Introduction to Real Analysis

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 421

Subsection 1

Existence of GLBs

Existence of GLBs

- In the axiomatization of $\mathbb{R}$, we assumed the existence of least upper bounds (completeness axiom).
- The existence of greatest lower bounds then follows:

Theorem (Existence of GLBs)

If A is a nonempty subset of $\mathbb{R}$ that is bounded below, then A has a greatest lower bound:

$$\inf A = -\sup(-A).$$

- The set $-A = \{-a : a \in A\}$ is nonempty and bounded above. Thus, it has a least upper bound by the completeness axiom. By a preceding proposition, $-(-A) = A$ has a greatest lower bound and $\inf A = -\sup(-A)$.

A Useful Corollary

Corollary

$$\inf \left\{ \frac{1}{n} : n \in \mathbb{P} \right\} = 0.$$

- Let $A = \{\frac{1}{n} : n \in \mathbb{P}\}$.
  - We know that A is bounded below by 0, so A has a greatest lower bound a and $0 \leq a$.
  - On the other hand, $a \leq \frac{1}{2n}$, for all positive integers n, so $2a$ is also a lower hound for A. It follows that $2a \leq a$, whence, $a \leq 0$.

This proves that $a = 0$.

Subsection 2

Archimedean Property

Archimedean Ordered Fields

- In every ordered field, $1 < 2 < 3 < \ldots$, therefore, $1 > \frac{1}{2} > \frac{1}{3} > \ldots$. For every $y > 0$, we thus have $y > \frac{y}{2} > \frac{y}{3} > \ldots$.
- As a result, we are expecting the elements $\frac{y}{n}$ $(n = 1, 2, 3, \ldots)$ to be "arbitrarily small" in the sense that, for every $x > 0$, there is an n for which $\frac{y}{n}$ is smaller than x.
- In actuality, there exist ordered fields in which it can happen that $\frac{y}{n} \geq x > 0$ for all n, i.e., the elements $\frac{y}{n}$ $(n = 1, 2, 3, \ldots)$ are "buffered away from 0" by the element x.
- The property at the heart of such considerations is the following:

Definition (Archimedean Ordered Field)

An ordered field is said to be **Archimedean** if, for each pair of elements x, y with $x > 0$, there exists a positive integer n such that $nx > y$. (If x is thought of as a "unit of measurement", then each element y can be surpassed by a sufficiently large multiple of the unit of measurement.)

$\mathbb{R}$ is Archimedean

Theorem

The field $\mathbb{R}$ of real numbers is Archimedean.

- Let x and y be real numbers, with $x > 0$.
  - If $y < 0$, then $1x > y$.
  - Assuming $y > 0$, we seek a positive integer n, such that $\frac{1}{n} < \frac{x}{y}$. The alternative is that $0 < \frac{x}{y} \leq \frac{1}{n}$, for every positive integer n. This is contrary to $\inf \{\frac{1}{n} : n \in \mathbb{P}\} = 0$.
- Example: The field $\mathbb{Q}(t)$ of rational forms over $\mathbb{Q}$ is not Archimedean.
- In fact, the completeness property implies the Archimedean property, but the converse statement fails:

$\mathbb{Q}$ is Archimedean but Not Complete

Theorem

The field $\mathbb{Q}$ of rational numbers is Archimedean but not complete.

- The Archimedean property for $\mathbb{Q}$ is an immediate consequence of the preceding theorem (since $\mathbb{Q}$ is a subfield of $\mathbb{R}$).

  We have to exhibit a nonempty subset A of $\mathbb{Q}$ that is bounded above but has no least upper bound in $\mathbb{Q}$. The core of the proof is the fact that 2 is not the square of a rational number. Let

$$A = \{r \in \mathbb{Q} : r > 0 \text{ and } r^2 < 2\}.$$

Since $1 \in A$, $A \neq \emptyset$. If $r \in \mathbb{Q}$ and $r \geq 2$ then $r^2 \geq 4 > 2$, so $r \notin A$, i.e., $r < 2$, for all $r \in A$, whence A is bounded above. Now we show that:

- A has no largest element;
- There is no smallest element r in $\mathbb{Q}$, with $r^2 > 2$;
- We conclude that A has no least upper bound in $\mathbb{Q}$.

