Introduction to Projective Geometry

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LSSU Math 400

Subsection 1

How a Hyperbolic Polarity Determines a Conic

Hyperbolic and Elliptic Polarities

- Analogously with involutions, we call a polarity **hyperbolic** or **elliptic** according as it does or does not admit a self-conjugate point.
- In the hyperbolic case it also admits a self-conjugate line: The polar of the point.

  Thus, any hyperbolic polarity can be described by a symbol

  $$(ABC)(Pp),$$

  where P lies on p.

Hyperbolic Polarities and Self-Conjugate Points

- This self-conjugate point P, whose existence suffices to make the polarity hyperbolic, is not the only self-conjugate point:

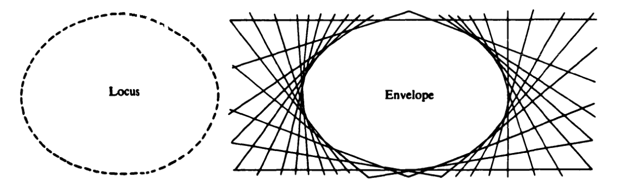
Proposition

A hyperbolic polarity has a self-conjugate point on every line through P except its polar p.

- By a previous result, the only self-conjugate point on a self-conjugate line is its pole. Dually, the only self-conjugate line through a self-conjugate point P is its polar p. By another result, it follows that every line through P, except p, is the kind of line that contains an involution of conjugate points. By another result, this involution, having one invariant point P, has a second invariant point Q which is, of course, another self-conjugate point of the polarity.

Conics and Tangents

- Thus the presence of one self-conjugate point implies the presence of many. Their locus is a **conic**, and their polars are its **tangents**.
- This simple definition exhibits the conic as a self-dual figure: The locus of self-conjugate points and also the envelope of self-conjugate lines.



  - In some geometries (e.g., complex geometry) every polarity is hyperbolic, that is, every polarity determines a conic.
  - In other geometries (e.g., real geometry) both kinds of polarity occur; then the theory of polarities is more general than the theory of conics.
- We will deal solely with hyperbolic polarities:
  - "Pole" will mean "pole with respect to a conic";
  - Instead of "conjugate for a polarity" we use "conjugate w.r.t. a conic".

Secant and Nonsecant Lines

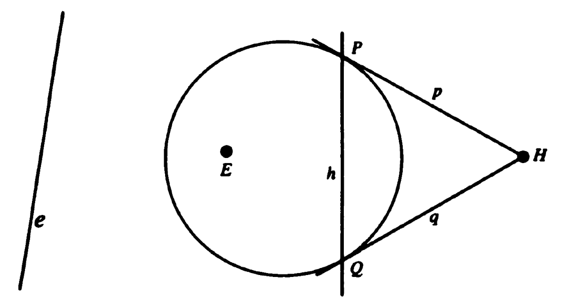
- A tangent meets the conic only at its pole: The **point of contact**.
- Any other line is called a **secant** or a **nonsecant** according as it meets the conic twice or not at all;
  - The involution of conjugate points on a secant is hyperbolic;
  - The involution of conjugate points on a nonsecant is elliptic.
- Thus:
  - Of the lines through any point P on the conic, one (namely p) is a tangent and all the others are secants;
  - If P and Q are any two distinct points on the conic, the line PQ is a secant.

Exterior and Interior Points

- Dually, a point not lying on the conic is said to be **exterior** or **interior** according as it lies on two tangents or on none;
  - The involution of conjugate lines through exterior points is hyperbolic;
  - The involution of conjugate lines through interior points is elliptic.

  Thus:
  - An exterior point H is the pole of a secant h;
  - An interior point E (if such a point exists) is the pole of a nonsecant e.
- We conclude that:
  - Of the points on a tangent p, one (namely P) is on the conic, and all the others are exterior.
  - If p and q are any two distinct tangents, the point $p \cdot q$ (which is the pole of the secant PQ) is exterior.

Illustration



- The figure helps to clarify some of these ideas.
- We must take care not to be unduly influenced by real geometry.

