Introduction to Projective Geometry

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 400

1 Triangles and Quadrangles
- Axioms
- Simple Consequences of the Axioms
- Perspective Triangles
- Quadrangular Sets
- Harmonic Sets

Subsection 1

Axioms

The Axioms

- We assume the three primitive concepts **point**, **line**, and **incidence**.
- We have already defined the words **plane**, **quadrangle**, and **projectivity** in terms of the primitive concepts.

Axiom 1 There exist a point and a line that are not incident.

Axiom 2 Every line is incident with at least three distinct points.

Axiom 3 Any two distinct points are incident with just one line.

Axiom 4 If A, B, C, D are four distinct points such that AB meets CD, then AC meets BD.

Axiom 5 If ABC is a plane, there is at least one point not in the plane ABC.

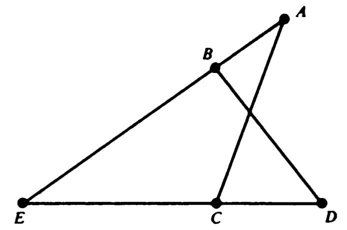
Axiom 6 Any two distinct planes have at least two common points.

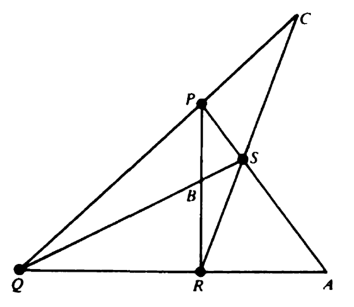
Axiom 7 The three diagonal points of a complete quadrangle are never collinear.

Axiom 8 If a projectivity leaves invariant each of three distinct points on a line, it leaves invariant all points on the line.

Illustration of Axioms 4 and 7

Axiom 4 If A, B, C, D are four distinct points such that AB meets CD, then AC meets BD.





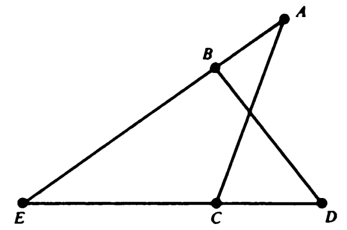
Axiom 7 The three diagonal points of a complete quadrangle are never collinear.

Subsection 2

Simple Consequences of the Axioms

On Axiom 4

- The first departure from Euclidean geometry appears in Axiom 4, which rules out the possibility that AC and BD might fail to meet by being "parallel".
- The axiom is Veblen's device for declaring that any two coplanar lines have a common point before defining a plane!



Suppose $AB \cdot CD = E$. Then B lies on AE and D lies on EC. Hence, BD lies on the plane AEC.

Pencils and Planes

- Given a triangle ABC, we can define a **pencil** of lines through C as consisting of all the lines CX, where X belongs to the range of points on AB.
- The first four axioms are all that we need in order to define the plane ABC as a certain set of points and lines, namely,
  - all the points on all the lines of the pencil;
  - all the lines that join pairs of such points.
- We then find that the same plane is determined when we replace C by another one of the points, and AB by one of the lines not incident with this point.

On Axioms 5-8

- Axiom 5 makes the geometry three-dimensional.
- Axiom 6 prevents it from being four-dimensional.

  In fact, four-dimensional geometry would admit a pair of planes having only one common point!
- It follows that two distinct planes, α and β, meet in a line, which we call the line $\alpha \cdot \beta$.
- By Axiom 7, the diagonal points of a quadrangle form a triangle, called the **diagonal triangle** of the quadrangle.
- The plausibility of Axiom 8 will appear later, when we will prove, on the basis of the remaining seven axioms, that a projectivity having three invariant points leaves invariant, if not the whole line, so many of its points that they have the "appearance" of filling the whole line.

Intersection of Two Distinct Lines

Theorem

Any two distinct lines have at most one common point.

- Suppose, if possible, that two given lines have two common points A and B. Axiom 3 tells us that each line is determined by these two points. Thus, the two lines coincide, contradicting our assumption that they are distinct.

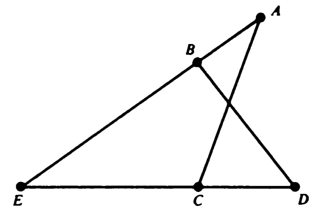
Intersection of Coplanar Lines

Theorem

Any two coplanar lines have at least one common point.

- Let E be a point coplanar with the two lines but not on either of them. Let AC be one of the lines.

  Since the plane ACE is determined by the pencil of lines through E that meet AC, the other one of the two given lines may be taken to join two points on distinct lines of this pencil, say B on EA, and D on EC.



  According to Axiom 4, the two lines AC and BD have a common point.

Intersecting Lines

Theorem

If two lines have a common point, they are coplanar.

