Introduction to Projective Geometry

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LSSU Math 400

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   - The Use of a Self-Polar Pentagon
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Subsection 1

Conjugate Points and Conjugate Lines

Polarities

- A **polarity** is a projective correlation of period 2.
- In general, a correlation transforms:
  - each point A into a line a';
  - transforms this line into a new point A''.

  When the correlation is of period 2, A'' always coincides with A and we can simplify the notation by omitting the prime.
- Thus a polarity relates A to a, and vice versa.

  We call a the **polar** of A, and A the **pole** of a.
- Since this is a projective correlation, the polars of all the points on a form a projectively related pencil of lines through A.

Conjugate Points and Conjugate Lines

- If A lies on b, the polar a passes through the pole B.
  In this case we say that A and B are **conjugate points**, and that a and b are **conjugate lines**.
- It may happen that A and a are incident, so that each is **self-conjugate**: A on its own polar, and a through its own pole.
- The occurrence of self-conjugate lines (and points) is restricted by the following

Theorem

The join of two self-conjugate points cannot be a self-conjugate line.

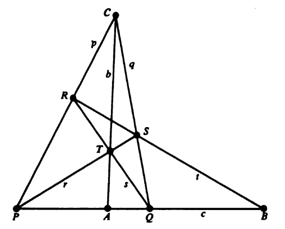
- If the join a of two self-conjugate points were a self-conjugate line, it would contain its own pole A and at least one other self-conjugate point, say B. The polar of B, containing both A and B, would coincide with a. Thus, two distinct points would both have the same polar. This is impossible, since a polarity is a one-to-one correspondence between points and lines.

Line and Self-Conjugate Points

Theorem

It is impossible for a line to contain more than two self-conjugate points.

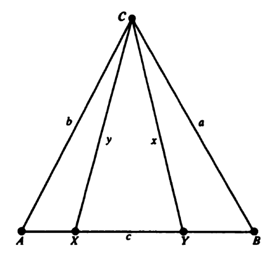
- Let p and q (through C) be the polars of two self-conjugate points P and Q on a line c. Let R be a point on p, distinct from C and P. Let its polar r meet q in S. Then $S = q \cdot r$ is the pole of $QR = s$, which meets r in T, say. Also $T = r \cdot s$ is the pole of $RS = t$, which meets c in B, say.



  Finally, $B = c \cdot t$ is the pole of $CT = b$, which meets c in A, the harmonic conjugate of B with respect to P and Q. The point B cannot coincide with Q or P. For, $B = Q$ would imply $R = C$; and $B = P$ would imply $S = C$, $r = p$, $R = P$; but we are assuming that R is neither C nor P. Hence, $A \neq B$, and B is not self-conjugate. On c, we have two self-conjugate points P, Q and a non-selfconjugate point B.

Line and Self-Conjugate Points (Cont'd)

- Since the polars of a range form a projectively related pencil, each point X on c determines a conjugate point Y on c, which is where its polar x meets c. This correspondence between X and Y is a projectivity: $X \overline{\wedge} x \overline{\wedge} Y$. When X is P, x is p, and Y is P again. Thus, P is an invariant point of this projectivity.



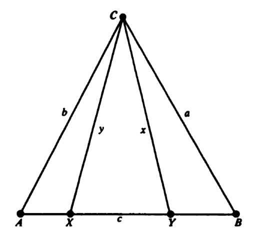
  Similarly, Q is another invariant point. But when X is B, Y is the distinct point A. Therefore, the projectivity is not the identity. By Axiom 8, P and Q are its only invariant points, that is, P and Q are the only self-conjugate points on c. This completes the proof that c cannot contain more than two self-conjugate points.

Polarities, Involutions and Self-Polar Triangles

Theorem

A polarity induces an involution of conjugate points on any line that is not self-conjugate.

- On a non-selfconjugate line c, the projectivity $X \barwedge Y$, where $Y = c \cdot x$ transforms any non-selfconjugate point B into another point $A = b \cdot c$, whose polar is BC. The same projectivity transforms A into B. Since it interchanges A and B, it must be an involution.