  E.g., in projective geometry, it is not the case that:
  - Every point on a nonsecant is exterior;
  - Every line through an interior point is a secant.

Drawing a Secant through a Given Point

- Consider the problem of drawing a secant through a given point A.
  - If A is interior, we simply join A to any point P on the conic. Since AP cannot be a tangent, it must be a secant.
  - If A is on the conic, we join it to another point on the conic.
  - Finally, if A is exterior, we join it to each of three points on the conic.

    At most two of the lines so drawn can be tangents.

    Hence, at least one must be a secant.

Secants and Harmonic Conjugate Pairs

- On a secant PQ, the involution of conjugate points is $(PP)(QQ)$. Hence, by a previous result,

Theorem

Any two conjugate points on a secant PQ are harmonic conjugates with respect to P and Q.

Theorem

On a secant PQ, any pair of harmonic conjugates with respect to P and Q is a pair of conjugate points with respect to the conic.

- Dually:

Theorem

Any two conjugate lines through an exterior point $p \cdot q$ are harmonic conjugates with respect to the two tangents p, q.
Any pair of harmonic conjugates with respect to p and q is a pair of conjugate lines with respect to the conic.

Subsection 2

The Polarity Induced by a Conic

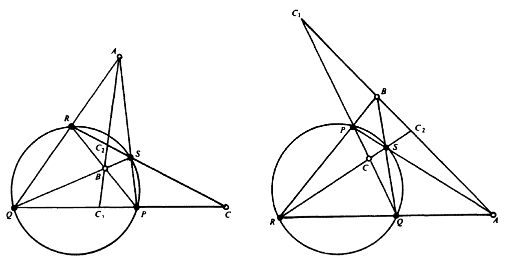
Quadrangle Inscribed in a Conic

- We saw that any hyperbolic polarity determines a conic.
- Conversely, any conic determines a hyperbolic polarity.

Theorem

If a quadrangle is inscribed in a conic, its diagonal triangle is self-polar.

- Assume the diagonal points of the inscribed quadrangle $PQRS$ be $A = PS \cdot QR$, $B = QS \cdot RP$, $C = RS \cdot PQ$. The line AB meets the sides PQ in C_1 and RS in C_2, such that $\mathrm{H}(PQ, CC_1)$ and $\mathrm{H}(RS, CC_2)$.



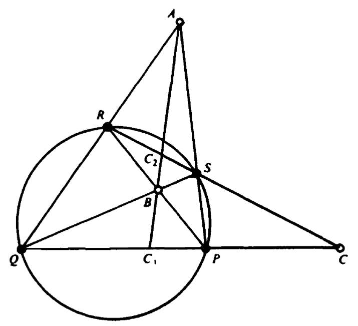
By a preceding result, both C_1 and C_2 are conjugate to C. Thus, the line AB, on which they lie, is the polar of C. Similarly, BC is the polar of A, and CA of B.

Hyperbolic Polarity Associated with a Conic

Theorem

To construct the polar of a given point C, not on the conic, draw any two secants PQ and RS through C; Then the polar joins the two points $QR \cdot PS$ and $RP \cdot QS$.

- We draw two secants through C to form an inscribed quadrangle with diagonal triangle ABC; Then the polar of C is AB.



The Dual Construction

- The dual construction is as follows:

Theorem

To construct the tangent at a given point P on the conic, join P to the pole of any secant through P.

- This construction presupposes that we know the tangents from any exterior point.
  - The tangents are not immediately apparent, since we are in the habit of dealing with loci rather than envelopes.
  - If we insist on regarding the conic as a locus, we can construct the pole of a given line as the common point of the polars of any two points on the line.
- The preceding constructions justify the statement that any conic determines a hyperbolic polarity whose self-conjugate points are the points on the conic.