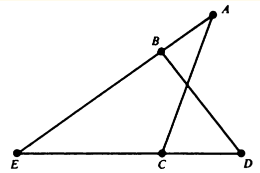
- If two lines have a common point C, we may name them AC, BC. We conclude that they lie in the plane ABC.

Four Coplanar Points in General Position

Theorem

There exist four coplanar points of which no three are collinear.

- By our first three axioms, there exist two distinct lines having a common point and each containing at least two other points, say lines EA and EC containing also B and D, respectively.



  The four distinct points A, B, C, D have the desired property of noncollinearity.

  For instance, if the three points A, B, C were collinear, E (on AB) would be collinear with all of them, and EA would be the same line as EC, contradicting our assumption that these two lines are distinct.
- Without this theorem, Axiom 7 might be "vacuous": it says that, if a complete quadrangle exists, its three diagonal points are not collinear.

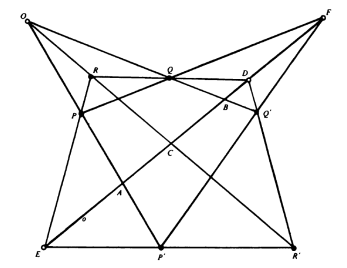
Subsection 3

Perspective Triangles

Perspective Figures

- Two ranges or pencils are said to be **perspective** if they are related by a perspectivity.
- Two plane figures involving more than one point and more than one line are said to be **perspective** if
  - their points can be put into one-to-one correspondence so that pairs of corresponding points are joined by concurrent lines, or
  - their lines can be put into one-to-one correspondence so that pairs of corresponding lines meet in collinear points.

Example: The two triangles PQR and $P'Q'R'$ are perspective: Corresponding vertices are joined by the three concurrent lines PP', QQ', RR', or since corresponding sides meet in the three collinear points $D = QR \cdot Q'R'$, $E = RP \cdot R'P'$, $F = PQ \cdot P'Q'$.



Perspectivities of Arbitrary Figures

- We will see that either kind of correspondence implies the other.
- For now, we say that two figures are:
  - **perspective from a point** O if pairs of corresponding points are joined by lines through O;
  - **perspective from a line** o if pairs of corresponding lines meet on o.
- It is sometimes convenient to call O the **center**, and o the **axis**.
- Whenever we speak of perspective figures we assume that the points, and also the lines, are all distinct;

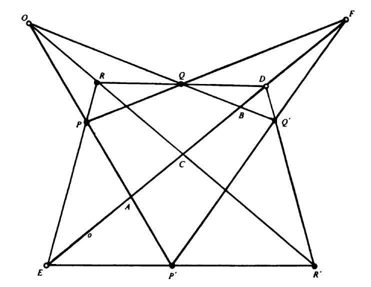
  e.g., in the case of a pair of triangles, we assume that there are six distinct vertices and six distinct sides.

Perspectivity of Triangles

Theorem

If two triangles are perspective from a line they are perspective from a point.

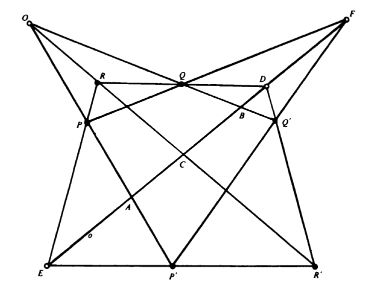
- Let two triangles, PQR and $P'Q'R'$, be perspective from a line o. In other words, let o contain three points D, E, F, such that D lies on both QR and $Q'R'$, E on both RP and $R'P'$, F on both PQ and $P'Q'$. We wish to prove that the three lines PP', QQ', RR' all pass through one point O.



  We distinguish two cases, according as the given triangles are in distinct planes or both in one plane.

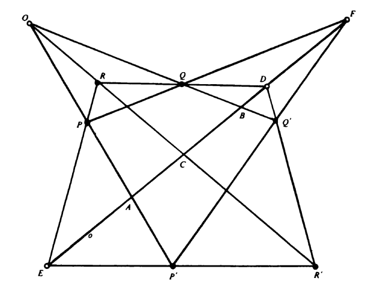
Perspectivity of Triangles Case (1)

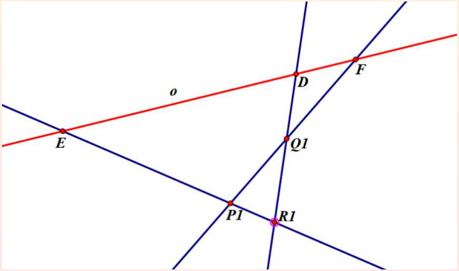
- According to Axiom 4, since QR meets $Q'R'$, QQ' meets RR'. Similarly, RR' meets PP', and PP' meets QQ'. Thus, the three lines PP', QQ', RR' all meet one another. If the planes PQR and $P'Q'R'$ are distinct, the three lines must be concurrent; for otherwise they would form a triangle, and this triangle would lie in both planes.



