Introduction to Projective Geometry

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

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Subsection 1

The Axiomatic Basis of Duality

Restricting to Two Dimensions

- The geometry of points on a line is said to be **one-dimensional**.
- The geometry of points and lines in a plane is said to be **two-dimensional**.
- The geometry of points, lines, and planes in space is said to be **three-dimensional**.
- Consideration of the third dimension may be avoided by regarding Desargues's theorem as a new axiom, replacing the three-dimensional Axioms 5 and 6.
- For a purely two-dimensional theory, we can replace the three Axioms 1, 2, 4 by the following two simpler statements:

  New Axiom 1 Any two lines are incident with at least one point.

  New Axiom 2 There exist four points of which no three are collinear.

Axioms for the Projective Plane

Axiom 3 Any two distinct points are incident with just one line.

New Axiom 1 Any two lines are incident with at least one point.

New Axiom 2 There exist four points of which no three are collinear.

Axiom 7 The three diagonal points of a complete quadrangle are never collinear.

Desargues If two triangles are perspective from a point they are perspective from a line.

Axiom 8 If a projectivity leaves invariant each of three distinct points on a line, it leaves invariant every point on the line.

Principle of Duality and Self-Dual Figures

- **The Two-Dimensional Principle of Duality**:

  Every definition remains significant, and every theorem remains true, when we interchange the words point and line (and consequently also certain other pairs of words such as join and meet, collinear and concurrent, vertex and side, and so forth).

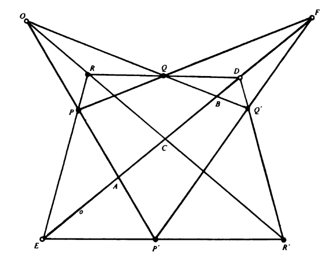
  Example: The dual of the point $AB \cdot CD$ is the line $(a \cdot b)(c \cdot d)$.

  Since duality interchanges joining and meeting, it requires not only the interchange of capital and small letters but also the removal of any dots that are present and the insertion of dots where they are absent.
- The dual of a quadrangle (with its three diagonal points) is a quadrilateral (with its three diagonal lines).
- The dual of a triangle (consisting of its vertices and sides) is again a triangle (consisting of its sides and vertices).

  Thus, a triangle is an instance of a **self-dual figure**.

Establishing the Principle of Duality

- Axiom 3 and New Axiom 1 clearly imply their duals.
- To prove the dual of New Axiom 2, consider the sides PQ, QR, RS, SP of the quadrangle $PQRS$ that is given by the axiom itself.
- Similarly, the duals of Axioms 7 and 8 present no difficulty.
- The dual of Desargues is its converse, which can be proved by applying Desargues to the triangles $PP'E$ and $QQ'D$ in the figure.
- Thus all the axioms for the projective plane imply their duals.



- After using the axioms and their consequences in proving a given theorem, we can immediately assert the dual theorem.
  A proof of the dual theorem mechanically by dualizing each step in the proof of the original theorem.

Subsection 2

The Desargues Configuration

Point-Line Configurations

- A set of m points and n lines in a plane is called a **configuration** (m_c, n_d) if:
  - c of the n lines pass through each of the m points;
  - d of the m points lie on each of the n lines.
- The four numbers are not independent but satisfy the equation

$$cm = dn.$$

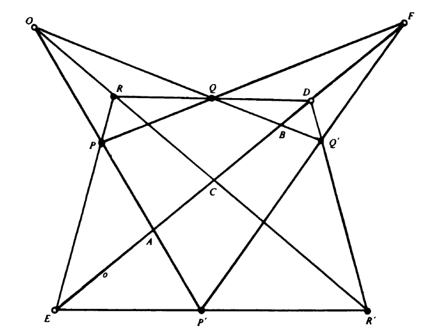
- The **dual** of (m_c, n_d) is (n_d, m_c).

  Example: $(4_3, 6_2)$ is a quadrangle and $(6_2, 4_3)$ is a quadrilateral.
- In the case of a **self-dual configuration**, we have $m = n$, $c = d$, and the symbol (n_d, n_d) is conveniently abbreviated to n_d.

  Example: 3_2 is a triangle.

A Self-Dual Configuration 10_3

- Desargues's theorem establishes the existence of a self-dual configuration 10_3: Ten points and ten lines, with three points on each line and three lines through each point.