Subsection 3

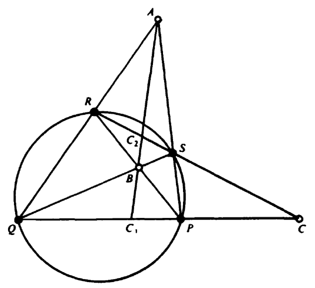
Projectively Related Pencils

Seydewitz's Theorem

Seydewitz's Theorem

If a triangle is inscribed in a conic, any line conjugate to one side meets the other two sides in conjugate points.

- Consider an inscribed triangle PQR. Any line c conjugate to PQ is the polar of some point C on PQ. Let RC meet the conic again in S. Consider the quadrangle $PQRS$. By a preceding theorem, its diagonal points form a self-polar triangle ABC. Thus, side c meets QR and RP on the conjugate points A and B.

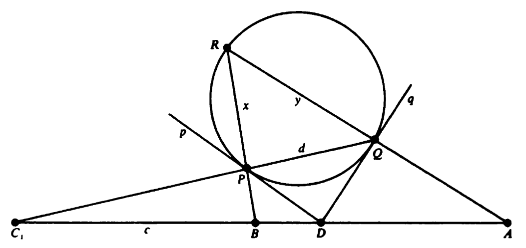


Steiner's Theorem

Steiner's Theorem

Suppose lines x and y join a variable point on a conic to two fixed points on the same conic. Then $x \overline{\wedge} y$.

- The tangents p and q, at the fixed points P and Q, meet in D, the pole of PQ.

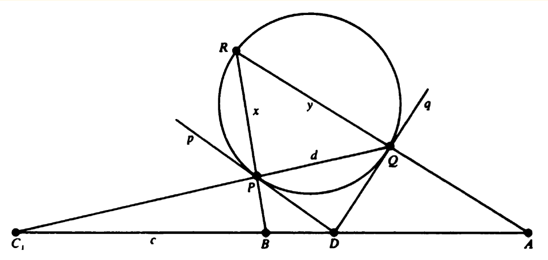


  Let c be a fixed line through D (but not through P or Q), meeting x in B, and y in A. By the preceding theorem, BA is a pair of the involution of conjugate points on c. Hence, when the point $R = x \cdot y$ varies on the conic, we have $x \overline{\wedge} B \overline{\wedge} A \overline{\wedge} y$.

Remarks

- The following remarks make it natural to include the tangents p and q as special positions for x and y.

  Write $d = PQ$ and $C_1 = c \cdot d$. Consider $R = P$ or $R = Q$.



  - If R is P, y is d, A is C_1, B is the conjugate point D, and, therefore, x is p.
  - If R is Q, x is d, B is C_1, A is D, and y is q.

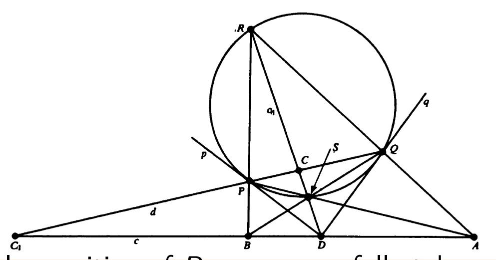
  Thus, when y is d, x is p, and when x is d, y is q.

Subsection 4

Conics Touching Two Lines as Given Points

Revisiting Steiner's Theorem

- In the proof of Steiner's Theorem, we chose an arbitrary line c through D (the pole of PQ).



For any particular position of R, we can usefully take c to be a side of the diagonal triangle ABC of the inscribed quadrangle $PQRS$, where S is on RD:
    - We define $C = PQ \cdot RS$;
    - We let c be its polar AB, which passes through D since C lies on d.

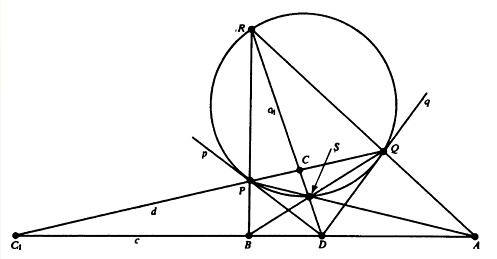
The point $C_1 = c \cdot d$, being the pole of the line CD, is the harmonic conjugate of C with respect to P and Q.

