Introduction to Projective Geometry

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 400

Subsection 1

Superposed Ranges

Classification of One-Dimensional Projectivities

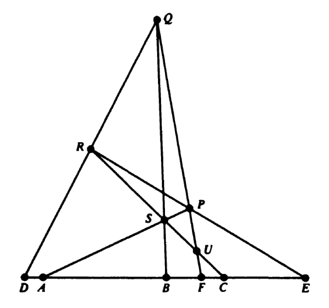
- Axiom 8 tells us that a projectivity relating two ranges on one line (that is, a projective transformation of the line into itself) cannot have more than two invariant points without being the identity, which relates each point to itself.
- The projectivity is said to be **elliptic**, **parabolic**, or **hyperbolic** according as the number of invariant points is 0, 1, or 2.
- We will see that both parabolic and hyperbolic projectivities always exist.

Hyperbolic and Parabolic Projectivities

- The figure suggests a simple construction for the hyperbolic or parabolic projectivity

$$AEC \barwedge BDC$$

with A, B, C, D, E any five collinear points and C invariant. We draw a quadrangle $PQRS$ as if we were looking for the sixth point of a quadrangular set.

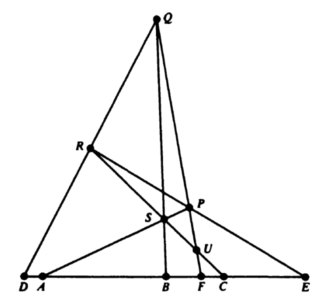


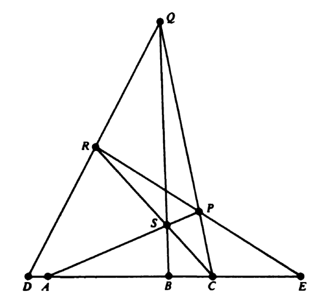
The given projectivity can be expressed as the product of two perspectivities

$$AEC \overset{P}{\overline{\overline{\wedge}}} SRC \overset{Q}{\overline{\overline{\wedge}}} BDC,$$

and it is easy to see what happens to any other point on the line.

Hyperbolic and Parabolic Projectivities (Cont'd)





- C (on RS) is invariant. If any other point is invariant, it must be collinear with the centers P and Q of the two perspectivities, i.e., it can only be F. Hence, the projectivity $AEC \barwedge BDC$ is:
  - hyperbolic if C and F are distinct; We write $AECF \barwedge BDCF$;
  - parabolic if they are coincident; We write $AECC \barwedge BDCC$, the repeated letter indicating the parabolic nature.

Using the Quadrangular Set Notation

Proposition

The two statements $AECF \barwedge BDCF$ and $(AD)(BE)(CF)$ are equivalent, not only when C and F are distinct, but also when they coincide.

- The statement $AECF \barwedge BDCF$ involves C and F symmetrically.

  So the statement $(AD)(BE)(CF)$ is equivalent to $(AD)(BE)(FC)$.

  Similarly, $(AD)(BE)(CF)$ is equivalent to to $(AD)(EB)(FC)$ and to $(DA)(EB)(FC)$.

  This is remarkable because, when the quadrangular set is derived from the quadrangle, the two triads ABC and DEF arise differently:

  - the first from three sides with a common vertex;
  - the second from three that form a triangle.

  Whereas one way of matching two quadrangles uses only Desargues's theorem, the other needs the fundamental theorem.

Subsection 2

Parabolic Projectivities

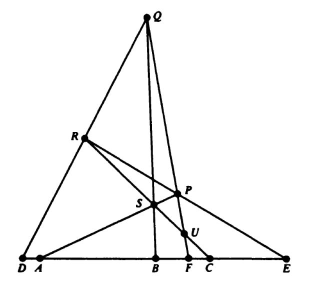
Hyperbolic Projectivities

- The fundamental theorem shows that a hyperbolic projectivity is determined when both its invariant points and one pair of distinct corresponding points are given.

  In fact, any four collinear points A, B, C, F determine such a projectivity $ACF \barwedge BCF$, with invariant points C and F.

  To construct it, we take a triangle QPS whose sides PS, SQ, QP pass, respectively, through A, B, F. Suppose the side through F meets CS in U. Then we have

  $$ACF \overset{P}{\barwedge} SCU \overset{Q}{\barwedge} BCF.$$

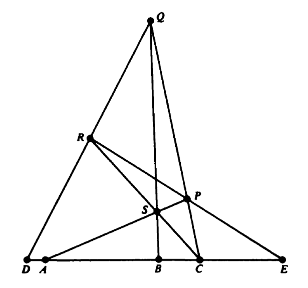


- If we regard E as an arbitrary point on the same line AB, this construction yields the corresponding point D.