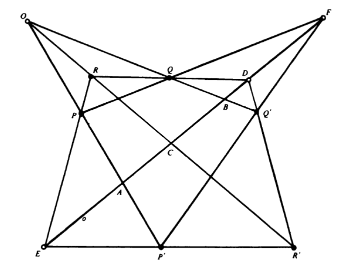
Perspectivity of Triangles Case (2)

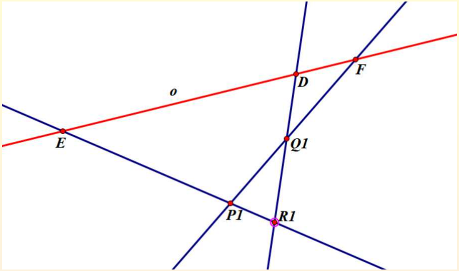
- If PQR and $P'Q'R'$ are in one plane, draw, in another plane through o, three nonconcurrent lines through D, E, F, respectively, so as to form a triangle $P_1Q_1R_1$, with Q_1R_1 through D, R_1P_1 through E, and P_1Q_1 through F. This triangle is perspective from o with both PQR and $P'Q'R'$.





Perspectivity of Triangles Case (2) (Cont'd)





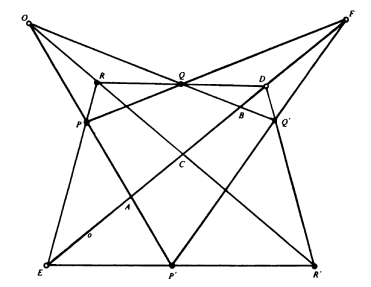
- By the result for noncoplanar triangles, the three lines PP_1, QQ_1, RR_1 all pass through one point S, and the three lines $P'P_1$, $Q'Q_1$, $R'R_1$ all pass through another point S'.
  The points S and S' are distinct; for otherwise P_1 would lie on PP' instead of being outside the original plane.
  Since P_1 lies on both PS and $P'S'$, Axiom 4 tells us that SS' meets PP'. Similarly SS' meets both QQ' and RR'. Hence, finally, the three lines PP', QQ', RR' all pass through the point $O = PQR \cdot SS'$.

Desargues's Theorem

Desargues's Theorem

If two triangles are perspective from a point they are perspective from a line.

- Let two triangles, PQR and $P'Q'R'$ (coplanar or noncoplanar) be perspective from a point O. We see from Axiom 4 that their three pairs of corresponding sides meet, say in D, E, F. It remains to be proved that these three points are collinear. Consider the two triangles $PP'E$ and $QQ'D$.



Since pairs of corresponding sides meet in the three collinear points R', R, O, these triangles are perspective from a line. Therefore, they are perspective from a point, namely, from the point $PQ \cdot P'Q' = F$. That is, the three points E, D, F are collinear.

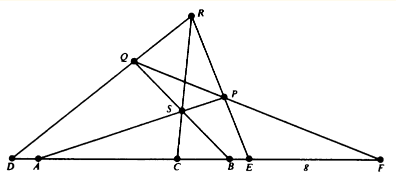
Subsection 4

Quadrangular Sets

Quadrangular Sets

- A **quadrangular set** is the section of a complete quadrangle by any line g that does not pass through a vertex.

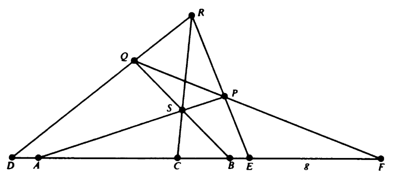
  It is thus, in general, a set of six collinear points, one point on each side of the quadrangle; but the number of points is reduced to five or four if the line happens to pass through one or two diagonal points.



- We use the symbol $(AD)(BE)(CF)$ to denote the statement that the six points A, B, C, D, E, F form a quadrangular set in the manner of the figure (that is, lying on the respective sides PS, QS, RS, QR, RP, PQ of the quadrangle), so that the first three points lie on sides through one vertex while the remaining three lie on the respectively opposite sides, which form a triangle.

The Quadrangular Set Notation

- This statement $(AD)(BE)(CF)$ is evidently unchanged if we apply any permutation to ABC and the same permutation to DEF.

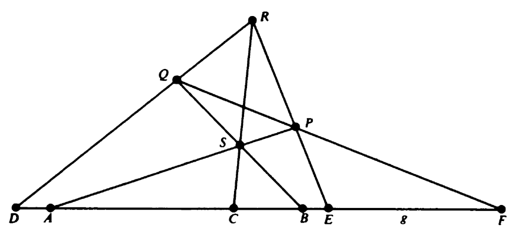


For instance, $(AD)(BE)(CF)$ has the same meaning as $(BE)(AD)(CF)$, since the quadrangle $PQRS$ can equally well be called $QPRS$.