A has no Largest Element

- We show that $A = \{r \in \mathbb{Q} : r > 0 \text{ and } r^2 < 2\}$ has no largest element.

  Given any element r of A, we produce a larger element of A. It suffices to find a positive integer n, such that $r + \frac{1}{n} \in A$, i.e., $(r + \frac{1}{n})^2 < 2$. Expand the square $r^2 + \frac{2r}{n} + \frac{1}{n^2} < 2$. Multiply both sides by $n > 0$: $nr^2 + 2r + \frac{1}{n} < 2n$. Rearrange: $2r + \frac{1}{n} < n(2 - r^2)$. Since $2 - r^2 > 0$, the Archimedean property yields a positive integer n, such that $n(2 - r^2) > 2r + 1$. But $2r + 1 > 2r + \frac{1}{n}$, so $n(2 - r^2) > 2r + \frac{1}{n}$ holds.

There is no smallest r in $\mathbb{Q}$, with $r^2 > 2$

- There are positive elements r of $\mathbb{Q}$, such that $r^2 > 2$ (e.g., $r = 2$). We show that there is no smallest such element r.
  Given any $r \in \mathbb{Q}$, with $r > 0$ and $r^2 > 2$, we shall produce a positive element of $\mathbb{Q}$, that is smaller than r but whose square is also larger than 2. It suffices to find a positive integer n such that $r - \frac{1}{n} > 0$ and $(r - \frac{1}{n})^2 > 2$, equivalently, $nr > 1$ and $n(r^2 - 2) > 2r - \frac{1}{n}$. Since $r > 0$ and $r^2 - 2 > 0$, the Archimedean property yields a positive integer n such that both $nr > 1$ and $n(r^2 - 2) > 2r$ (choose an n for each inequality, then take the larger of the two). But $2r > 2r - \frac{1}{n}$, so the required conditions are verified.

A has no LUB in $\mathbb{Q}$

- We assert that $A = \{r \in \mathbb{Q} : r > 0 \text{ and } r^2 < 2\}$ has no least upper bound in $\mathbb{Q}$.

  Assume to the contrary that A has a least upper bound t in $\mathbb{Q}$. We know that $t^2 \neq 2$ (2 is not the square of a rational number) and $t > 0$ (because $1 \in A$). Let us show that each of the possibilities $t^2 < 2$ and $t^2 > 2$ leads to a contradiction.
  - If $t^2 < 2$, then $t \in A$. But then t would be the largest element of A, contrary to our earlier observation that no such element exists.
  - If $t^2 > 2$, then, as observed above, there exists a rational number s, such that $0 < s < t$ and $s^2 > 2$. Since t is supposedly the least upper bound of A and s is smaller than t, s cannot be an upper bound for A. This means that there exists an element r of A with $s < r$. But then $s^2 < r^2 < 2$, contrary to $s^2 > 2$.

Subsection 3

Bracket Function

Uniqueness of Bracket

- A useful application of the Archimedean property is that every real number can be sandwiched between a pair of successive integers:

Theorem

For each real number x, there exists a unique integer n such that $n \leq x < n+1$.

- Uniqueness: The claim is that a real number x cannot belong to the interval $[n, n+1)$ for two distinct values of n.
  If m and n are distinct integers, say $m < n$, then $n - m$ is an integer and is > 0. Therefore $n - m \geq 1$. Thus, $m + 1 \leq n$ and it follows that the intervals $[m, m+1)$ and $[n, n+1)$ can have no element x in common.

Existence of Bracket

- Existence: Let $x \in \mathbb{R}$. By the Archimedean property, there exists a positive integer j such that $j \cdot 1 > -x$, that is, $j + x > 0$. It will suffice to find an integer k such that $j + x \in [k, k+1)$: This would imply that $x \in [k-j, k-j+1)$. Changing notation, we can suppose that $x > 0$. Let $S = \{k \in \mathbb{P} : k \cdot 1 > x\}$.
  - By the Archimedean property, S is nonempty;
  - So S has a smallest element m by the "well-ordering principle".