Parabolic Projectivities

- The preceding construction remains effective when C, F and U coincide, i.e., when the line AB passes through the diagonal point $U = PQ \cdot RS$ of the quadrangle. The relations become

$$ACC \stackrel{P}{\overline{\wedge}} SCC \stackrel{Q}{\overline{\wedge}} BCC.$$



- Thus a parabolic projectivity is determined when its single invariant point and one pair of distinct corresponding points are given.

  We naturally call it the **projectivity** $ACC \overline{\wedge} BCC$.

Transitivity of Parabolic Projectivities

Theorem

The product of two parabolic projectivities having the same invariant point is another such parabolic projectivity (if it is not the identity).

- Clearly, the common invariant point C of the two projectivities is still invariant for the product.

  The product is therefore either parabolic or hyperbolic.

  Claim: The latter possibility is excluded.

  If any other point A were invariant for the product, the first parabolic projectivity would take A to some different point B, and the second would take B back to A. Thus, the first would be $ACC \barwedge BCC$, the second would be its inverse $BCC \barwedge ACC$. The product, thus, would not be properly hyperbolic, but merely the identity.

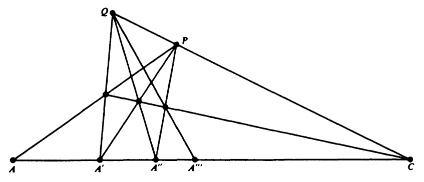
Iterating Parabolic Projectivities

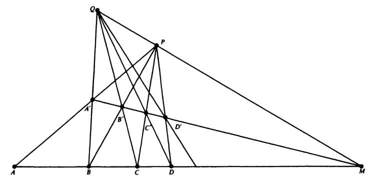
- The product of $ACC \,\overline{\wedge}\, A'CC$ and $A'CC \,\overline{\wedge}\, A''CC$ is $ACC \,\overline{\wedge}\, A''CC$;
- Moreover, we can safely write out strings of parabolic relations, such as

$$ABCC \,\overline{\wedge}\, A'B'CC \,\overline{\wedge}\, A''B''CC.$$

- In particular, by "iterating" a parabolic projectivity $ACC \,\overline{\wedge}\, A'CC$, we obtain a sequence of points $A, A', A'', \ldots$, such that

$$CCAA'A'' \ldots \,\overline{\wedge}\, CCA'A''A''' \ldots.$$

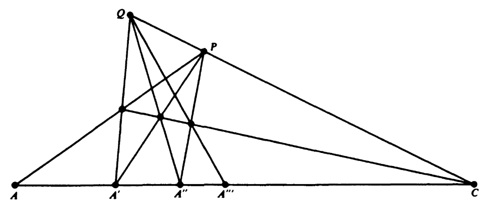




- Comparing this with the figure on the right we see that $AA'A'' \ldots$ is a harmonic sequence.

Projectivities and Quadrangular Relations

- The statements $AECF \,\overline{\wedge}\, BDCF$ and $(AD)(BE)(CF)$ are equivalent.



Setting $B = E$ and $C = F$, we deduce the equivalence of $ABCC \,\overline{\wedge}\, BDCC$ and $\mathrm{H}(BC, AD)$.

Theorem

The projectivity $AA'C \,\overline{\wedge}\, A'A''C$ is parabolic if $\mathrm{H}(A'C, AA'')$, and hyperbolic otherwise.

- In other words, the parabolic projectivity $ACC \,\overline{\wedge}\, A'CC$ transforms A' into the harmonic conjugate of A with respect to A' and C.

Subsection 3

Involutions

Involutions

- An **involution** is a projectivity of period two, that is, a projectivity which interchanges pairs of points.

Theorem

Any projectivity that interchanges two distinct points is an involution.

- Let $X \barwedge X'$ be the given projectivity which interchanges two distinct points A and A', so that $AA'X \barwedge A'AX'$, where X is an arbitrary point on the line AA'. By a previous theorem, there is a projectivity in which $AA'XX' \barwedge A'AX'X$. By the fundamental theorem, this projectivity, which interchanges X and X', is the same as the given projectivity. Since X was arbitrarily chosen, the given projectivity is an involution.

Notation on Convolutions

- Any four collinear points A, A', B, B' determine a projectivity

$$AA'B \barwedge A'AB',$$

which we now know to be an involution.