Conics Through Three Points and Two Tangents

Theorem

A conic is determined when three points on it and the tangents at two of them are given.

- If we are not given the conic, but only the points P, Q, R, D, we can still construct $C = PQ \cdot RD$ and its harmonic conjugate C_1. Then c is the line C_1D, which meets QR and RP in A and B.



The conic itself can be described as the locus of self-conjugate points (and the envelope of self-conjugate lines) in the polarity $(ABC)(Pp)$, where $p = PD$.

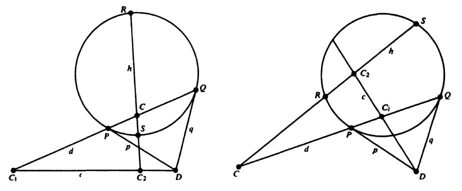
Conics with Double Contact

- Retaining P, p, Q, q, but letting R vary, we obtain a "pencil" of conics touching p at P and q at Q, said to have **double contact**.

Theorem

Of the conics that touch two given lines at given points, those which meet a third line (not through either of the points) do so in pairs of an involution.

- Let one of them meet a fixed line h in R and S. Let h meet the fixed line $d = PQ$ in C. Let c (the polar of C) meet d in C_1, and h in C_2.



The line c is fixed, since it joins $D = p \cdot q$ to C_1, which is the harmonic conjugate of C with respect to P and Q. So the fixed point $C = d \cdot h$ has the same polar for all the conics.

Thus $C_2 = c \cdot h$ is another fixed point. Hence RS is always a pair of the hyperbolic involution $(CC)(C_2C_2)$ on h.

Subsection 5

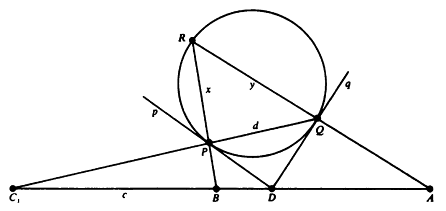
Steiner's Definition for a Conic

Steiner's Definition for a Conic

Theorem

Let variable lines x and y pass through fixed points P and Q in such a way that $x \barwedge y$ but not $x \doublebarwedge y$. Then the locus of $x \cdot y$ is a conic through P and Q. If the projectivity has the effect $pdx \barwedge dqy$, where $d = PQ$, then p and q are the tangents at P and Q.

- Since the projectivity $x \barwedge y$ is not a perspectivity, the line $d = PQ$ does not correspond to itself. Hence, there exist lines p and q, such that the projectivity relates p to d, and d to q.



  Thus, there is a unique conic touching p at P, q at Q, and passing through any other particular position of $x \cdot y$. This conic determines a projectivity relating all the lines through P to all the lines through Q. By the fundamental theorem, the two projectivities must coincide.

Subsection 6

The Conic Touching Five Given Lines

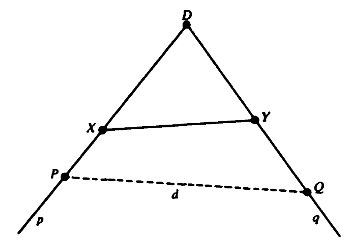
The Dual of Steiner's Definition of a Conic

- Dualizing Steiner's Definition, we obtain

Theorem

Let points X and Y vary on fixed lines p and q in such a way that $X \overline{\wedge} Y$, but not $X \overline{\overline{\wedge}} Y$. Then the envelope of XY is a conic touching p and q.

If the projectivity has the effect $PDX \overline{\wedge} DQY$, where $D = p \cdot q$, then P and Q are the points of contact of p and q.



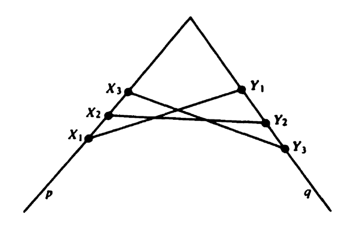
Conic Determined by Five Lines

Theorem

Any five lines, of which no three are concurrent, determine a unique conic touching them.