  Since $m \in S$, we have $m > x$.
  - If $m = 1$, then $0 < x < 1$ and the assertion is proved with $n = 0$.
  - If $m > 1$, then $m - 1$ is a positive integer smaller than m, so it cannot belong to S. This means that $m - 1 \leq x$. Thus, $x \in [m-1, m)$ and $n = m - 1$ is the required integer.

Definition (Bracket Function)

The integer n is denoted $[x]$ and the function $\mathbb{R} \to \mathbb{Z}$ defined by $x \mapsto [x]$ is called the **bracket function** (or the **greatest integer function**, since $[x]$ is the largest integer that is $\leq x$).

Subsection 4

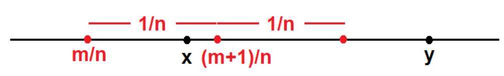
Density of the Rationals

Density of Rationals

- Between any two reals, there is a rational:

Theorem (Density of Rationals)

If x and y are real numbers such that $x < y$, then there exists a rational number r, such that $x < r < y$.



- Since $y - x > 0$, by the Archimedean property, there exists a positive integer n such that $n(y-x) > 1$, i.e., $\frac{1}{n} < y - x$. Think of $\frac{1}{n}$ as a "unit of measurement", small enough for the task at hand. We find a multiple of $\frac{1}{n}$ that lands between x and y.

  Let $m = [nx]$. Then $m \leq nx < m + 1$. Hence $\frac{m}{n} \leq x$ and $x < \frac{m+1}{n} = \frac{m}{n} + \frac{1}{n} \leq x + \frac{1}{n} < x + (y - x) = y$, so $r = \frac{m+1}{n}$ meets the requirements of the theorem.

Irrational Numbers

- The conclusion of the theorem is expressed by saying that the rational field $\mathbb{Q}$ is everywhere dense in $\mathbb{R}$.
- There are "lots" of rational numbers, but are there any real numbers that are not rational?

  The answer is yes: The set $A = \{r \in \mathbb{Q} : r > 0 \text{ and } r^2 < 2\}$ is nonempty and bounded above, so it has a least upper bound u in $\mathbb{R}$ by completeness. If u were rational, then it would be a least upper bound for A in the ordered field $\mathbb{Q}$, contrary to what we proved.

Definition (Irrational Numbers)

A real number that is not rational is called an **irrational number**. Thus, the irrational numbers are the elements of the difference set $\mathbb{R} - \mathbb{Q} = \{x \in \mathbb{R} : x \notin \mathbb{Q}\}$.

Subsection 5

Monotone Sequences

Sequences

Definition (Sequence)

If X is a set and if, for each positive integer n, an element x_n of X is given, we say that we have a **sequence** of elements of X, or "a sequence in X", whose n-**th term** is x_n.

- Various notations are used to indicate sequences, for example
$$(x_n), \quad (x_n)_{n\in\mathbb{P}}, \quad (x_n)_{n\geq 1}, \quad (x_n)_{n=1,2,3,\ldots}.$$
- Informally, a sequence of elements of a set is an unending list $x_1, x_2, x_3, \ldots$ of (not necessarily distinct) elements of the set.
- Formally, it is a function $f : \mathbb{P} \to X$, where we write x_n instead of $f(n)$ for the element of X corresponding to the positive integer n.
- Another notation that stresses the functional aspect of a sequence: $n \mapsto x_n$, $n \in \mathbb{P}$.
- In the notation (x_n), the integers n are called the **indices**.
- Sometimes index sets other than $\mathbb{P}$ are appropriate, as, for example, $(a_n)_{n\in\mathbb{N}}$ for the coefficients of a power series $\sum_{n=0}^{\infty} a_n x^n$.

Increasing and Decreasing Sequences

Definition (Increasing/Decreasing Sequence)

A sequence (a_n) in $\mathbb{R}$ is said to be:

- **increasing** if $a_1 \leq a_2 \leq a_3 \leq \cdots$, i.e., if $a_n \leq a_{n+1}$, for all $n \in \mathbb{P}$;
- **strictly increasing** if $a_n < a_{n+1}$, for all n;
- **decreasing** if $a_1 \geq a_2 \geq a_3 \geq \cdots$;
- **strictly decreasing** if $a_n > a_{n+1}$, for all n.