Similarly, the statement $(AD)(BE)(CF)$ is equivalent to each of $(AD)(EB)(FC)$, $(DA)(BE)(FC)$, $(DA)(EB)(CF)$.

Five Collinear Points

- Any five collinear points A, B, C, D, E may be regarded as belonging to a quadrangular set.



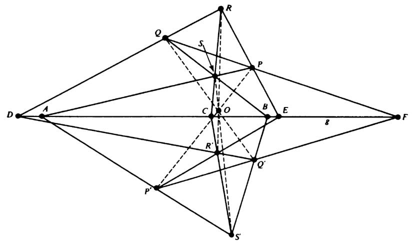
Draw a triangle QRS whose sides RS, SQ, QR pass, respectively, through C, B, D. (These sides may be any three nonconcurrent lines through C, B, D.) We can now construct $P = AS \cdot ER$ and $F = g \cdot PQ$.

Five Points Determine a Quadrangular Set

Theorem

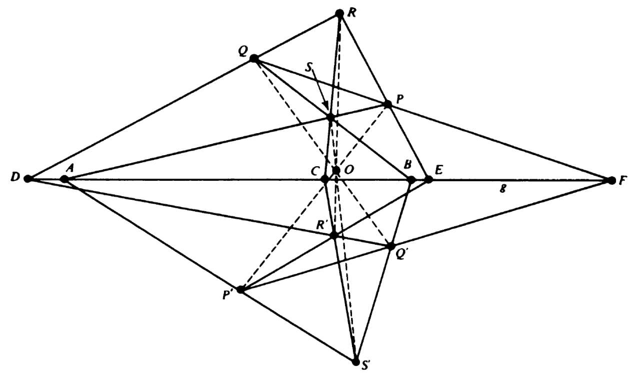
Each point of a quadrangular set is uniquely determined by the remaining points.

- To show that F is uniquely determined by A, B, C, D, E we set up another quadrangle $P'Q'R'S'$ whose first five sides pass through the same five points on g. Since the two triangles PRS and $P'R'S'$ are perspective from g, they are also perspective from a point.



  Thus, the line PP' passes through the point $O = RR' \cdot SS'$. Similarly, the perspective triangles QRS and $Q'R'S'$ show that QQ' passes through this same point O. (In other words, $PQRS$ and $P'Q'R'S'$ are perspective quadrangles.)

Five Points Determine a Quadrangular Set (Cont'd)



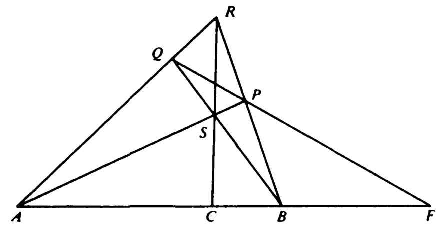
The triangles PQR and $P'Q'R'$, which are perspective from the point O, are also perspective from the line DE, which is g; that is, the sides PQ and $P'Q'$ both meet g in the same point F.

Subsection 5

Harmonic Sets

Harmonic Sets

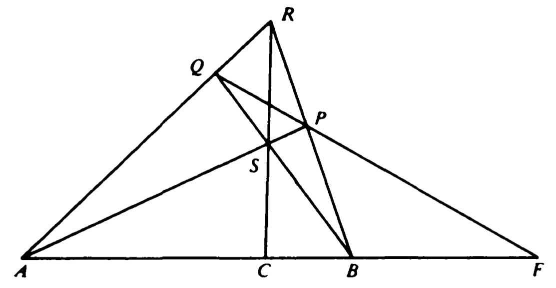
- A **harmonic set** of four collinear points may be defined to be the special case of a quadrangular set when the line g joins two diagonal points of the quadrangle.



- Because of the importance of this special case, we write the relation $(AA)(BB)(CF)$ in the abbreviated form $H(AB, CF)$.
- This has the same meaning as $H(BA, CF)$ or $H(AB, FC)$ or $H(BA, FC)$, namely that A and B are two of the three diagonal points of a quadrangle while C and F lie, respectively, on the sides that pass through the third diagonal point.
- We call F the **harmonic conjugate of C with respect to A and B**. Of course also C is the harmonic conjugate of F.

Harmonic Sets Determined by Three Points

- By a preceding theorem, F is uniquely determined by A, B, C.



For a simple construction, draw a triangle QRS whose sides QR, QS, RS pass through A, B, C. Then $P = AS \cdot BR$ and $F = AB \cdot PQ$.

The Fourth Point of a Harmonic Set

Theorem

If A, B, C are all distinct, the relation $H(AB, CF)$ implies that F is distinct from C.

- Axiom 7 implies that C and F are distinct, except in the degenerate case when they coincide with A or B.
- It follows that there must be at least four points on every line.