- Dually, the lines x and CX are paired in the involution of conjugate lines through C.
- Such a triangle ABC, in which each vertex is the pole of the opposite side (so that any two vertices are conjugate points, and any two sides are conjugate lines), is called a **self-polar triangle**.

Subsection 2

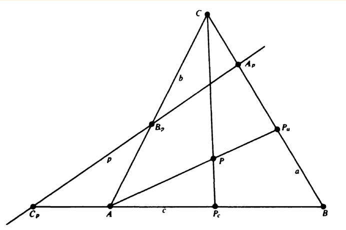
The Use of a Self-Polar Triangle

Correlations, Triangles and Polarities

Theorem

Any projective correlation that relates the three vertices of one triangle to the respectively opposite sides is a polarity.

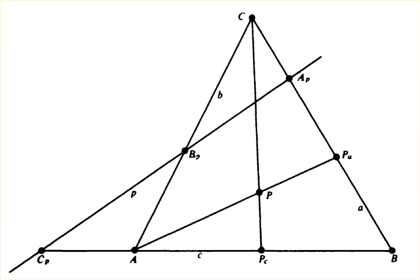
- Consider the correlation $ABCP \to abcp$, where a, b, c are the sides of the given triangle ABC and P is a point not on any of them. Then p is a line not through any of A, B, C. The point P and line p determine 6 points on the sides of the triangle: $P_a = a \cdot AP$, $P_b = b \cdot BP$, $PC = c \cdot CP$, $A_p = a \cdot p$, $B_p = b \cdot p$, $C_p = c \cdot p$. The correlation, transforming A, B, C into a, b, c, also transforms $a = BC$ into $b \cdot c = A$, AP into $a \cdot p = A_p$, $P_a = a \cdot AP$ into AA_p, and so on.



Thus, it transforms the triangle ABC in the manner of a polarity. We next show, besides transforming P into p, it also transforms p into P.

Correlations, Triangles and Polarities (Cont'd)

- The correlation transforms each point X on c into a certain line which intersects c in Y, say. Since it is a projective correlation, we have $X \barwedge Y$.
  - When X is A, Y is B;
  - When X is B, Y is A.

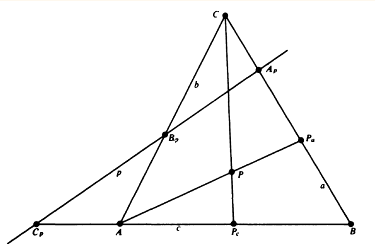


Thus the projectivity $X \barwedge Y$ interchanges A and B, and is an involution. Since the correlation transforms P_c into CC_p, the involution includes $P_c C_p$, as one of its pairs. Hence, the correlation transforms C_p into CP_c, which is CP. Similarly, it transforms A_p into AP, and B_p into BP. Therefore, it transforms $p = A_p B_p$ into $AP \cdot BP = P$, as required.

The Construction of the Polar

- We proved that the correlation $ABCP \rightarrow abcp$ is a polarity.

  An appropriate symbol, analogous to the symbol $(AB)(PQ)$ for an involution, is $(ABC)(Pp)$.
- Thus any triangle ABC, any point P not on a side, and any line p not through a vertex, determine a definite polarity $(ABC)(Pp)$, in which the polar x of an arbitrary point X can be constructed by incidences.
- This construction could be carried out by adapting the notation of the figure: $X_a = a \cdot AX$, $X_b = b \cdot BX$, $A_x = a \cdot x$, $B_x = b \cdot x$. Then A_x is the mate of X_a in the involution $(BC)(P_aA_p)$, B_x is the mate of X_b in $(CA)(P_bB_p)$, and x is A_xB_x.

