be verified as follows: Two points $(y) = (y_1, y_2, y_3)$ and $(z) = (z_1, z_2, z_3)$ are joined by the line

$$\left[\begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix}, \begin{vmatrix} y_3 & y_1 \\ z_3 & z_1 \end{vmatrix}, \begin{vmatrix} y_1 & y_2 \\ z_1 & z_3 \end{vmatrix} \right].$$

  Two lines $[Y] = [Y_1, Y_2, Y_3]$ and $[Z] = [Z_1, Z_2, Z_3]$ meet in the point

$$\left(\begin{vmatrix} Y_2 & Y_3 \\ Z_2 & Z_3 \end{vmatrix}, \begin{vmatrix} Y_3 & Y_1 \\ Z_3 & Z_1 \end{vmatrix}, \begin{vmatrix} Y_1 & Y_2 \\ Z_1 & Z_3 \end{vmatrix} \right).$$

New Axiom 2 and Axiom 7

- To verify New Axiom 2 and Axiom 7, we consider a quadrangle $PQRS$ whose first three vertices $(p),(q),(r)$ satisfy $\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \neq 0$. Since the side PS joins (p) to the diagonal point $A = QR \cdot PS$, we may take:
  - A (on, QR, but distinct from Q and R) to be $(q+r)$;
  - S (on PA, but distinct from P and A) to be $(p+q+r)$, meaning $(p_1+q_1+r_1, p_2+q_2+r_2, p_3+q_3+r_3)$.

  Then B, on both RP and QS, must be $(r+p)$.

  C, on both PQ and RS, must be $(p+q)$.

  The three diagonal points A, B, C are noncollinear since

$$\begin{vmatrix} q_1+r_1 & q_2+r_2 & q_3+r_3 \\ r_1+p_1 & r_2+p_2 & r_3+p_3 \\ p_1+q_1 & p_2+q_2 & p_3+q_3 \end{vmatrix} = 2 \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \neq 0.$$

Axiom 8

- Let the first triangle PQR and the center of perspective O be $(p)(q)(r)$ and (u). There is no loss of generality in taking the second triangle $P'Q'R'$ to be $(p+u)(q+u)(r+u)$.

  The point $D = QR \cdot Q'R'$, being collinear with (q) and (r) and also with $(q+u)$ and $(r+u)$, can only be $(q-r)$.

  Similarly, E is $(r-p)$, and F is $(p-q)$.

  These points D, E, F are collinear since

$$\begin{vmatrix} q_1 - r_1 & q_2 - r_2 & q_3 - r_3 \\ r_1 - p_1 & r_2 - p_2 & r_3 - p_3 \\ p_1 - q_1 & p_2 - q_2 & p_3 - q_3 \end{vmatrix} = 0$$

  or, more simply, since $(q_i - r_i) + (r_i - p_i) + (p_i - q_i) = 0$, $i = 1, 2, 3$.

Axiom 8 (Cont'd)

- When a range of points P arises as a section of a pencil of lines p, the "elementary correspondence" $P \barwedge p$ may be described as relating three positions of P, say $(y),(z),(y+z)$ to three positions of p, say $[Y],[Z],[Y+Z]$.
  Claim: From the information that P and p are incident in these three cases, we can deduce that, when P is $(\mu y+z)$, p is $[\mu Y+Z]$ with the same μ.
  Since
  $$\{(Y+Z)(y+z)\} = \{Yy\}+\{Yz\}+\{Zy\}+\{Zz\},$$
  the three given incidences imply $\{Yy\}=0$, $\{Zz\}=0$, $\{Yz\}+\{Zy\}=0$, whence
  $$\{(\mu Y+Z)(\mu y+z)\} = \mu^2\{Yy\}+\mu(\{Yz\}+\{Zy\})+\{Zz\}=0,$$
  Thus, the line $[\mu Y+Z]$ is indeed incident with the point $(\mu y+z)$.