- A sequence that is either increasing or decreasing is said to be **monotone**; more precisely, one speaks of sequences that are "monotone increasing" or "monotone decreasing".
- If (a_n) is an increasing sequence, we write $a_n \uparrow$, and if it is a decreasing sequence we write $a_n \downarrow$ (no special notation is offered for "strictly monotone" sequences.)

Suprema and Infima of Monotone Sequences

Definition (Supremun and Infimum of Monotone Sequences)

- If (a_n) is an increasing sequence in $\mathbb{R}$, such that $A = \{a_n : n \in \mathbb{P}\}$ is bounded above, and if $a = \sup A$, then we write $a_n \uparrow a$.
- Similarly, $a_n \downarrow a$ means that (a_n) is a decreasing sequence, the set $A = \{a_n : n \in \mathbb{P}\}$ is bounded below, and $a = \inf A$.

- Example: $\frac{1}{n} \downarrow 0$:
  - The sequence $(\frac{1}{n})$ is decreasing;
  - $\inf \{\frac{1}{n} : n \in \mathbb{P}\} = 0$.
- Example: If $0 < c < 1$, then the sequence of powers (c^n) is strictly decreasing and $c^n \downarrow 0$:
  - (c^n) is strictly decreasing since $0 < c < 1$ implies $0 < c^2 < c$ implies $0 < c^3 < c^2$ etc.
  - Let $a = \inf \{c^n : n \in \mathbb{P}\}$. We know that $a \geq 0$ and $c^n \downarrow a$. Now $a \leq c^{n+1}$ implies $\frac{a}{c} \leq c^n$, for all n. It follows that $\frac{a}{c} \leq a$, whence $a(1 - c) \leq 0$ and, therefore, $a \leq 0$, which gives $a = 0$.

Properties of Infima and Suprema of Monotone Sequences

Theorem

If $a_n \uparrow a$ and $b_n \uparrow b$, then:

(i) $a_n + b_n \uparrow a + b$;

(ii) $-a_n \downarrow -a$;

(iii) $a_n + c \uparrow a + c$, for every real number c.

(i) It is clear that $(a_n + b_n)$ is an increasing sequence. Moreover, it is bounded above by $a + b$. To show that $a + b$ is the least upper bound, suppose $a_n + b_n \leq c$, for all n. We have to show that $a + b \leq c$, i.e., $a \leq c - b$. Given any index m, it is enough to show that $a_m \leq c - b$, i.e., $b \leq c - a_m$. Thus, given any index n, we need only show that $b_n < c - a_m$, i.e, $a_m + b_n \leq c$. Indeed, if p is the larger of m and n then $a_m + b_n \leq a_p + b_p \leq c$, by the assumed monotonicity.

(ii) This follows from $\inf\{-a_n\} = -\sup\{a_n\}$.

(iii) This is a special case of (i), with $b_n = c$, for all n.

Subsection 6

Theorem on Nested Intervals

Nested Intervals

- A sequence of intervals (I_n) of $\mathbb{R}$ is said to be **nested** if $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. As the intervals "shrink" with increasing n, there is no assurance that there is any point that belongs to every I_n.
- Example: If $I_n = (0, \frac{1}{n}]$, then there is no point belonging to all I_n.
- However, if the intervals are closed, we can be sure that there is at least one survivor:

Theorem (Sequence of Nested Closed Intervals)

If (I_n) is a nested sequence of closed intervals, then the intersection of the I_n is nonempty. More precisely, if $I_n = [a_n, b_n]$, where $a_n \leq b_n$ and $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, and if $a = \sup\{a_n : n \in \mathbb{P}\}$, $b = \inf\{b_n : n \in \mathbb{P}\}$, then $a \leq b$ and $\bigcap_{n=1}^{\infty}[a_n, b_n] = [a, b]$.