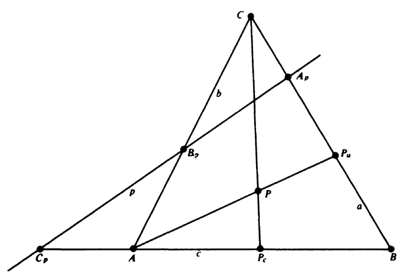


Involution Determined by Quadrangles

Theorem

In a polarity $(ABC)(Pp)$, where P is not on p, the involution of conjugate points on p is the involution determined on p by the quadrangle $ABCP$.

- Consider a polarity $(ABC)(Pp)$, in which P does not lie on p. The polars of the points $A_p = a \cdot p$, $B_p = b \cdot p$, $C_p = c \cdot p$, are AP, BP, CP. So the pairs of opposite sides of the quadrangle $ABCP$ meet the line p in pairs of conjugate points.



Subsection 3

Polar Triangles

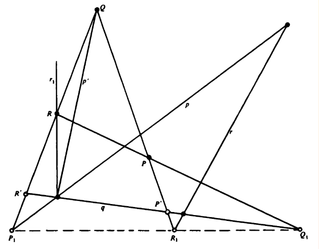
Chasles's Theorem

- From any given triangle we can derive a **polar triangle** by taking the polars of the three vertices, or the poles of the three sides.

Chasles's Theorem

If the polars of the vertices of a triangle do not coincide with the respectively opposite sides, they meet these sides in three collinear points.

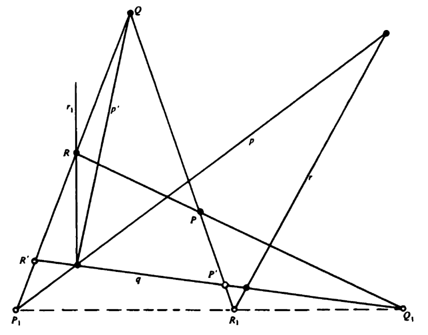
- Let PQR be a triangle whose sides QR, RP, PQ meet the polars p, q, r of its vertices in points P_1, Q_1, R_1. The polar of $R_1 = PQ \cdot r$ is $r_1 = (p \cdot q)R$. Define the extra points $P' = PQ \cdot q$, $R' = QR \cdot q$, and the polar $p' = (p \cdot q)Q$ of the former.



By a previous theorem, $R_1PP'Q \barwedge PR_1QP' \barwedge pr_1qp' \barwedge P_1RR'Q$. Since Q is invariant, $R_1PP' \doublebarwedge P_1RR'$. The center of the perspectivity, namely $PR \cdot P'R' = Q_1$, must lie on the line R_1P_1. So P_1, Q_1, R_1 are collinear.

The Exceptional Cases

- This proof breaks down if P_1 or Q lies on q.



  - In the former case, $P_1(=R')$ and $R_1(=P')$ are collinear with Q_1.
  - In the latter (when Q lies on q) we can permute the names of P, Q, R (and correspondingly p, q, r), or call the first triangle pqr and the second PQR, in such a way that the new Q and q are not incident. It is evidently impossible for each triangle to be inscribed in the other.

Subsection 4

A Construction for the Polar of a Point

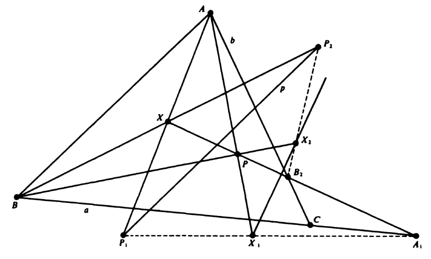
Construction for the Polar of a Point

Theorem

The polar of a point X (not on AP, BP, or p) in the polarity $(ABC)(Pp)$ is the line X_1X_2 determined by