Axiom 8 (Conclusion)

- Repeated application of the claim shows that the relation

$$(y)(z)(y+z)(\mu y+z)\,\overline{\wedge}\,[Y][Z][Y+Z][\mu Y+Z]$$

holds not only for an elementary correspondence but for any projectivity from a range to a pencil;

And of course we have also

$$\begin{array}{rcl}(y)(z)(y+z)(\mu y+z) & \overline{\wedge} & (y')(z')(y'+z')(\mu y'+z'),\\ [Y][Z][Y+Z][\mu Y+Z] & \overline{\wedge} & [Y'][Z'][Y'+Z'][\mu Y'+Z'].\end{array}$$

This is the algebraic version of the Fundamental Theorem.

From this, Axiom 8 can be deduced as a special case.

Subsection 4

Projective Collineations

Barycentric Coordinates of a Point

- The condition $\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \neq 0$ makes $(p),(q),(r)$ noncollinear, so that they form a triangle.

  This triangle enables us to describe the position of any point by means of **barycentric coordinates** λ, μ, ν, which are the coefficients in the expression $(\lambda p + \mu q + \nu r)$.
  - Points on a side of the triangle can be included by allowing $\lambda\mu\nu = 0$;
  - When $\mu = \nu = 0$, we have the point (p) itself.
- When $(p)(q)(r)$ and $(p+q+r)$ are the triangle of reference and unit point, $(\lambda p + \mu q + \mu r)$ is (λ, μ, ν), and the barycentric coordinates are the same as the ordinary coordinates.

The General Projective Collineation

- The correspondence $(x) \to (x')$, where $\begin{cases} x'_1 = p_1x_1 + q_1x_2 + r_1x_3 \\ x'_2 = p_2x_1 + q_2x_2 + r_3x_3 \\ x'_3 = p_3x_1 + q_3x_2 + r_3x_3 \end{cases}$ transforms (λ, μ, ν) into $(\lambda p + \mu q + \nu r)$ having same barycentric coordinates referred to $(p)(q)(r)$ instead of $(1,0,0)(0,1,0)(0,0,1)$.
  - Under the hypothesis $\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \neq 0$, the preceding system can be solved for the xs in terms of the x's, whence this is a point-to-point correspondence;
  - Since $\{X'x'\} = 0$ is equivalent to $\{X'p\}x_1 + \{X'q\}x_2 + \{X'r\}x_3 = 0$, it is a collineation;
  - Since it transforms $(0, \mu, \nu)$ into $(\mu q + \nu r)$, it is projective.

It transforms $(1,0,0)(0,1,0)(0,0,1)(1,1,1)$ into $(p)(q)(r)(p+q+r)$, which may be identified with any given quadrangle by a suitable choice of p's, q's and r's.

So it is the general projective collineation.

Alibis and Aliases

- The general projective collineation, which shifts the points (x) to new positions (x'), is the **active** or **alibi** aspect of the linear transformation.
- A **passive** or **alias** aspect is a coordinate transformation that gives a new name to each point:

  We may regard $(p)(q)(r)$ as a new triangle of reference, with respect to which the point that we have been calling (x') has coordinates (x_1, x_2, x_3), whereas its coordinates with respect to the original triangle are, of course, (x_1', x_2', x_3').

Usefulness of Aliases

- Practical consequences of the "alias" aspect:
  - A triangle and a point of general position may be taken to be the triangle of reference and unit point.
  - A given quadrangle may be taken to have vertices $(1, \pm 1, \pm 1)$, so that the six sides have equations $x_i \pm x_j = 0$, $i < j$, and the diagonal triangle is the triangle of reference.
  - A given quadrilateral may be taken to have sides $[1, \pm 1, \pm 1]$ and vertices $X_i \pm X_j = 0$.