Proof of the Theorem

- The notation $\bigcap_{n=1}^{\infty}[a_n, b_n]$ means the intersection $\bigcap \mathcal{S}$ of the set $\mathcal{S}$ of all the intervals $[a_n, b_n]$.
- From $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ we see that it follows that the sequence (a_n) is increasing and bounded above (for example by b_1). On the other had, (b_n) is decreasing and bounded below (for example by a_1). If a and b are defined as in the statement of the theorem, we have $a_n \uparrow a$ and $b_n \downarrow b$. By the preceding theorem (and its "dual") we have $-b_n \uparrow -b$, so $a_n + (-b_n) \uparrow a + (-b)$. Therefore, $b_n - a_n \downarrow b - a$. Since $b_n - a_n \geq 0$, for all n, it follows that $b - a \geq 0$. Then $a_n \leq a \leq b \leq b_n$, whence $[a, b] \subseteq [a_n, b_n]$, for all n, and, therefore, $[a, b] \subseteq \bigcap_{n=1}^{\infty}[a_n, b_n]$.
  Conversely, if x belongs to every $[a_n, b_n]$ then $a_n \leq x \leq b_n$, for all n, and, therefore, $a \leq x \leq b$ showing that $\bigcap_{n=1}^{\infty}[a_n, b_n] \subseteq [a, b]$.

Theorem on Nested Intervals

- The following corollary is known as the **Theorem on Nested Intervals**:

Corollary (Theorem on Nested Intervals)

Suppose $I_n = [a_n, b_n]$, where $a_n \leq b_n$ and $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. Suppose, also, that $\inf (b_n - a_n) = 0$. Then $\bigcap_{n=1}^{\infty}[a_n, b_n] = \{c\}$, where $c = a = b$, with $a = \sup \{a_n : n \in \mathbb{P}\}$, $b = \inf \{b_n : n \in \mathbb{P}\}$.

- As shown in the proof of the theorem, $b_n - a_n \downarrow b - a$. By hypothesis, $b_n - a_n \downarrow 0$, so $b = a$ and $\bigcap_{n=1}^{\infty}[a_n, b_n] = [a, a] = \{a\}$.
- A surprising corollary is a nonconstructive proof of the existence of irrational numbers.

Subsection 7

Dedekind Cut Property

Dedekind Cuts

Definition (Dedekind Cut)

A **cut** (or **Dedekind cut**) of the real field $\mathbb{R}$ is a pair (A, B) of nonempty subsets of $\mathbb{R}$, such that (a) every real number belongs to either A or B and (b) $a < b$, for all $a \in A$ and $b \in B$. In symbols, $A \neq \emptyset$, $B \neq \emptyset$, $\mathbb{R} = A \cup B$, $a < b$, $a \in A, b \in B$. (It follows from the latter property that $A \cap B = \emptyset$.)

- Examples: If $\gamma \in \mathbb{R}$ and
$$A = \{x \in \mathbb{R} : x \leq \gamma\}, \quad B = \{x \in \mathbb{R} : x > \gamma\},$$
then (A, B) is a cut of $\mathbb{R}$. Note that A has a largest element but B has no smallest.
The pair
$$A = \{x \in \mathbb{R} : x < \gamma\}, \quad B = \{x \in \mathbb{R} : x \geq \gamma\}$$
also defines a cut of $\mathbb{R}$. Here, B has a smallest element but A has no largest.
- The key fact about cuts of $\mathbb{R}$ is that there are no other examples.

Uniqueness of γ

Theorem

If (A, B) is a cut of $\mathbb{R}$, then there exists a unique real number γ, such that either

(i) $A = \{x \in \mathbb{R} : x \leq \gamma\}$ and $B = \{x \in \mathbb{R} : x > \gamma\}$, or

(ii) $A = \{x \in \mathbb{R} : x < \gamma\}$ and $B = \{x \in \mathbb{R} : x \geq \gamma\}$.

- Uniqueness: The number γ is uniquely determined by the property of being either the largest element of A or the smallest element of B, according as case (i) or case (ii) holds.