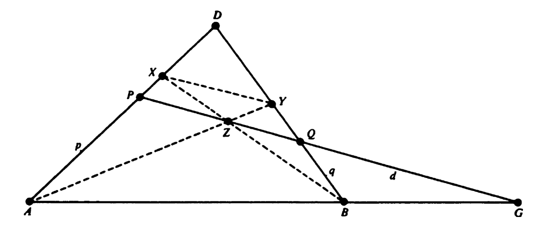
- Let X_1, X_2, X_3 be three positions of X on p, and Y_1, Y_2, Y_3 the corresponding positions of Y on q.

  By a preceding result, there is a unique projectivity $X_1X_2X_3X \barwedge Y_1Y_2Y_3Y$. By a previous theorem, the envelope of XY is a conic, provided no three of the five lines X_iY_i, p, q are concurrent.



  Conversely, if five such lines all touch a conic, any other tangent XY satisfies $X_1X_2X_3X \barwedge Y_1Y_2Y_3Y$.

The Variable Triangle XYZ



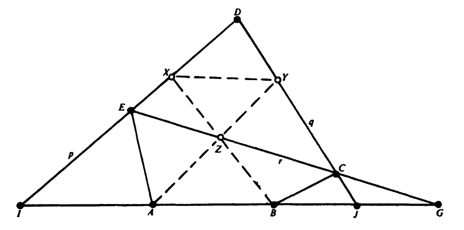
- By a previous theorem, the line $d = PQ$ is the axis of the projectivity $X \overline{\wedge} Y$, that is, if A is a particular position of X and B is the corresponding position of Y, the point $Z = AY \cdot BX$ always lies on this fixed line d. In fact, if AB meets d in G, we have an expression for the projectivity as the product of two perspectivities : $APDX \overset{B}{\overline{\overline{\wedge}}} GPQZ \overset{A}{\overline{\overline{\wedge}}} BDQY$.
  We may regard XYZ as a variable triangle whose vertices run along fixed lines p, q, d while the two sides YZ and ZX pass through fixed points A and B. We have seen that the envelope of XY is a conic touching p at P, and q at Q.

Enveloping the Conic

Theorem

If the vertices of a variable triangle lie on three fixed nonconcurrent lines p, q, r, while two sides pass through fixed points A and B, not collinear with $p \cdot q$, then the third side envelops a conic.

- Let XYZ be the variable triangle, whose vertices X, Y, Z run along the fixed lines p, q, r while the sides YZ and ZX pass through points A and B (not necessarily on p or q).



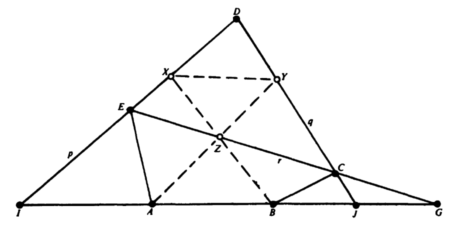
Then $X \stackrel{B}{\barwedge} Z \stackrel{A}{\barwedge} Y$. Since neither r nor AB passes through $D = p \cdot q$, the projectivity $X \barwedge Y$ is not a perspectivity. By a previous theorem, the envelope of XY is a conic touching p and q.

Special Positions of Z

Theorem

Let Z be a variable point on the diagonal CE of a given pentagon $ABCDE$. Then the two points $X = ZB \cdot DE$ and $Y = ZA \cdot CD$ determine a line XY whose envelope is the inscribed conic.

- In the figure, each position for Z on r yields a corresponding position for the tangent XY. Define $C = q \cdot r$, $E = p \cdot r$, $G = AB \cdot r$, $I = AB \cdot p$, $J = AB \cdot q$. When Z is E, X also is E,



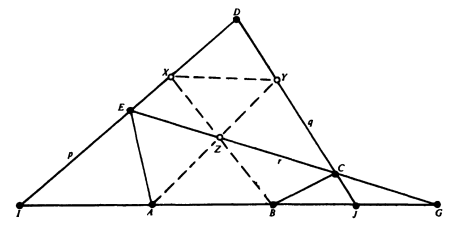
  AY is AE, and XY also is AE. Similarly, when Z is C, Y also is C, BX is BC, and XY also is BC. Finally, when Z is G, X is I, Y is J, and XY is AB. Thus, the lines AE, BC, AB, like p and q, are special positions for XY. In other words, all five sides of the pentagon $ABCDE$ are tangents of the conic.