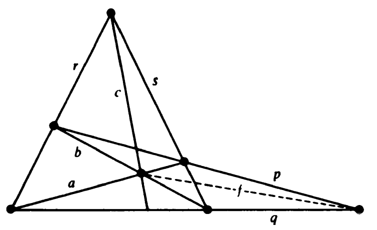
In fact, the ten points P, Q, R, P', Q', R', D, E, F, O lie by threes on ten lines, as follows: $DQ'R'$, $ER'P'$, $FP'Q'$, DQR, ERP, FPQ, OPP', OQQ', ORR', DEF.

Subsection 3

The Invariance of the Harmonic Relation

Harmonic Conjugate of a Line

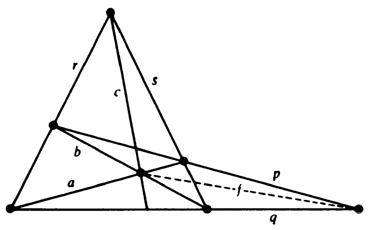
- Any three concurrent lines a, b, c determine a fourth line f, concurrent with them, which we call the **harmonic conjugate of c with respect to a and b.**
- To construct it, we draw a triangle qrs with vertices $q \cdot r$ on a, $q \cdot s$ on b, and $r \cdot s$ on c. Then $p = (a \cdot s)(b \cdot r)$, $f = (a \cdot b)(p \cdot q)$.

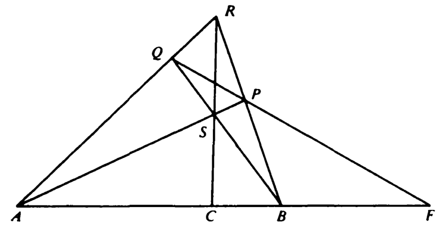


- The quadrilateral $pqrs$ has a and b for two of its three diagonal lines.
- The lines c and f, respectively, pass through the vertices that would be joined by the third diagonal line.

Harmonic Points to Harmonic Lines

- By identifying the lines p, q, r, s, a, b, c on the left with the lines $PQ, AB, QR, RP, PS, QS, RS$ on the right,





we see how the harmonic set of points $ABCF$ arises as a section of the harmonic set of lines $abcf$. Such a figure can be derived from any harmonic set of points and any point S outside their line.

Theorem

A harmonic set of points is projected from any point outside the line by a harmonic set of lines.

The Dual: Harmonic Lines to Harmonic Points

Theorem

Any section of a harmonic set of lines, by a line not passing through the point of concurrence, is a harmonic set of points.

- Combining these two dual statements, we deduce:

Theorem (Perspectivities Preserve the Harmonic Relation)

If $ABCF \doublebarwedge A'B'C'F'$, and $H(AB, CF)$, then $H(A'B', C'F')$.

- By repeated application of this principle, we deduce:

Theorem (Projectivities Preserve the Harmonic Relation)

If $ABCF \barwedge A'B'C'F'$ and $H(AB, CF)$, then $H(A'B', C'F')$.

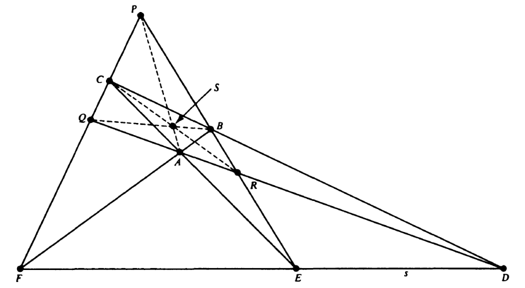
- By a previous theorem in the form $ABCF \barwedge FCBA$, we can assert: If $H(AB, CF)$ then $H(FC, BA)$, and, therefore, also $H(CF, BA)$, $H(CF, AB)$, $H(FC, AB)$.

Subsection 4

Trilinear Polarity

Poncelet's Trilinear Polar

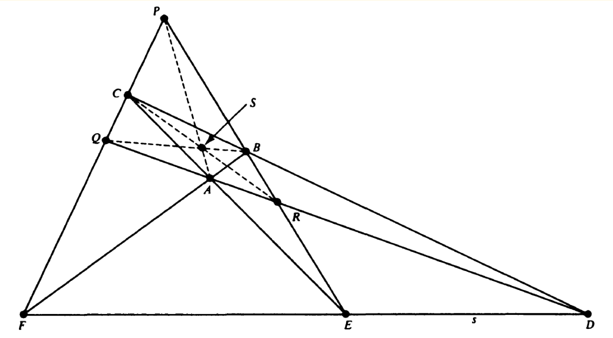
- Let PQR be a triangle, and S a point in general position. The **Cevians** SP, SQ, SR determine points A, B, C on the sides QR, RP, PQ, as in the figure.



