Introduction to Projective Geometry

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Subsection 1

How Three Pairs Determine a Projectivity

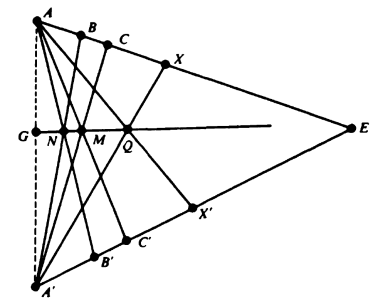
Constructing a Projectivity

- Given four distinct points A, B, C, X on one line, and three distinct points A', B', C' on the same or another line, there are many possible ways in which we may proceed to construct a point X' (on $A'B'$), such that $ABCX \barwedge A'B'C'X'$.

  Example: If the two lines are distinct, one way is

  $$ABCX \overset{A'}{\barwedge} GNMQ \overset{A}{\barwedge} A'B'C'X'.$$

  This can be varied by using B' and B (or C' and C), instead of A' and A, as centers of the two perspectivities.

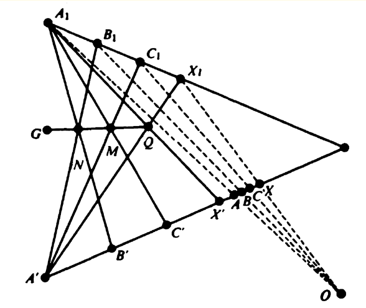


Constructing a Projectivity: Points on a Single Line

- If, on the other hand, the given points are all on one line,

  we can use:

  - an arbitrary perspectivity $ABCX \overset{=}{\barwedge} A_1 B_1 C_1 X_1$ to obtain four points on another line;
  - relate $A_1 B_1 C_1 X_1$ to $A'B'C'X'$.



Altogether, we have

$$ABCX \overset{O}{\overset{=}{\barwedge}} A_1 B_1 C_1 X_1 \overset{A'}{\overset{=}{\barwedge}} GNMQ \overset{A_1}{\overset{=}{\barwedge}} A'B'C'X'.$$

The Fundamental Theorem of Projective Geometry

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A projectivity is determined when three collinear points and the corresponding three collinear points are given.

- It suffices to show that seven points A, B, C, X, A', B', C' (with the first four, and likewise the last three, collinear and distinct) determine a unique eighth point X', such that

  $$ABCX \,\overline{\wedge}\, A'B'C'X'.$$

  If not, there must be two distinct chains of perspectivities yielding, respectively,

  $$ABCX \,\overline{\wedge}\, A'B'C'X' \quad \text{and} \quad ABCX \,\overline{\wedge}\, A'B'C'X'',$$

  where $X'' \neq X'$. Proceeding backwards along the first chain and then forwards along the second, we would have $A'B'C'X' \,\overline{\wedge}\, A'B'C'X''$, contradicting Axiom 8.

The Dual of the Fundamental Theorem

- Either of the sets of "three collinear points" (or both) can be replaced by "three concurrent lines".

  Thus each of the relations

  $$ABC \barwedge A'B'C', \quad ABC \barwedge abc, \quad abc \barwedge ABC, \quad abc \barwedge a'b'c'$$

  suffices to specify uniquely a particular projectivity.
- On the other hand, each of the relations

  $$ABCD \barwedge A'B'C'D', \quad ABCD \barwedge abcd, \quad abcd \barwedge a'b'c'd'$$

  expresses a special property of eight points, or of four points and four lines, or of eight lines, of such a nature that any seven of the eight will uniquely determine the remaining one.
- A previous theorem asserted that, if a projectivity interchanges A and B while transforming C into D, it also transforms D into C, that is, it interchanges C and D.

Subsection 2

Some Special Projectivities

Projectivity Relating Given Harmonic Sets

Theorem

Any two harmonic sets (of four collinear points or four concurrent lines) are related by a unique projectivity.

- If $\mathrm{H}(AB, CF)$ and $\mathrm{H}(A'B', C'F')$, the projectivity $ABC \barwedge A'B'C'$ transforms F into a point F'', such that $\mathrm{H}(A'B', C'F'')$. But the harmonic conjugate of C' with respect to A' and B' is unique. Therefore, F'' coincides with F'.

  The same reasoning can be used when either of the harmonic sets consists of lines instead of points.

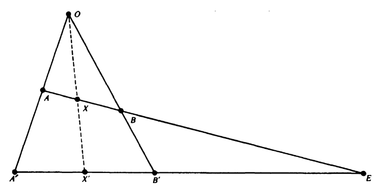
Characterization of Perspectivities

Theorem

A projectivity relating ranges on two distinct lines is a perspectivity if and only if the common point of the two lines is invariant.

- A perspectivity obviously leaves invariant the common point of the two lines.

  On the other hand, if a projectivity relating ranges on two distinct lines has an invariant point E, this point, belonging to both ranges, must be the common point of the two lines.