Theorem

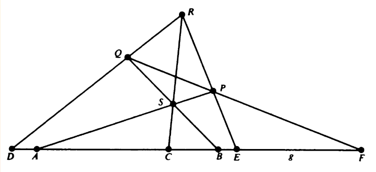
An involution is determined by any two of its pairs.

- Accordingly, it is convenient to denote the involution $AA'B \barwedge A'AB'$ by $(AA')(BB')$ or $(A'A)(BB')$, or $(BB')(AA')$, and so forth.
- This notation remains valid when B' coincides with B, i.e., the involution $AA'B \barwedge A'AB$, for which B is invariant, may be denoted by $(AA')(BB)$.

Involutions and Quadrangular Relations

Theorem

The three pairs of opposite sides of a complete quadrangle meet any line (not through a vertex) in three pairs of an involution. Conversely, any three collinear points, along with their mates in an involution, form a quadrangular set.



- Suppose $(AD)(BE)(CF)$. Consider the projectivity $AECF \overline{\wedge} BDCF$. Combine it with the involution $(BD)(CF)$. We get $AECF \overline{\wedge} BDCF \overline{\wedge} DBFC$. This shows that there is a projectivity in which $AECF \overline{\wedge} DBFC$. Since this interchanges C and F, it is an involution, $(BE)(CF)$ or $(CF)(AD)$ or $(AD)(BE)$. Thus, the quadrangular relation $(AD)(BE)(CF)$ is equivalent to the statement that the projectivity $ABC \overline{\wedge} DEF$ is an involution, or that $ABCDEF \overline{\wedge} DEFABC$.

Equivalence of Involutions

- We saw that CF is a pair of the involution $(AD)(BE)$ if and only if $AECF \barwedge BDCF$.
- Using other letters in the same context, we get, e.g., MN is a pair of the involution $(AB')(BA')$ if and only if $AA'MN \barwedge BB'MN$.
- Since $(AB')(BA')$ is the same as $(AB')(A'B)$, it follows that the two statements
$$AA'MN \barwedge BB'MN \quad \text{and} \quad ABMN \barwedge A'B'MN$$
are equivalent (here it is only the statements that are equivalent, the two projectivities being, of course, distinct).
- If two involutions, $(AA')(BB')$ and $(AA_1)(BB_1)$, have a common pair MN, we deduce $A'B'MN \barwedge BAMN \barwedge A_1B_1MN$.

Theorem

If MN is a pair of each of the involutions $(AA')(BB')$ and $(AA_1)(BB_1)$, it is also a pair of $(A'B_1)(B'A_1)$.

Equivalence of Involutions in the Parabolic Case

- The equivalences remain valid when M and N coincide, so that we are dealing with parabolic (instead of hyperbolic) projectivities.
- Thus, M is an invariant point of the involution $(AB')(BA')$ if and only if $AA'MM \barwedge BB'MM$, i.e., if and only if $ABMM \barwedge A'B'MM$.
- If M is an invariant point of each of the involutions $(AA')(BB')$ and $(AA_1)(BB_1)$, it is also an invariant point of $(A'B_1)(B'A_1)$.

Decomposition as a Product of Involutions

- If two involutions have a common pair MN, their product is evidently hyperbolic, with invariant points M and N.

  By watching their effect on A, M, N in turn, we see that the product of $(AB)(MN)$ and $(BC)(MN)$ is $AMN \barwedge CMN$.

Theorem

Any one-dimensional projectivity is expressible as the product of two involutions.

- Let the given projectivity be $ABC \barwedge A'B'C'$, where neither A nor B is invariant. Suppose D is the mate of C in $(AB')(BA')$. Consider the product of the two involutions $(AB')(BA')$ and $(A'B')(C'D)$.
  - $A \rightarrow B' \rightarrow A'$;
  - $B \rightarrow A' \rightarrow B'$;
  - $C \rightarrow D \rightarrow C'$.

  Thus $ABC \barwedge A'B'C'$ has the same effect as this product.

Subsection 4

Hyperbolic Involutions

Involutions with an Invariant Point

Theorem

Any involution that has an invariant point B has another invariant point A, which is the harmonic conjugate of B with respect to any pair of distinct corresponding points.

- Any involution that has an invariant point B (and a pair of distinct corresponding points C and C') may be expressed as $BCC' \barwedge BC'C$ or $(BB)(CC')$.