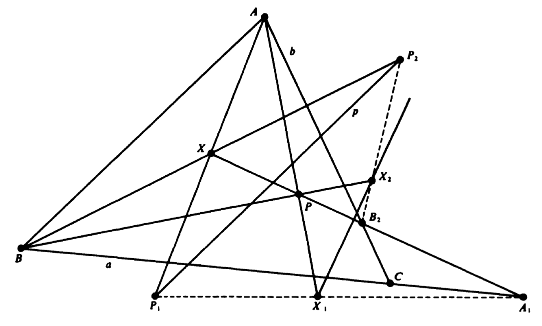
$$A_1 = a \cdot PX, \quad P_1 = p \cdot AX, \quad X_1 = AP \cdot A_1P_1,$$
$$B_2 = b \cdot PX, \quad P_2 = p \cdot BX, \quad X_2 = BP \cdot B_2P_2.$$

- Applying Chasles' Theorem to the triangle PAX, we deduce that its sides AX, XP, PA meet the polars p, a, x of its vertices in three collinear points, the first two of which are P_1, and A_1.



Hence x must meet PA in a point lying on P_1A_1, namely, in the point $PA \cdot P_1A_1 = X_1$. That is, x passes through X_1. Similarly, (by using triangle PBX instead of PAX), x passes through X_2.

Construction of the Polar: Special Case 1



- The construction fails when X lies on AP.

  Then A_1P_1 coincides with AP, and X_1, is no longer properly defined.

  However, since X_2 can still be constructed as above, the polar of X is now A_pX_2 (where $A_p = a \cdot p$).

  Similarly, when X is on BP, its polar is X_1B_p.

Construction for the Polar: Special Case 2

- Finally, to locate the polar of a point X on p, we can apply the dual of the above construction to locate the pole Y of a line y through X.

  This y may be any line through X except p or PX.

  It is convenient to choose $y = AX$ or, if this happens to coincide with PX, to choose $y = BX$.

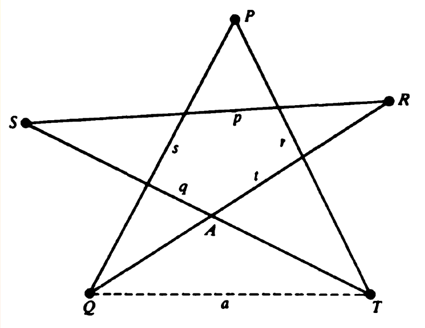
  Then the desired polar is $x = PY$.

Subsection 5

The Use of a Self-Polar Pentagon

Self-Polar Pentagons

- Instead of describing a polarity as $(ABC)(Pp)$, we can equally well describe it in terms of a **self-polar pentagon**, i.e., a pentagon in which each of the five vertices is the pole of the "opposite" side.

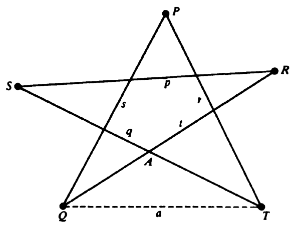


The Use of a Self-Polar Pentagon

Theorem (von Staudt)

The projective correlation that transforms four vertices of a pentagon into the respectively opposite sides is a polarity and transforms the remaining vertex into the remaining side.

- The correlation that transforms vertices Q, R, S, T of $PQRST$ into the four sides $q = ST$, $r = TP$, $s = PQ$, $t = QR$ also transforms the three sides $t = QR$, $p = RS$, $q = ST$ into the three vertices $T = q \cdot r$, $P = r \cdot s$, $Q = s \cdot t$, and the "diagonal point" $A = q \cdot t$ into the "diagonal line" $a = QT$.



Thus, it transforms each vertex of the triangle AQT into the opposite side. By the triangle Theorem, this is a polarity, namely (since it transforms p into P), the polarity $(AQT)(Pp)$.

Subsection 6

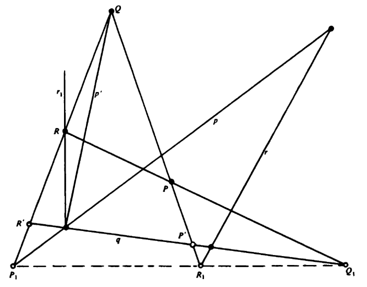
A Self-Conjugate Quadrilateral

Hesse's Theorem

Hesse's Theorem

If two pairs of opposite vertices of a complete quadrilateral are pairs of conjugate points (in a given polarity), then the third pair of opposite vertices is likewise a pair of conjugate points.