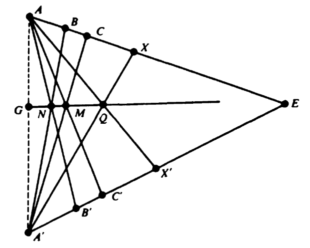
  Let A and B be two other points of the first range, A' and B' the corresponding points of the second. The fundamental theorem tells us that the perspectivity $ABE \overset{O}{\overline{\wedge}} A'B'E$, where $O = AA' \cdot BB'$, is the same as the given projectivity $ABE \,\overline{\wedge}\, A'B'E$.

Subsection 3

The Axis of a Projectivity

Perspectivities in a Projectivity Determined by 3 Pairs

- By the fundamental theorem, there is only one projectivity $ABC \barwedge A'B'C'$ relating three distinct points on one line to three distinct points on the same or any other line.
- The construction indicated in the figure shows how, when the lines AB and $A'B'$ are distinct, this unique projectivity can be expressed as the product of two perspectivities whose centers may be any pair of corresponding points (in reversed order) of the two related ranges.



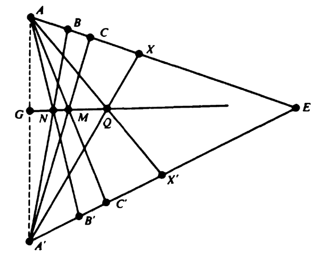
- Would different choices of the two centers yield different positions for the line MN of the intermediate range?

Uniqueness of the Axis

Theorem

Every projectivity relating ranges on two distinct lines determines another special line, the "axis", which contains the intersection of the cross-joins of any two pairs of corresponding points.

- The perspectivities from A' and A determine the axis MN, which contains the common point of the "cross-joins" of the pair AA' and any other pair; e.g., N is the common point of the cross joins (AB' and BA') of the two pairs AA' and BB'.



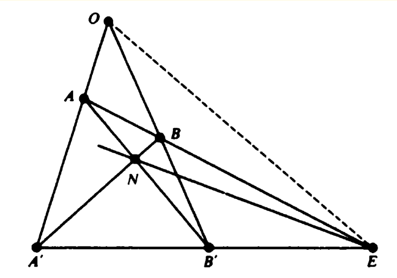
  We must prove the uniqueness of this axis, i.e. that another choice of the two pairs of corresponding points (such as BB' and CC') will yield another "crossing point" on the same axis.

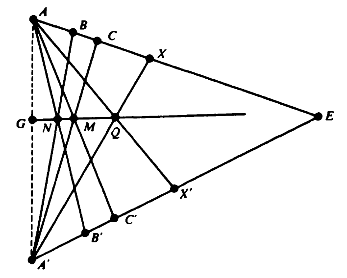
  For this purpose, we seek a new specification for the axis, independent of the particular pair AA'.

Uniqueness of the Axis (Invariant Common Point)

- Let E be the common point of the two lines.

  Suppose first that E is invariant, as in the figure on the left:

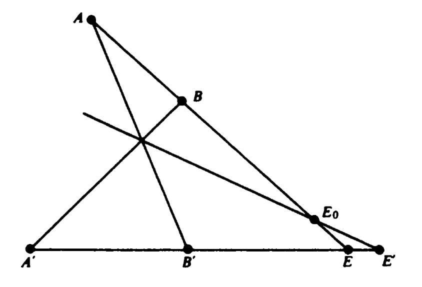


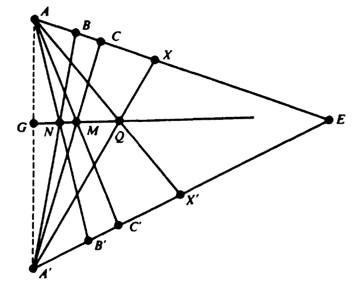


  Referring to the figure on the right, we observe that, when X coincides with E, so also do Q and X'. The axis EN is independent of AA', since it can be described as the harmonic conjugate of EO with respect to the given lines EB and EB'.

Uniqueness of the Axis (Non-Invariant Common Point)

- If the common point E is noninvariant, as in the figure on the left,



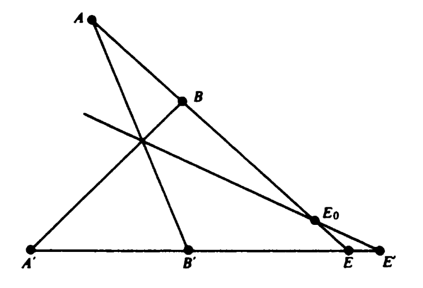


it is still a point belonging to both ranges. Referring to the figure on the right, we observe that, when X coincides with E, Q and X' both coincide with the corresponding point E'. Hence the axis passes through E'. For a similar reason the axis also passes through the point E_0 of the first range for which the corresponding point of the second is E. Hence, the axis can be described as the join E_0E'.

A Final Remark

Proposition

If E_0EE' is a triangle, the axis of the projectivity $AE_0E \barwedge A'EE'$ is the line E_0E'.