  Let A denote the harmonic conjugate of B with respect to C and C'. Then, the two harmonic sets $ABCC'$ and $ABC'C$ are related by a unique projectivity $ABCC' \barwedge ABC'C$. The fundamental theorem identifies this with the given involution.
- Thus any involution that is not elliptic is hyperbolic: there are no "parabolic involutions".

Revisiting Harmonic Conjugates

- Any two distinct points A and B are the invariant points of a unique hyperbolic involution, which is simply the correspondence between harmonic conjugates with respect to A and B.

  This is naturally denoted by $(AA)(BB)$.
- The **harmonic conjugate of C with respect to any two distinct points A and B** may now be redefined as the mate of C in the involution $(AA)(BB)$.
- This new definition remains meaningful when C coincides with A or B:

Theorem

Any point is its own harmonic conjugate with respect to itself and any other point.

Subsection 5

Projective Collineations

Two-Dimensional Projectivities

- The one-dimensional projectivity $ABC \barwedge A'B'C'$ has two different analogues in two dimensions:
  - One relates points to points and lines to lines;
  - The other relates points to lines and lines to points.
- The names **collineation** and **correlation** were introduced by Möbius in 1827, but some special collineations (such as translations, rotations, reflections, and dilatations) were considered much earlier.
- Another example is Poncelet's "homology":

  This is the relation between the central projections of a plane figure onto another plane from two different centers of perspective.

Transformations and Collineations

- By a **point-to-point transformation** $X \to X'$ we mean a rule for associating every point X with every point X' so that there is exactly one X' for each X and exactly one X for each X'.
- A **line-to-line transformation** $x \to x'$ is defined similarly.
- A **collineation** is a point-to-point and line-to-line transformation that preserves the relation of incidence.

  Thus, it transforms ranges into ranges, pencils into pencils, quadrangles into quadrangles, and so on.
- Clearly,
  - collineation is a self-dual concept;
  - the inverse of a collineation is a collineation;
  - the product of two collineations is again a collineation.

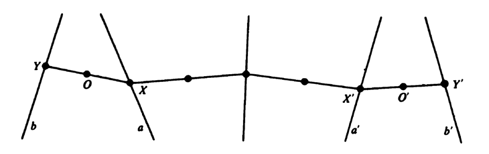
Projective Collineations

- A **projective collineation** is a collineation that transforms every one-dimensional form (range or pencil) projectively. If it transforms the points Y on a line b into the points Y' on the corresponding line b', the relation between Y and Y' is a projectivity $Y \barwedge Y'$.

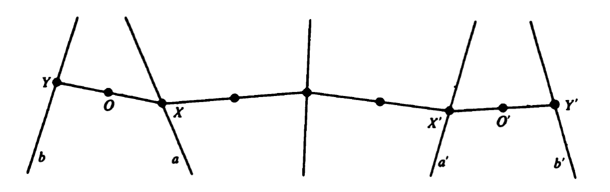
Theorem

Any collineation that transforms one range projectively is a projective collineation.

- Let a and a' be the corresponding lines that carry the projectively related ranges. We must establish the same kind of relationship between any other pair of corresponding lines, say b and b'.



Projective Collineations



- Let Y be a variable point on b, and O a fixed point on neither a nor b. Let OY meet a in X. The given collineation transforms O into a fixed point O' (on neither a' nor b'), and OY into a line $O'Y'$ that meets a' in X'. Since X is on the special line a, we have $X \barwedge X'$. Thus,

$$Y \overset{O}{\doublebarwedge} X \barwedge X' \overset{O'}{\doublebarwedge} Y',$$

so the collineation induces a projectivity $Y \barwedge Y'$ between b and b'.

Characterization of the Identity

Theorem

The only projective collineation that leaves invariant 4 lines forming a quadrilateral, or 4 points forming a quadrangle, is the identity.

- Suppose the sides of a quadrilateral are 4 invariant lines. Then the vertices (where the sides intersect in pairs) are 6 invariant points, 3 on each side. Since the relation between corresponding sides is projective, every point on each side is invariant. Any other line contains invariant points where it meets the sides and is consequently invariant. Thus, the collineation must be the identity.

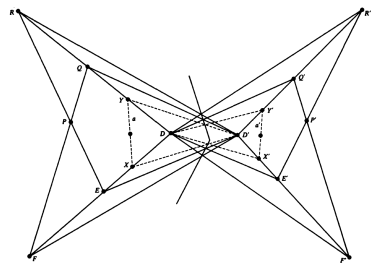
  The dual argument gives the same result when there is an invariant quadrangle.