Briancho n's Theorem

Brianchon's Theorem

If a hexagon is circumscribed about a conic, the three diagonals are concurrent.

- In the figure, we see a hexagon $ABCYXE$ whose six sides all touch a conic. The three lines AY, BX, CE, which join pairs of opposite vertices, are called **diagonals** of the hexagon.

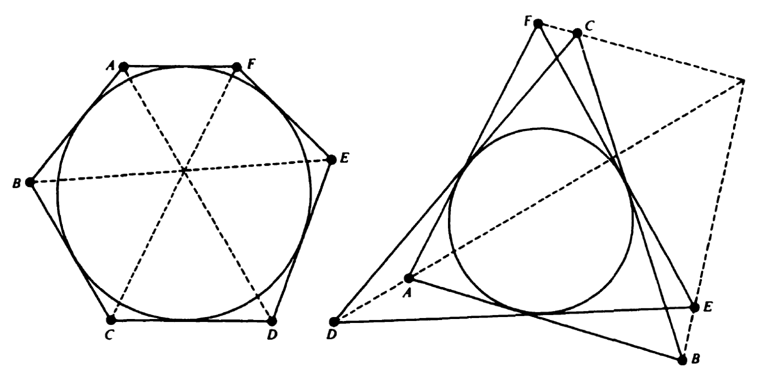


  The preceding theorem tells us that, if the diagonals of a hexagon are concurrent, the six sides all touch a conic.

  Conversely, if all the sides of a hexagon touch a conic, five of them can be identified with the lines DE, EA, AB, BC, CD. Since the given conic is the only one that touches these fixed lines, the sixth side must coincide with one of the lines XY for which $BX \cdot AY$ lies on CE.

Illustration of Brianchon's Theorem

- The figure illustrates Brianchon's Theorem in a more natural notation: the Brianchon hexagon is $ABCDEF$ and its diagonals are AD, BE, CF.



Subsection 7

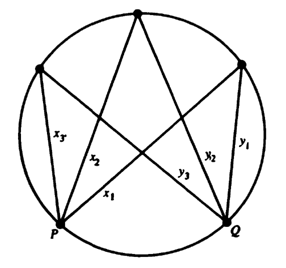
The Conic Through Five Given Points

The Conic Determined by Five Given Points

- We saw that any five lines, of which no three are concurrent, determine a unique conic touching them.
- Dualizing, we obtain

Theorem

Any five points, of which no three are collinear, determine a unique conic through them.

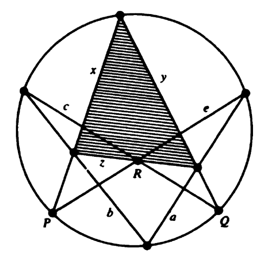


The Braikenridge-MacLaurin Theorem

- We saw that if the vertices of a variable triangle lie on three fixed nonconcurrent lines p, q, r, while two sides pass through fixed points A and B, not collinear with $p \cdot q$, then the third side envelops a conic.
- The dual was proved independently by Braikenridge and MacLaurin:

The Braikenridge-MacLaurin Theorem

If the sides of a variable triangle pass through three fixed noncollinear points P, Q, R, while two vertices lie on fixed lines a and b, not concurrent with PQ, then the third vertex describes a conic.



- This enables us to locate any number of points on the conic through five given points (the variable triangle is shaded in the figure).