Line-to-Line Aspect of Collineations and Incidence

- A collineation is not only a point-to-point transformation but also a line-to-line transformation. This aspect is expressed by

  $$X_1 = \{pX'\}, \quad X_2 = \{qX'\}, \quad X_3 = \{rX'\}.$$

  A more systematic notation for the same two sets of equations is

  $$\begin{array}{rcl} \rho x'_i & = & c_{i1}x_1 + c_{i2}x_2 + c_{i3}x_3 = \sum c_{ij}x_j, \quad i = 1,2,3, \\ \sigma X_j & = & c_{1j}X'_1 + c_{2j}X'_2 + c_{3j}X'_3 = \sum c_{ij}X'_i, \quad j = 1,2,3, \end{array}$$

  where $\rho\sigma \neq 0$ and $\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} \neq 0.$
- The preservation of incidence is verified as follows:

  $$\rho\{X'x'\} = \rho \sum X'_i x'_i = \sum\sum c_{ij}X'_i x_j = \sigma \sum X_j x_j = \sigma\{Xx\}.$$
- Since our coordinates are homogeneous, there are many occasions when we can omit the ρ and σ, i.e., set $\rho = \sigma = 1$.

Invariance

- Since $\rho x_i' = \sum c_{ij} x_j$, $i = 1, 2, 3$, the invariant points $x_i' = x_i$ are given by

$$\rho x_i = \sum c_{ij} x_j, \quad i = 1, 2, 3.$$

Eliminating the x's from these three equations, we obtain

$$\begin{vmatrix} c_{11} - \rho & c_{12} & c_{13} \\ c_{21} & c_{22} - \rho & c_{23} \\ c_{31} & c_{32} & c_{33} - \rho \end{vmatrix} = 0.$$

Any root ρ of this **characteristic equation** makes the three equations for the x's consistent, and then we can solve any two of them to obtain the coordinates of an invariant point.

Example: A Homology

- Consider the collineation

$$\rho x_1' = x_1, \quad \rho x_2' = x_2, \quad \rho x_3' = \mu^{-1} x_3, \quad \mu \neq 1.$$

It has the characteristic equation

$$(\rho - 1)^2 (\rho - \mu^{-1}) = 0.$$

The double root $\rho = 1$ yields the range of invariant points $(x_1, x_2, 0)$;
The remaining root $\rho = \mu^{-1}$ yields the isolated invariant point $(0, 0, 1)$.
By preceding results, this collineation is a homology with center $(0, 0, 1)$ and axis $[0, 0, 1]$.

Example: An Elation

- Consider the collineation

$$\rho x_1' = x_1 + a_1 x_3, \quad \rho x_2' = x_2 + a_2 x_3, \quad \rho x_3' = x_3.$$

It has the characteristic equation

$$(\rho - 1)^3 = 0.$$

If a_1 and a_3 are not both zero, the triple root $\rho = 1$ yields the range of invariant points $(x_1, x_2, 0)$ and no others.

By preceding results, this collineation is an elation with axis $[0,0,1]$.

Since the equation $a_2 x_1' - a_1 x_2' = 0$ implies $a_2 x_1 - a_1 x_2 = 0$, there is an invariant line (other than $[0,0,1]$) through the point $(a_1, a_2, 0)$. Hence, this point is the center of the elation.

Homology and Elation as Line-to-Line Transformations

- We compare the two parts of

$$\begin{aligned} \rho x_i' &= c_{i1}x_1 + c_{i2}x_2 + c_{i3}x_3 = \sum c_{ij}x_j, \quad i = 1,2,3, \\ \sigma X_j &= c_{1j}X_1' + c_{2j}X_2' + c_{3j}X_3' = \sum c_{ij}X_i', \quad j = 1,2,3. \end{aligned}$$

  - The expression for the homology

    $$\rho x_1' = x_1, \quad \rho x_2' = x_2, \quad \rho x_3' = \mu^{-1}x_3, \quad \mu \neq 1,$$

    as a line-to-line transformation is

    $$\sigma X_1 = X_1', \quad \sigma X_2 = X_2', \quad \sigma X_3 = \mu^{-1}X_3'$$

    or, taking $\sigma = 1$, for convenience, and solving,

    $$X_1' = X_1, \quad X_2' = X_2, \quad X_3' = \mu X_3.$$

  - The elation

    $$\rho x_1' = x_1 + a_1x_3, \quad \rho x_2' = x_2 + a_2x_3, \quad \rho x_3' = x_3,$$

    qua line-to-line transformation, is

    $$X_1' = X_1, \quad X_2' = X_2, \quad X_3' = X_3 - a_1X_1 - a_2X_2.$$

Subsection 5

Polarities

The General Projective Correlation

- The product of two correlations (e.g., a polarity and another projective correlation) is a collineation.