Analog of the Fundamental Theorem

Theorem

Given any two complete quadrilaterals (or quadrangles), with their four sides (or vertices) named in a corresponding order, there is just one projective collineation that will transform the first into the second.

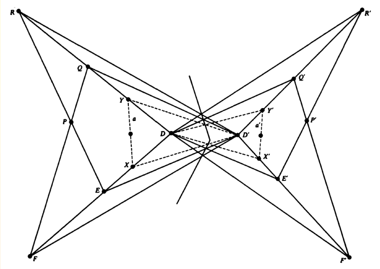
- Let $DEFPQR$ and $D'E'F'P'Q'R'$ be the two given quadrilaterals. Consider an arbitrary line a. There are certainly two sides of the first quadrilateral that meet a in two distinct points. For definiteness, suppose a is XY, with X on DE and Y on DQ.



The projectivities $DEF \barwedge D'E'F'$ and $DQR \barwedge D'Q'R'$ determine a line $a' = X'Y'$, where $DEFX \barwedge D'E'F'X'$ and $DQRY \barwedge D'Q'R'Y'$.

Analog of the Fundamental Theorem (Cont'd)

- To prove that the correspondence $a \to a'$ is a collineation, we have to verify that it also relates points to points in such a way that incidences are preserved. Let a vary in a pencil, so that $X \overline{\overline{\wedge}} Y$. By our construction for a', we now have $X' \overline{\wedge} X \overline{\overline{\wedge}} Y \overline{\wedge} Y'$.



Since D is the invariant point of the perspectivity $X \overline{\overline{\wedge}} Y$, D' must be an invariant point of the projectivity $X' \overline{\wedge} Y'$. Hence, this projectivity is again a perspectivity. Thus a', like a, varies in a pencil, i.e., concurrent lines yield concurrent lines. We have not only a line-to-line transformation but also a point-to-point transformation, preserving incidences, namely, a collineation. The projectivity $X \overline{\wedge} X'$ suffices to make it a projective collineation.

Analog of the Fundamental Theorem (Uniqueness)

- There is no other projective collineation transforming $DEFPQR$ into $D'E'F'P'Q'R'$:

  If another transformed a into a_1, the inverse of the latter would take a_1 to a. Since the original collineation takes a to a', altogether we would have a projective collineation leaving $D'E'F'P'Q'R'$ invariant and taking a_1 to a'. Thus, this combined collineation can only be the identity. So, for every a, a_1 coincides with a': the "other" collineation is really the old one over again. In other words, the projective collineation $a \to a'$ is unique.

Remarks on the Fundamental Theorem

- In the statement of the theorem, we used the phrase "named in a corresponding order".

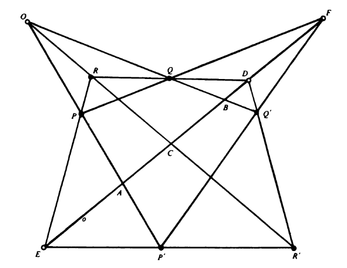
  In general, we could have permuted the sides of one of the quadrilaterals in any one of $4! = 24$ ways, obtaining not just one collineation but 24 collineations.
- We happened to use quadrilaterals, but the dual argument would immediately yield the same result for quadrangles.

Subsection 6

Perspective Collineations

Perspective Collineations

- We obtain Desargues configuration by taking two triangles PQR and $P'Q'R'$, perspective from O. There is just one projective collineation that transforms the quadrangle $DEPQ$ into $DEP'Q'$. This collineation, transforming the line $o = DE$ into itself and PQ into $P'Q'$, leaves invariant the point $o \cdot PQ = F = o \cdot P'Q'$.

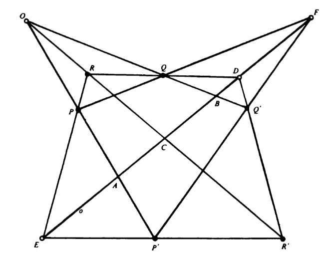


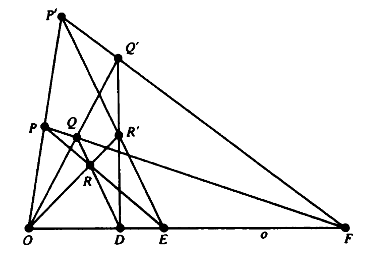
  By Axiom 8, it leaves invariant every point on o. The join of any two distinct corresponding points meets o in an invariant point, and is therefore an invariant line. The two invariant lines PP' and QQ' meet in an invariant point, namely O. The point $R = DQ \cdot EP$ is transformed into $DQ' \cdot EP' = R'$.