The cross-join of the pairs AA' and EE_0 is E_0.
The cross-join of the pairs AA' and EE' is E'.
Hence, the axis of projectivity is the line E_0E'.

Subsection 4

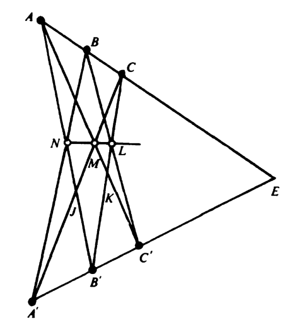
Pappus and Desargues

Pappus's Theorem

Pappus's Theorem

If the six vertices of a hexagon lie alternately on two lines, the three pairs of opposite sides meet in collinear points.

- Let the hexagon be $AB'CA'BC'$. Since alternate vertices are collinear, there is a projectivity $ABC \barwedge A'B'C'$. The pairs of opposite sides, namely $B'C, BC'$; $C'A, CA'$; $A'B, AB'$, are just the cross-joins of the pairs of corresponding points BB', CC'; CC', AA'; AA', BB'. By a previous theorem, their points of intersection $L = B'C \cdot BC'$, $M = C'A \cdot CA'$, $N = A'B \cdot AB'$ all lie on the axis of the projectivity.



Pappus's Theorem (Alternate Proof)

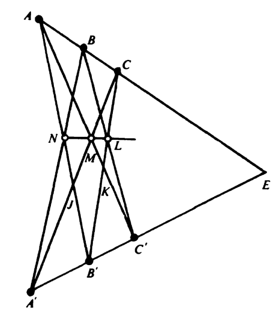
- Using further points $J = AB' \cdot CA'$, $E = AB \cdot A'B'$, $K = AC' \cdot CB'$, we have

$$ANJB' \overset{A'}{\overline{\overline{\wedge}}} ABCE \overset{C'}{\overline{\overline{\wedge}}} KLCB'.$$

Thus B' is an invariant point of the projectivity $ANJ \overline{\wedge} KLC$. By a previous theorem, this projectivity is a perspectivity, namely

$$ANJ \overset{M}{\overline{\overline{\wedge}}} KLC.$$

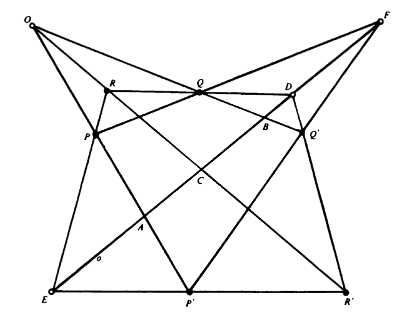
Thus M lies on NL, i.e., L, M, N are collinear.



Second Proof of Deargues' Theorem

- Recall that, without using Desargues, it was shown that a projectivity relating ranges on two distinct lines is a perspectivity if and only if the common point of the two lines is invariant.
- Assume that the lines PP', QQ', RR' all pass through O. Define $D = QR \cdot Q'R'$, $E = RP \cdot R'P'$, $F = PQ \cdot P'Q'$. Further, define points $A = OP \cdot DE$, $B = OQ \cdot DE$, $C = OR \cdot DE$. Then

$$OPAP' \overset{E}{\overline{\overline{\wedge}}} ORCR' \overset{D}{\overline{\overline{\wedge}}} OQBQ'.$$

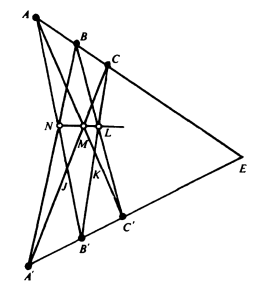


So, O is an invariant point of the projectivity $PAP' \overline{\wedge} QBQ'$. Thus, this projectivity is a perspectivity. Its center, F, lies on AB, which is DE. Thus, D, E, F are collinear.

The Self-Dual Configuration 9_3

- Desargues's theorem is a selfdual configuration 10_3.
- Somewhat analogously, the figure for Pappus's theorem is a self-dual configuration 9_3: nine points and nine lines, with three points on each line and three lines through each point.

  This fact becomes evident as soon as we have made the notation more symmetrical by calling the nine points $A_1 = A$, $B_1 = B$, $C_1 = C$, $A_2 = A'$, $B_2 = B'$, $C_2 = C'$, $A_3 = L$, $B_3 = M$, $C_3 = N$ and the nine lines $a_1 = BL$, $b_1 = AM$, $c_1 = A'B'$, $a_2 = CM$, $b_2 = CL$, $c_2 = AB$, $a_3 = AN$, $b_3 = BN$, $c_3 = LM$.



- The three triangles

$$A_1B_1C_2, A_2B_2C_3, A_3B_3C_1 \text{ or } a_1b_1c_2, a_3b_3c_1, a_2b_2c_3$$

  provide an instance of **Graves triangles**: A cycle of three triangles, each inscribed in the next.