Let D, E, F be the harmonic conjugates of these points A, B, C, so that $H(QR, AD)$, $H(RP, BE)$, $H(PQ, CF)$. These points D, E, F are the intersections of pairs of corresponding sides of the two triangles PQR and ABC, namely, $D = QR \cdot BC$, $E = RP \cdot CA$, $F = PQ \cdot AB$. Since these two triangles are perspective from S, D, E, F lie on a line s. This line is called the **trilinear polar** of S.

Poncelet's Trilinear Pole

- Given triangle PQR and any line s, not through a vertex, we can define A, B, C to be the harmonic conjugates of the points D, E, F in which s meets the sides QR, RP, PQ.



The three lines PA, QB, RC all pass through a point S, which is the **trilinear pole** of s.

Subsection 5

Harmonic Sets

Harmonic Nets

- A point P is said to be **harmonically related** to three distinct collinear points A, B, C if P belongs to a sequence of points beginning with A, B, C and proceeding according to this rule:

  Each point (after C) forms a harmonic set with three previous points. (Any three previous points can be used, in any order.)
- The set of all points harmonically related to A, B, C is called a **harmonic net** or "**net of rationality**", and is denoted by $\mathrm{R}(ABC)$.
- The same harmonic net is also denoted by $\mathrm{R}(BAC)$ or $\mathrm{R}(BCA)$, etc.
- Since a projectivity transforms any harmonic set into a harmonic set, it also transforms any harmonic net into a harmonic net.

Theorem

If a projectivity leaves invariant each of three distinct points A, B, C on a line, it leaves invariant every point of the harmonic net $\mathrm{R}(ABC)$.

- This result, which we have deduced from our first seven axioms, may be regarded as making Axiom 8 plausible.

Characterization of a Harmonic Net

Theorem

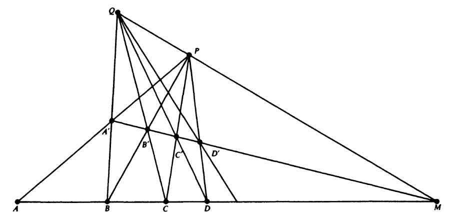
A harmonic net is equally well determined by any three distinct points belonging to it.

- By a previous theorem, any four collinear points A, B, C, D satisfy $ABCD \barwedge DCBA$. If D belongs to $\mathrm{R}(ABC)$, A must belong to $\mathrm{R}(DCB)$, which is the same as $\mathrm{R}(BCD)$. It follows that not only A, but every point of $\mathrm{R}(ABC)$, belongs also to $\mathrm{R}(BCD)$. Hence, if D belongs to $\mathrm{R}(ABC)$, $\mathrm{R}(ABC) = \mathrm{R}(BCD)$.

  By a repeated application, we see that, if K, L, M are any three distinct points of $\mathrm{R}(ABC)$, $\mathrm{R}(ABC) = \mathrm{R}(BCK) = \mathrm{R}(CKL) = \mathrm{R}(KLM)$.
- Thus, a harmonic net is the smallest set of at least three collinear points which includes, with every three of its members, the harmonic conjugate of each with respect to the other two.

Harmonic Sequences

- When the points of a line in rational geometry, or the points of a harmonic net in real geometry, are represented by the rational numbers, the natural numbers (that is, positive integers) represent a **harmonic sequence** $ABCD\ldots$, which is derived from three collinear points A, B, M by the following special procedure:
- Take P and Q on another line through M. Construct $A' = AP \cdot BQ, B' = BP \cdot A'M,$ $C = B'Q \cdot AM, C' = CP \cdot A'M,$ $D = C'Q \cdot AM, D' = DP \cdot A'M,$ and so on.



- In view of $H(BM, AC)$, $H(CM, BD)$, $\ldots$, the sequence $ABC\ldots$ depends only on the given points A, B, M, and is independent of our choice of the auxiliary points P and Q.