  By the dual of Axiom 8, every line through O is invariant.

Perspective Collineations: Homologies and Elations

- This collineation, relating two perspective triangles, is naturally called a **perspective collineation**. The point O and line o, from which the triangles are perspective, are the **center** and **axis**.





  - If O and o are nonincident, the collineation is a **homology** (Poncelet).
  - If O and o are incident, it is an **elation** (Lie).

Theorem

Any two perspective triangles are related by a perspective collineation, an elation or a homology according as the center and axis are or not incident.

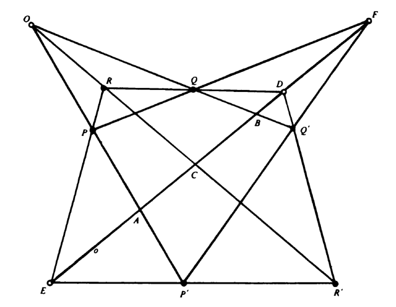
Determining a Homology

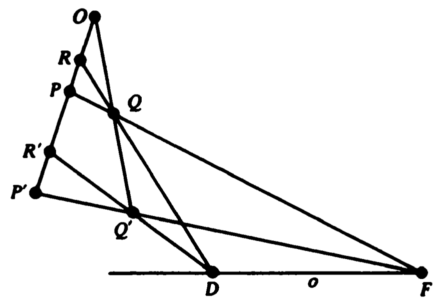
Theorem

A homology is determined when its center and axis and one pair of corresponding points (collinear with the center) are given.

- Let O be the center, o the axis, P and P' (collinear with O) the given corresponding points. We set up a construction whereby each point R yields a definite corresponding point R'.
    - If R coincides with O or lies on o, it is, of course, invariant, that is, R' coincides with R.

Determining a Homology (Cont'd)





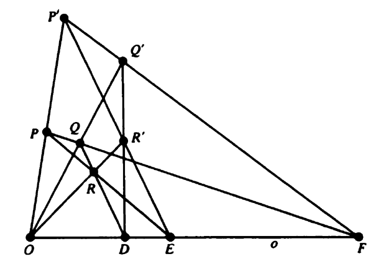
- If R is neither on o nor on OP (as on the left), take $E = o \cdot PR$ and set $R' = EP' \cdot OR$.
- If R is on OP (as on the right), we use an auxiliary pair of points Q, Q' (of which the former is arbitrary while the latter is derived from it the way we derived R' from R). Again take $D = o \cdot QR$ and set $R' = DQ' \cdot OP$.

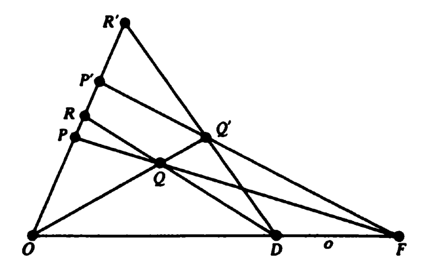
Determining an Elation

Theorem

An elation is determined when its axis and one pair of corresponding points are given.

- Let o be the axis, P and P' the given pair. Since the collineation is known to be an elation, its center is $o \cdot PP'$. We proceed as in the proof of Homology, using:





- The elation, with center $o \cdot PP'$, is denoted by $[o; P \to P']$.

Perspective Collineations

Theorem

Any collineation that has one range of invariant points (but not more than one) is perspective.

- Since the identity is (trivially) a projectivity, any such collineation is projective. There cannot be more than one invariant point outside the line o whose points are all invariant: two such would form, with two arbitrary points on o, a quadrangle left invariant, yielding the identity. If there is one invariant point O outside o, every line through O meets o in another invariant point. Hence, every line through O is invariant.
  - Any noninvariant point P lies on such a line and is therefore transformed into another point P' on this line OP. Hence, the collineation is a homology.
  - If, on the other hand, all the invariant points lie on o, any two distinct joins PP' and QQ' (of pairs of corresponding points) must meet o in the same point O. Thus, the collineation is an elation.

Invariant Points of Collineations

Corollary

If a collineation has a range of invariant points, it has a pencil of invariant lines.

Corollary

All the invariant points of an elation lie on its axis.

Corollary

For a homology, the center is the only invariant point not on the axis.