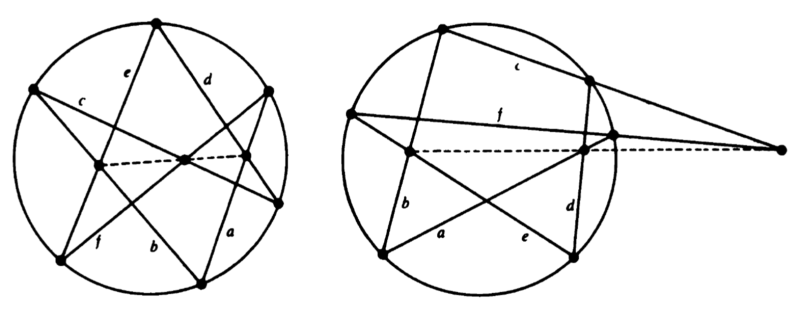
Pascal's Theorem

- The dual of Brianchon's Theorem is Pascal's Theorem:

Pascal's Theorem

If a hexagon is inscribed in a conic, the three pairs of opposite sides meet in collinear points.

- The hexagon is *abcdef* and the 3 collinear points are $a \cdot d$, $b \cdot e$, $c \cdot f$.



Subsection 8

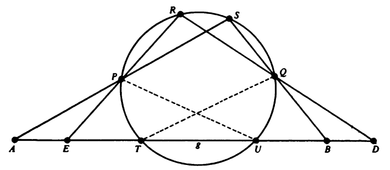
Conics Through Four Given Points

Desargues's Involution Theorem

Desargues's Involution Theorem

Of the conics that can be drawn through the vertices of a quadrangle, those which meet a given line (not through a vertex) do so in pairs of an involution.

- Let $PQRS$ be the given quadrangle, and g the given line, meeting the sides PS, QS, QR, PR in A, B, D, E, and any one of the conics in T and U.



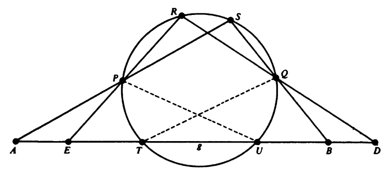
By regarding S, R, T, U as four positions of a variable point on this conic, we see that the four lines joining them to P are projectively related to the four lines joining them to Q. Hence $AETU \barwedge BDTU$. Since $BDTU \barwedge DBUT$, it follows that $AETU \barwedge DBUT$. Hence TU is a pair of the involution $(AD)(BE)$. Since this involution depends only on the quadrangle, all those conics of the pencil which intersect g (or touch g) determine pairs (or invariant points) of the same involution.

Conics Touching Lines and Pairs of an Involution

Theorem

Of the conics that can be drawn to touch a given line at a given point while passing also through two other given points, those which meet another given line (not through any of the three given points) do so in pairs of an involution.

- We observe that, when S and Q coincide, the line SQ (which determines B) is replaced by the tangent at Q.



- By letting R and P coincide, we obtain an alternative proof for:

Theorem

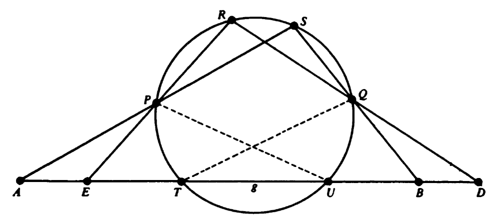
Of the conics that touch two given lines at given points, those which meet a third line (not through either of the points) do so in pairs of an involution.

Subsection 9

Two Self-Polar Triangles

Self-Polar Triangles on Conics

- Combining Desargues' Involution with the result on determining involutions through self-polar triangles, we see that the involution determined on g by the quadrangle $PQRS$



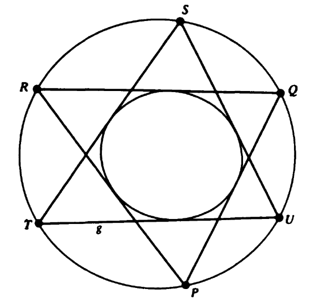
is not only the Desargues involution determined by conics through P, Q, R, S but also the involution of conjugate points on g for the polarity $(PQR)(Sg)$:

Theorem

If two triangles have six distinct vertices, all lying on a conic, there is a polarity for which both triangles are self-polar.