  Thus, any given projective correlation can be exhibited as the product of an arbitrary polarity and a suitable projective collineation.
- The most convenient polarity for this purpose is the one that transforms each point (or line) into the line (or point) that has the same coordinates (it is obviously projective, and of period 2).

The General Projective Correlation: Coordinates

- Combining the general projective collineation

$$\begin{aligned} \rho x_i' &= c_{i1}x_1 + c_{i2}x_2 + c_{i3}x_3 = \sum c_{ij}x_j, \quad i = 1,2,3, \\ \sigma X_j &= c_{1j}X_1' + c_{2j}X_2' + c_{3j}X_3' = \sum c_{ij}X_i', \quad j = 1,2,3. \end{aligned}$$

with the polarity that interchanges X_i' and x_i', we obtain the general projective correlation in the form

$$\begin{aligned} \rho X_i' &= c_{i1}x_1 + c_{i2}x_2 + c_{i3}x_3 = \sum c_{ij}x_j, \quad i = 1,2,3, \\ \sigma X_j &= c_{1j}x_1' + c_{2j}x_2' + c_{3j}x_3' = \sum c_{ij}x_i', \quad j = 1,2,3. \end{aligned}$$

where again the coefficients satisfy the nonzero determinant property.

- Incidences are dualized in the proper manner for a correlation, since

$$\rho\{X'x'\} = \rho \sum X_i' x_i' = \sum \sum c_{ij} x_i' x_j = \sigma \sum X_j x_j = \sigma \{Xx\}.$$

Incidence and Polarities

- The projective correlation is a **polarity** if it is equivalent to the inverse correlation $\sigma X_j' = \sum c_{ij} x_i$ or (interchanging i and j)

$$\sigma X_i' = \sum c_{ji} x_j.$$

This gives $c_{ji} = \frac{\sigma}{\rho} c_{ij}$, with the same $\frac{\sigma}{\rho}$, for all i and j.
Hence, since the c_{ij} are not all zero, $c_{ij} = \frac{\sigma}{\rho} c_{ij} = (\frac{\sigma}{\rho})^2 c_{ij}$.
So $(\frac{\sigma}{\rho})^2 = 1$, whence $\frac{\sigma}{\rho} = \pm 1$.
The minus sign is inadmissible, as that would make $c_{ji} = -c_{ij}$, and

$$\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = \begin{vmatrix} 0 & c_{12} & -c_{31} \\ -c_{12} & 0 & c_{23} \\ c_{31} & -c_{23} & 0 \end{vmatrix} = 0.$$

Hence $\sigma = \rho$ and $c_{ji} = c_{ij}$.
In other words, a projective correlation is a polarity if and only if the matrix of coefficients c_{ij} is **symmetric**.

Polar and Pole

- By the nature of a polarity, no confusion can be caused by omitting the prime

$$X_i = \sum c_{ij} x_j, \quad i = 1, 2, 3,$$

where $c_{ij} = c_{ji}$ and $\det(c_{ij}) = \Delta \neq 0$.

These equations give us the polar $[X]$ of a given point (x).

- Solving them, we obtain the pole (x) of a given line $[X]$ in the form

$$\Delta x_i = \sum C_{ij} X_j, \quad i = 1, 2, 3,$$

where C_{ij} is the cofactor of c_{ij} in the determinant Δ.