Existence of γ

- Existence: Note that A is bounded above (by any element of B) and B is bounded below (by any element of A). Let $\alpha = \sup A$, $\beta = \inf B$. If $a \in A$, then $a < b$, for all $b \in B$, whence $a \leq \beta$. Since $a \in A$ is arbitrary, $\alpha \leq \beta$. In fact $\alpha = \beta$, for if $\alpha < \beta$, then any number in the gap between α and β would be too large to belong to A and too small to belong to B, which would contradict $\mathbb{R} = A \cup B$. Write γ for the common value of α and β. By assumption, γ must belong to either A or B.
  - (i) Case 1: $\gamma \in A$. We have $A \subseteq \{x \in \mathbb{R} : x \leq \gamma\}$, $B \subseteq \{x \in \mathbb{R} : x > \gamma\}$: The first inclusion follows from $\gamma = \sup A$. The second inclusion follows from $\gamma = \inf B$ and the fact that $\gamma \in B$ is ruled out by $\gamma \in A$. These imply that both inclusions are actually equalities: if $x \leq \gamma$ then necessarily $x \in A$. The alternative $x \in B$ is unacceptable because it would imply $x > \gamma$.
  - (ii) Case 2: $\gamma \in B$. In this case, a similar argument shows that the other pair of formulas hold.

Subsection 8

Square Roots

Uniqueness of Square Roots

Theorem

Every positive real number has a unique positive square root. That is, if $c \in \mathbb{R}, c > 0$, then there exists a unique $x \in \mathbb{R}, x > 0$, such that $x^2 = c$.

- Uniqueness: If x and y are positive real numbers such that $x^2 = c = y^2$, then $0 = x^2 - y^2 = (x+y)(x-y)$ and $x + y > 0$, whence $x - y = 0$, i.e., $x = y$.

Existence of Square Roots

- Existence: Given $c \in \mathbb{R}, c > 0$, the strategy is to construct a cut (A, B) of $\mathbb{R}$ for which the γ of the preceding theorem satisfies $\gamma^2 = c$. Let
$$A = \{x \in \mathbb{R} : x \leq 0\} \cup \{x \in \mathbb{R} : x > 0 \text{ and } x^2 < c\},$$
$$B = \{x \in \mathbb{R} : x > 0 \text{ and } x^2 \geq c\}.$$
Then $A \neq \emptyset$, $B \neq \emptyset$ $(c + 1 \in B)$ and $A \cup B = \mathbb{R}$. Moreover, if $a \in A$ and $b \in B$, then $a < b$:
  - If $a \leq 0$, this is trivial.
  - If $a > 0$, then $a^2 < c \leq b^2$ implies $a < b$.

  In summary, (A, B) is a cut of $\mathbb{R}$. Let γ be the real number that defines the cut.
  Note that A contains numbers > 0:
  - If $c > 1$ then $\frac{1}{2} \in A$ (because $\frac{1}{4} < 1 \leq c$).
  - If $0 < c < 1$, then $c \in A$ (because $c^2 < c$).

  It follows that $\gamma > 0$.

Existence (Cont'd)

- Next, we assert that $\gamma \in B$. By the arguments in the preceding section, we need only show that A has no largest element. Assuming $a \in A$, we find a larger element of A.
  - If $a \leq 0$, then any positive element of A will do.
  - Suppose $a > 0$. We know that $a^2 < c$. It will suffice to find a positive integer n, such that $(a + \frac{1}{n})^2 < c$. The existence of such an n is due to the Archimedean Property applied to $n(c - a^2) > 2a + 1 \geq 2a + \frac{1}{n}$.

  We now know that $A = \{x \in \mathbb{R} : x < \gamma\}$, $B = \{x \in \mathbb{R} : x \geq \gamma\}$. Since $\gamma \in B$, we have $\gamma^2 \geq c$. It remains only to show that $\gamma^2 \leq c$, i.e., $\gamma^2 - c \leq 0$.