Subsection 7

Involutory Collineations

Periodic Transformations and Harmonic Homologies

- Suppose a given transformation relates a point X to X', X' to X'', X'' to X''', ..., $X^{(n-1)}$ to $X^{(n)}$. If, for every position of X, $X^{(n)}$ coincides with X itself, the transformation is said to be **periodic** and the smallest n for which this happens is called the **period**.
- The identity is of period 1.
- An involution is (by definition) of period 2.
- The projectivity $ABC \barwedge BCA$ (for any three distinct collinear points A, B, C, is of period 3.
- We know that a homology is determined by its center O, axis o, and one pair of corresponding points P, P'.

  In the special case when the harmonic conjugate of O with respect to P and P' lies on o, we speak of a **harmonic homology**.

Periodicity and Harmonicity

Theorem

A harmonic homology is determined when its center and axis are given.

- For any point P, the corresponding point P' is simply the harmonic conjugate of P with respect to O and $o \cdot OP$. Thus a harmonic homology is of period 2.

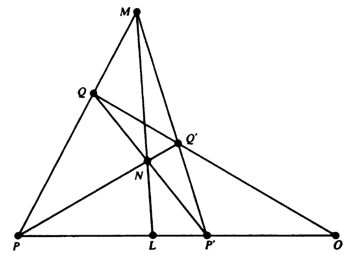
Theorem

Every projective collineation of period 2 is a harmonic homology.

- Given a projective collineation of period 2, suppose it interchanges the pair of distinct points PP' and also another pair QQ' (not on the line PP'). By a previous result, it is the only projective collineation that transforms the quadrangle $PP'QQ'$ into $P'PQ'Q$. The invariant lines PP' and QQ' meet in an invariant point O.

Periodicity and Harmonicity (Cont'd)

- The invariant lines PP' and QQ' meet in an invariant point O. The collineation interchanges the pair of lines PQ, $P'Q'$. Likewise, it interchanges the pair PQ', $P'Q$. So the two points $M = PQ \cdot P'Q'$ and $N = PQ' \cdot P'Q$ are invariant. Moreover, the two invariant lines PP' and MN meet in a third invariant point L on MN.



By Axiom 8, every point on MN is invariant. Thus, the collineation is perspective. Since, by Axiom 7, its center O does not lie on its axis MN, it is a homology. Finally, since $\mathrm{H}(PP', OL)$, it is a harmonic homology.

Subsection 8

Projective Correlations

Correlations

- We have considered the elementary point-to-line correspondence that relates a range to a pencil when the former is a section of the latter.
- We will now extend this to a transformation $X \to x'$ relating all the points in a plane to all the lines in the same plane, and its dual $x \to X'$ which relates all the lines to all the points.
- A **correlation** is a point-to-line and line-to-point transformation that preserves the relation of incidence in accordance with the principle of duality.
- Thus, a correlation transforms ranges into pencils, pencils into ranges, quadrangles into quadrilaterals, and so on.
- We have:
  - A correlation is a self-dual concept;
  - The inverse of a correlation is again a correlation;
  - The product of two correlations is a collineation.

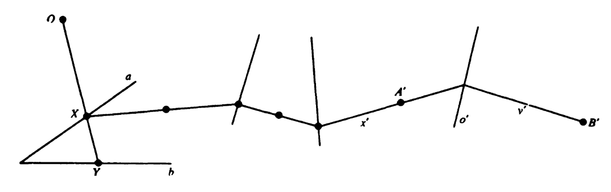
Projective Correlations

- A **projective correlation** is a correlation that transforms every one-dimensional form projectively, so that, if it transforms the points Y on a line b into the lines y' through the corresponding point B', the relation between Y and y' is a projectivity $Y \,\overline{\wedge}\, y'$.

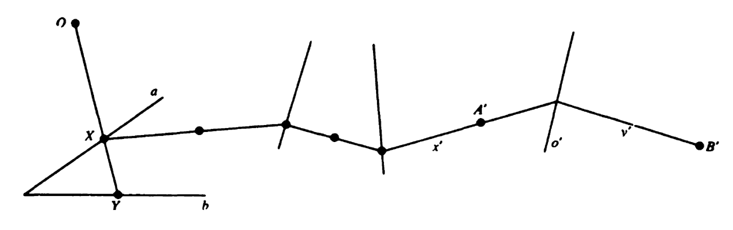
Theorem

Any correlation that transforms one range projectively is a projective correlation.

- Let a and A' be the corresponding line and point that carry the projectively related range and pencil $X \,\overline{\wedge}\, x'$. We establish the same kind of relationship between any other corresponding pair, say b, B'.