- Let $PQRP_1Q_1R_1$ be a quadrilateral, with P conjugate to P_1, and Q to Q_1. The polars p and q (of P and Q) pass through P_1 and Q_1, respectively. By Chasles's Theorem, the polar of R meets PQ in a point that lies on P_1Q_1, namely in the point $PQ \cdot P_1Q_1 = R_1$.



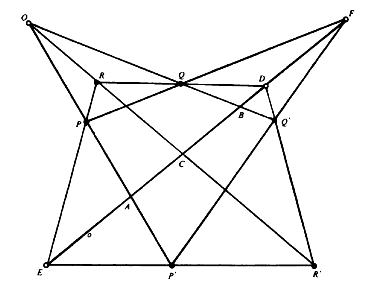
  Therefore, the polar of R passes through R_1. That is, R is conjugate to R_1.

Subsection 7

The Product of Two Polarities

A Homology as a Product of Two Polarities

- The figure shows the homology with center O and axis $o = DF$ that transforms P into P' (and consequently Q into Q'). Let p be any line not passing through a vertex of the triangle ODF. Then the given homology may be expressed as the product of two polarities $(ODF)(Pp)$ and $(ODF)(P'p)$.

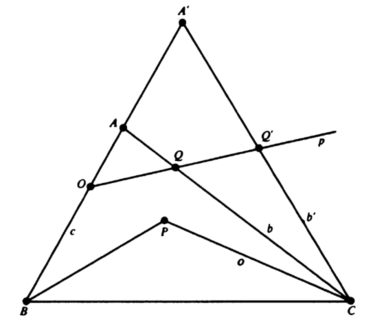


It suffices to observe that the homology and the product of polarities both transform the quadrangle $ODFP$ into $ODFP'$.

Unfortunately, this expression for a homology as the product of two polarities cannot in any simple way be adapted to an elation. We mention a subtler expression that applies equally well to either kind of perspective collineation.

A Collineation as a Product of Two Polarities

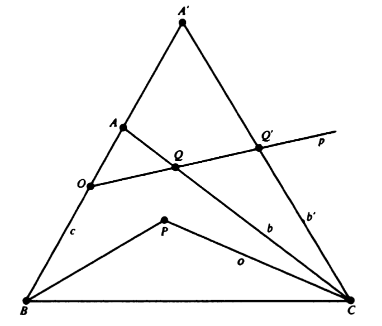
- The figure shows the homology or elation with center O and axis $o = CP$ that transforms A into another point A' on the line $c = OA$. Here C and P are arbitrary points on the axis o (passing through O if the collineation is an elation). Let p be any line through O, meeting $b = CA$ in Q and $b' = CA'$ in Q'. Let B be any point on c.



A Collineation as a Product of Two Polarities (Cont'd)

- Claim: The given perspective collineation is the product of the polarities $(ABC)(Pp)$, $(A'BC)(Pp)$.

  In fact, the first polarity transforms the four points A, P, $O = c \cdot p$, $Q = b \cdot p$ into the four lines BC, p, CP, BP; and the second transforms these lines into the four points A', P, $c \cdot p = O$, $b' \cdot p = Q'$. Thus, their product transforms the quadrangle $APOQ$ into $A'POQ'$. By a preceding result, this product is the same as the given perspective collineation.



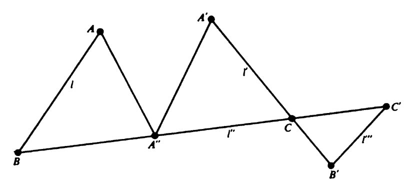
Projective Collineations as Products of Polarities

Theorem

Any projective collineation is expressible as the product of two polarities.