Conjugates

- Two points (x) and (y) are conjugate if (x) lies on the polar $[Y]$ of (y). Since $Y_i = \sum c_{ij} y_j$, the condition $\{Yx\} = 0$ or $\sum Y_i x_i = 0$ becomes

$$\sum \sum c_{ij} x_i y_j = 0,$$

  which we shall sometimes write in the abbreviated form $(xy) = 0$.
- Letting (x) vary, we see that this is the equation of the polar of (y).
- Dually, the condition for lines $[X]$ and $[Y]$ to be conjugate, or the equation of the pole of $[Y]$, is $[XY] = 0$, where

$$[XY] = \sum \sum C_{ij} X_i Y_j.$$

Obtaining a Canonical Form

- As a particular case of $(xy) = 0$, the condition for $(0,1,0)$ and $(0,0,1)$ to be conjugate is $c_{23} = 0$.

  Thus the triangle of reference is self-polar if and only if $c_{23} = c_{31} = c_{12} = 0$.

  By choosing any self-polar triangle as triangle of reference, we reduce a given polarity to its **canonical form**

  $$X_i = c_{ii}x_i, \quad c_{11}c_{22}c_{33} \neq 0,$$

  or, more conveniently,

  $$X_1 = ax_1, \quad X_2 = bx_2, \quad X_3 = cx_3, \quad abc \neq 0.$$

  This is the polarity $(ABC)(Pp)$, where ABC is the triangle of reference, P is $(1,1,1)$, and p is $[a,b,c]$.

Subsection 6

Conics

General Equation for Conics

- Consider the polarity

$$X_i = \sum c_{ij}x_j, \quad i = 1,2,3, \quad c_{ij} = c_{ji},\ \det(c_{ij}) = \Delta \neq 0.$$

- The condition for a point (x) to be self-conjugate is $(xx) = 0$, or

$$c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2 + 2c_{23}x_2x_3 + 2c_{31}x_3x_1 + 2c_{12}x_1x_2 = 0.$$

- Dually, the condition for a line $[X]$ to be self-conjugate is $[XX] = 0$, or

$$C_{11}X_1^2 + C_{22}X_2^2 + C_{33}X_3^2 + 2C_{23}X_2X_3 + 2C_{31}X_3X_1 + 2C_{12}X_1X_2 = 0.$$

- Hence every conic (locus or envelope) has such an equation.
- Using the standard form

$$X_1 = ax_1, \quad X_2 = bx_2, \quad X_3 = cx_3,$$

every conic for which the triangle of reference is self-polar has an equation of the form

$$ax_1^2 + bx_2^2 + cx_3^2 = 0 \quad \text{or} \quad a^{-1}X_1^2 + b^{-1}X_2^2 + c^{-1}X_3^2 = 0.$$

Existence of Elliptic Polarities

- The polarity
$$X_i = \sum c_{ij} x_j, \quad i = 1,2,3, \quad c_{ij} = c_{ji},\ \det(c_{ij}) = \Delta \neq 0,$$
is hyperbolic or elliptic according as the equation $(xx) = 0$ does or does not have a solution (other than $x_1 = x_2 = x_3 = 0$).
- The distinction depends on the coordinate field.
  - If this is the field of complex numbers, every such equation can be solved.
    Example: The equation $x_1^2 + x_2^2 + x_3^2 = 0$ is satisfied by $(1, i, 0)$.
    Over such a field, every polarity is hyperbolic.
  - In the case of the field of real numbers, on the other hand, the quadratic form (xx) may be "definite", in which case the polarity (for instance, $X_i = x_i$) is elliptic.
- Some particular equations represent conics regardless of the field.
  Example: The equation $x_1^2 + x_2^2 - x_3^2 = 0$, being satisfied by $(1, 0, 1)$, cannot fail to represent a conic.

Triangle and Circumscribed Conic

- Since the condition for the conic $(xx) = 0$ to pass through $(1,0,0)$ is $c_{11} = 0$, the most general conic circumscribing the triangle of reference is

$$c_{23}x_2x_3 + c_{31}x_3x_1 + c_{12}x_1x_2 = 0.$$

- The coordinate transformation

$$x_1 \to c_{23}x_1, \quad x_2 \to c_{31}x_2, \quad x_3 \to c_{12}x_3$$

  converts this into

$$x_2x_3 + x_3x_1 + x_1x_2 = 0.$$

- Thus, in any problem concerning a triangle and a circumscribed conic, the conic can be expressed in this simple form.