Projective Correlations



- Let Y be a variable point on b, and O a fixed point on neither a nor b. Let OY meet a in X. The given correlation transforms O into a fixed line o' (through neither A' nor B'), and OY into a point $o' \cdot y'$ which is joined to A' by a line x'. Since $Y \overset{O}{\overline{\overline{\wedge}}} X \,\overline{\wedge}\, x' \overset{o'}{\overline{\overline{\wedge}}} y'$, the correlation induces a projectivity $Y \,\overline{\wedge}\, y'$ between b and B'.

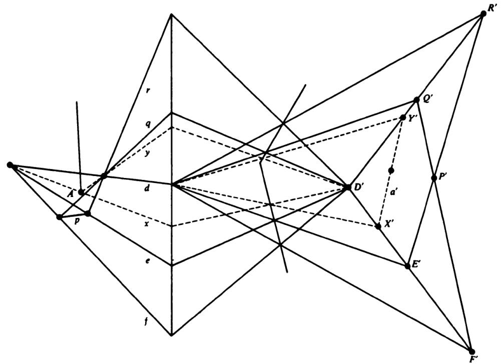
  To obtain the dual result for a pencil and the corresponding range, we regard the range of points Y on b as a section of the given pencil. This pencil yields a range which is a section of the pencil of lines y' through B'.

Determining a Projective Correlation

Theorem

A quadrangle and a quadrilateral, with the four vertices of the former associated in a definite order with the four sides of the latter, are related by just one projective correlation.

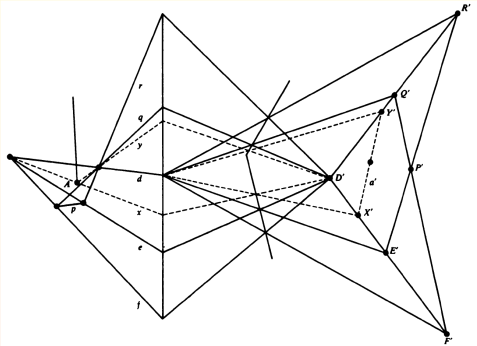
- Let $defpqr$, $D'E'F'P'Q'R'$ be the quadrangle and the quadrilateral. What effect should such a correlation have on an arbitrary point A? For definiteness, suppose A is $x \cdot y$, with
  - x through $d \cdot e$;
  - y through $d \cdot q$.



Projective Correlation: Incidence and Projectivity

- The projectivities $def \barwedge D'E'F'$ and $dqr \barwedge D'Q'R'$ determine a line $a' = X'Y'$, where $defx \barwedge D'E'F'X'$, $dqry \barwedge D'Q'R'Y'$.

  To prove that this correspondence $A \to a'$ is a correlation, we have to verify that it also relates lines to points in such a way that incidences are preserved. Let A vary in a range, so that $x \doublebarwedge y$. By our construction for a', we now have $X' \barwedge x \doublebarwedge y \barwedge Y'$.

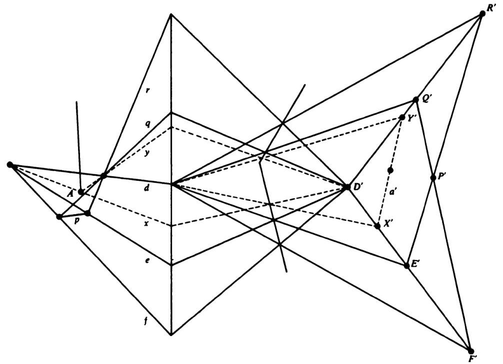


  Since d is an invariant line of the perspectivity $x \doublebarwedge y$, D' must be an invariant point of the projectivity $X' \barwedge Y'$. Thus a' varies in a pencil.

Projective Correlation (Cont'd)

- We showed that collinear points yield concurrent lines.

  We have not only a point - to - line transformation but also a line-to-point transformation, dualizing incidences.



  Hence, we obtain a correlation.

  Finally, the projectivity $x \barwedge X'$ suffices to make it a projective correlation.

Determining a Projective Correlation: Uniqueness

- There is no other projective correlation transforming *defpqr* into $D'E'F'P'Q'R'$:

  If another transformed A into a_1, the inverse of the latter would take a_1 to A. Since the original correlation takes A to a', altogether we would have a projective collineation leaving $D'E'F'P'Q'R'$ invariant and taking a_1 to a'. This establishes the uniqueness of the correlation $A \to a'$.

  The dual construction yields a projective correlation transforming a given quadrilateral into a given quadrangle.