Conics by Self-Polar Triangles

- Converse to the preceding theorem:



Theorem

If two triangles, with no vertex of either on a side of the other, are self-polar for a given polarity, their six vertices lie on a conic and their six sides touch another conic.

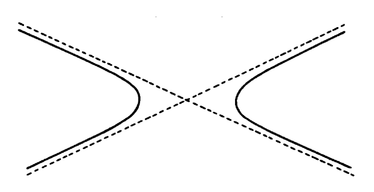
Subsection 10

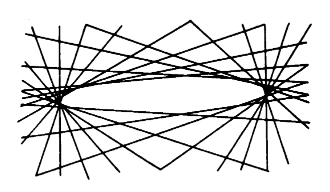
Degenerate Conics

Degenerate Conics

- It is sometimes convenient to admit, as **degenerate conics**, a pair of lines (regarded as a locus) or a pair of points (regarded as an envelope: the set of all lines through one or both).

  Example: A hyperbola may differ as little as we please from a pair of lines (its asymptotes),

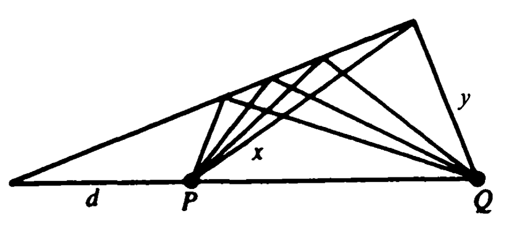




  and the set of tangents of a very thin ellipse is hardly distinguishable from the lines through one or other of two fixed points.

Degenerate Conics: Allowing Two Lines

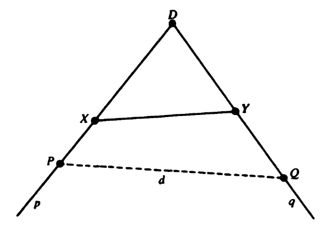
- By omitting the phrase "but not $x \overline{\overline{\wedge}} y$" from the statement of Steiner's construction, we could allow the locus to consist of two lines:
    - The axis of the perspectivity $x \overline{\overline{\wedge}} y$;
    - The line PQ.

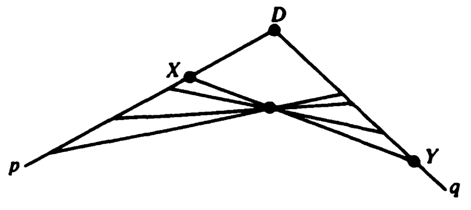


- Any point of the locus PQ is joined to P and Q by "corresponding lines" of the two pencils, namely by the invariant line PQ itself.

Degenerate Conics: Allowing Two Points

- Dually, when the points P and Q coincide with D,

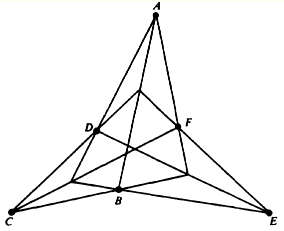




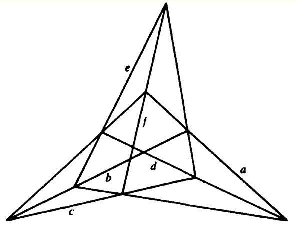
we have a degenerate conic envelope consisting of two points, regarded as two pencils:
  - The various positions of the line XY when X and Y are distinct;
  - The pencil of lines through D.
- In the same spirit we can say that a conic is determined by five points, no *four* collinear, or by five lines, no *four* concurrent.

Degenerate Forms of Brianchon's and Pascal's Theorems

- The degenerate forms of Brianchon's theorem and Pascal's theorem are as follows:



If AB, CD, EF are concurrent and DE, FA, BC are concurrent, then AD, BE, CF are concurrent.



If $a \cdot b$, $c \cdot d$, $e \cdot f$ are collinear and $d \cdot e$, $f \cdot a$, $b \cdot c$ are collinear, then $a \cdot d$, $b \cdot e$, $c \cdot f$ are collinear.

- Both these statements are equivalent to Pappus's theorem.