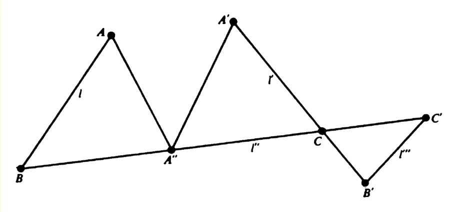
- By the preceding remarks, this is certainly true if the given collineation is perspective. We look at nonperspective collineations.

Let A be a noninvariant point, and ℓ a noninvariant line through A. Suppose the given collineation transforms A into A', A' into A'', ℓ into ℓ', ℓ' into ℓ'', and ℓ'' into ℓ'''.



Since the collineation is not perspective, we may choose A and ℓ, so that AA' is not an invariant line and $\ell \cdot \ell'$ is not an invariant point. So A'' does not lie on ℓ, nor A' on any of the three lines ℓ, ℓ'', ℓ'''. Consequently, A does not lie on ℓ' nor on ℓ''.

Projective Collineations as Products of Polarities (Cont'd)



- Let ℓ'' meet ℓ in B, ℓ' in C. The polarity $(AA''B)(A'\ell')$ transforms the four points $A, A', B, C = \ell' \cdot \ell''$ into the four lines $A''B = \ell'' = A''C$, $\ell' = CA'$, $A''A$, $A'A$. The polarity $(A'A''C)(A\ell''')$ transforms these lines into the four points A', A'', $\ell' \cdot \ell''' = B'$, $\ell'' \cdot \ell''' = C'$. Hence, their product is the same as the given collineation.

Corollary

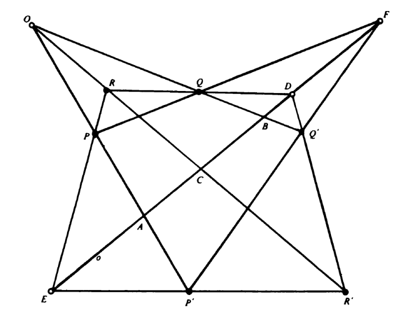
In any projective collineation, the invariant points and invariant lines form a self-dual figure.

Subsection 8

The Self-Polarity of the Desargues Configuration

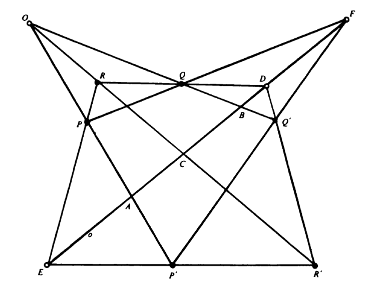
The Self-Polarity of the Desargues Configuration

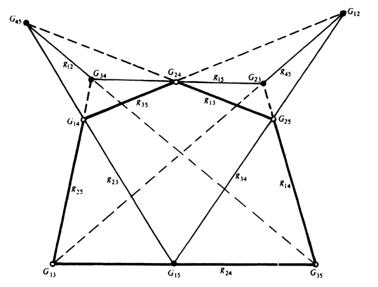
- The Desargues configuration 10_3 can be regarded as a pair of mutually inscribed pentagons, such as $FDROP'$ and $EPQQ'R'$. Any pentagon determines a polarity for which each vertex is the pole of the opposite side.



Consider the polarity for which $FDROP'$ is such a self-polar pentagon, having sides $f = RO$, $d = OP'$, $r = P'F$, $o = FD$, $p' = DR$. Since d passes through A, and f through C, the involution of pairs of conjugate points on o is $(AD)(CF)$. The quadrangle $OPQR$ yields the quadrangular relation $(AD)(BE)(CF)$. This indicates that e (the polar of E) is OB.

The Self-Polarity of the Desargues Configuration (Cont'd)





- Since Q' is $r \cdot e$, q' is RE; since P is $d \cdot q'$, p is DQ'; since R' is $f \cdot p$, r' is FP; and since Q is $p' \cdot r'$, q is $P'R'$. Thus $EPQQ'R'$ is another self-polar pentagon. Also the perspective triangles PQR and $P'Q'R'$ are polar triangles. We obtain:

Theorem

There is a unique polarity for which G_{ij} is the pole of g_{ij}.