Envelop, Conic Inscribed in a Triangle

- Working out the cofactors in the determinant, we obtain the envelope equation
$$X_1^2+X_2^2+X_3^2-2X_2X_3-2X_3X_1-2X_1X_2=0$$
or
$$X_1^{1/2}\pm X_2^{1/2}\pm X_3^{1/2}=0.$$

- Dually, a conic inscribed in the triangle of reference is
$$X_2X_3+X_3X_1+X_1X_2=0$$
or
$$x_1^{1/2}\pm x_2^{1/2}\pm x_3^{1/2}=0.$$

Subsection 7

The Analytic Geometry PG$(2, q)$

The Projective Plane PG$(2,q)$

- If q is any power of a prime, there is a field having just q elements.

  A famous theorem tells us that a finite field must necessarily be commutative.
- We saw that our coordinates can belong to any such field, provided q is odd (i.e., not a power of 2), since, if $q = 2^k$, the determinant of the coordinates of A, B, C, being twice the determinant of the coordinates of P, Q, R, must be zero.
- We also saw that all the points on the general line $(y)(z)$ can be expressed in the form $(\mu y + z)$, where μ runs over all the elements of the field and the extra element ∞, which yields (y).

  Hence the field with q elements (q a power of an odd prime) yields the finite projective plane with $q+1$ points on each line, that is, PG$(2,q)$.
- In an n-dimensional geometry such as PG(n, q), a point has n + 1 coordinates.

Number of Points and Number of Lines

- In PG(2, q), the number of coordinate symbols (x_1, x_2, x_3), with q possible values for each x_i, is q^3.

  From this number we subtract 1, since the symbol (0,0,0) has no geometric meaning.

  Moreover, each point (x) is the same as (λx), for $q-1$ values of λ, namely all the nonzero elements of the field.

  We compute again that the number of points in the plane is

$$\frac{q^3-1}{q-1} = q^2 + q + 1.$$

- For each point (x), there is a corresponding line $[x]$;

  So the number of lines is the same.

Subsection 8

Cartesian Coordinates

Affine Geometry

- The analytic treatment can also be carried out in two dimensions.
- Consider the affine plane, that is, the ordinary plane of elementary geometry, in which two lines are said to be **parallel** if they do not meet.
- We regard this as **part** of the projective plane, namely, the projective plane minus one line: "the line at infinity".
- Two lines are said to be **parallel** if they meet on this special line.
- The apparent inconsistency, of saying that parallel lines meet and yet do not meet, is resolved by regarding the affine plane as being derived from the projective plane by omitting the special line (and all its points) while retaining the consequent concept of parallelism.

  This modification of projective geometry is called **affine geometry**.

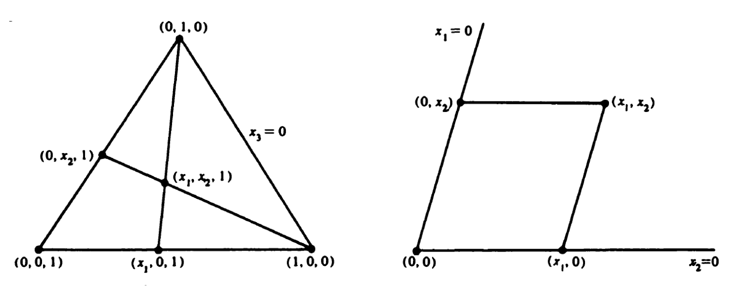
Affine Coordinates

- When coordinates are used, it is convenient to take the line at infinity to be $[0,0,1]$ or $x_3 = 0$, so that the points at infinity are just all the points (x) for which the third coordinate is zero.
- The points of the affine plane are thus all the points (x) for which the third coordinate is not zero.
- By a suitable multiplication (if necessary), any such point can be expressed in the form $(x_1, x_2, 1)$, which can be shortened to (x_1, x_2).
- The two numbers x_1 and x_2 are called the **affine coordinates** of the point.
- In other words, if $x_3 \neq 0$, the point (x_1, x_2, x_3) of the projective plane can be regarded as the point $(\frac{x_1}{x_3}, \frac{x_2}{x_3})$ of the affine plane.

Equations of Loci

- The equation of any locus can be made into the corresponding equation in affine coordinates by setting $x_3 = 1$.
- In particular:
  - The line $[X]$ has the equation $X_1 x_1 + X_2 x_2 + X_3 = 0$;
  - A pencil of parallel lines is obtained by fixing X_1 and X_2 (or, more precisely, the ratio $X_1 : X_2$) while allowing X_3 to take various values.

The Triangle of Reference



- The first two sides of the triangle of reference have become the coordinate axes:
  - The x_2-axis $x_1 = 0$ (along which x_2 varies);
  - The x_1-axis $x_2 = 0$ (along which x_1 varies).

  The third side, $x_3 = 0$, is the line at infinity.
- The first two vertices are the points at infinity on the axes: $(1,0,0)$ on the x_1-axis; $(0,1,0)$ on the x_2-axis.

  The third vertex is the origin $(0,0)$, where the two axes meet.

Central Dilatations and Translations

- The homology $\rho x_1' = x_1,\ \rho x_2' = x_2,\ \rho x_3' = \mu^{-1} x_3$, becomes a transformation of affine coordinates
$$x_1' = \mu x_1, \quad x_2' = \mu x_2,$$
when we set $x_3 = x_3' = 1$, which requires $\rho = \mu^{-1}$.
- Similarly, the elation $\rho x_1' = x_1 + a_1 x_3,\ \rho x_2' = x_2 + a_2 x_3,\ \rho x_3' = x_3$, with $\rho = 1$, becomes
$$x_1' = x_1 + a_1, \quad x_2' = x_2 + a_2.$$
- In either case, every line is transformed into a parallel line, i.e., directions are preserved.
    - The homology leaves the origin invariant and multiplies the coordinates of every point by μ; we call this a **central dilatation**.
    - The elation, leaving no (proper) point invariant, is a **translation** (or "**parallel displacement**").
- These two affine transformations enable us to define relative distances along one line, or along parallel lines; but affine geometry provides no comparison for distances in different directions.

Hyperbolas, Parabolas and Ellipses

- A conic is called a **hyperbola**, a **parabola**, or an **ellipse**, according as the line at infinity is a secant, a tangent, or a nonsecant.
- This agrees with the classical definitions:
  - A hyperbola "goes off to infinity" in two directions;
  - A parabola in one direction;
  - The ellipse not at all.
- The pole of the line at infinity is called the center of the conic.
- In the case of a hyperbola, this is an exterior point, and the two tangents that can be drawn from it are the **asymptotes**, whose points of contact are at infinity.
- We have:
  - $x_1x_2 = 1$ is a hyperbola;
  - $x_1^2 - x_2^2 = 1$ is also a hyperbola;
  - $x_2^2 = x_1$ is a parabola;
  - In real geometry, $x_1^2 + x_2^2 = 1$ is an ellipse.

From Affine to Euclidean Geometry

- To pass from affine geometry to **Euclidean geometry** we select, among all the ellipses centered at the origin, a particular one, and call it the **unit circle**.

  This provides units of measurement in all directions.
- To pass from affine coordinates to Cartesian coordinates we choose, as unit circle, the ellipse $x_1^2 + x_2^2 = 1$.
  - The dilatation $x_1' = \mu x_1$, $x_2' = \mu x_2$ transforms this into a circle of radius μ:
  $$x_1^2 + x_2^2 = \mu^2.$$
  - The translation $x_1' = x_1 + a_1$, $x_2' = x_2 + a_2$ then yields the general circle
  $$(x_1 - a_1)^2 + (x_2 - a_2)^2 = \mu^2.$$