  By the Archimedean property, choose a positive integer N such that $N\gamma > 1$. For every integer $n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} < \gamma$, so $\gamma - \frac{1}{n} > 0$. Since $\gamma - \frac{1}{n}$ belongs to A, it follows that $(\gamma - \frac{1}{n})^2 < c$, whence $\gamma^2 - c < \frac{2\gamma}{n} - \frac{1}{n^2} < \frac{2\gamma}{n}$. Thus, $\frac{\gamma^2 - c}{2\gamma} < \frac{1}{n}$, for all $n \geq N$, and a fortiori also for $1 \leq n < N$. Consequently, $\frac{\gamma^2 - c}{2\gamma} \leq \inf\{\frac{1}{n} : n \in \mathbb{P}\} = 0$. Since $2\gamma > 0$, we conclude that $\gamma^2 - c \leq 0$.

Definition of Square Root

Definition (Square Root)

If $c \in \mathbb{R}, c > 0$, then the unique $x \in \mathbb{R}, x > 0$, such that $x^2 = c$ is called the **square root** of c and is denoted $\sqrt{c}$. We also define $\sqrt{0} = 0$.

- It follows by the theorem that every nonnegative real number has a unique nonnegative square root.

Subsection 9

Absolute Value

Absolute Value and Basic Properties

Definition (Absolute Value)

The **absolute value** of a real number a is the nonnegative real number $|a|$ defined as follows:

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a \leq 0 \end{cases}$$

Theorem (Properties of the Absolute Value)

For real numbers a, b, c, x,

(1) $|a| \geq 0$.

(2) $|a|^2 = a^2$.

(3) Properties (1) and (2) characterize $|a|$: if $x \geq 0$ and $x^2 = a^2$, then $x = |a|$.

(4) $|a| = 0 \Leftrightarrow a = 0$; $|a| > 0 \Leftrightarrow a \neq 0$.

(5) $|a| = |b| \Leftrightarrow a^2 = b^2 \Leftrightarrow a = \pm b$.

(6) $|-a| = |a|$.

(7) $|ab| = |a||b|$.

(8) $-|a| \leq a \leq |a|$.

(9) $|x| \leq c \Leftrightarrow -c \leq x \leq c$.

(10) $|a + b| \leq |a| + |b|$.

(11) $||a| - |b|| \leq |a - b|$.

Proof of the Absolute Value Properties

- (1) $|a| \geq 0$, (2) $|a|^2 = a^2$ and (4) $|a| = 0$ iff $a = 0$ and $|a| > 0$ iff $a \neq 0$ are obvious from the definition of absolute value.

(3) If $x \geq 0$ and $x^2 = a^2$, that is, $x^2 = |a|^2$, then $x = |a|$, by a previous theorem.

- (5) and (6) follow easily from (1)-(3).

(7) If $x = |a||b|$, then $x^2 = |a|^2|b|^2 = a^2b^2 = (ab)^2$, whence $x = |ab|$, by (3).

(8) If $a \geq 0$, then $-|a| = -a \leq 0 \leq a = |a|$. If $a \leq 0$, then $-|a| = -(-a) = a \leq 0 \leq |a|$.

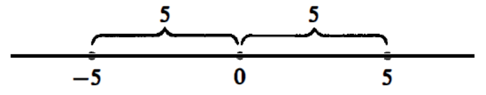
(9) If $-c \leq x \leq c$, then both $-x \leq c$ and $x \leq c$. But $|x|$ is either x or $-x$, so $|x| \leq c$. Conversely, if $|x| \leq c$, then $-c \leq -|x| \leq x \leq |x| \leq c$.

(10) Addition of the inequalities $-|a| \leq a \leq |a|$, $-|b| \leq b \leq |b|$ yields $-(|a| + |b|) \leq a + b \leq |a| + |b|$. So $|a + b| \leq |a| + |b|$ by (9).

(11) Let $x = |a| - |b|$. Then $|a| = |(a - b) + b| \leq |a - b| + |b|$, whence $x \leq |a - b|$. Interchanging a and b, we have $-x \leq |b - a| = |a - b|$, and, hence, $|x| \leq |a - b|$.

Distance Between Real Numbers

- $|a|$ may be interpreted as the distance from the origin to the point a.
- Example: $|\pm 5| = 5$ means that either of the points labeled 5 and -5 has distance 5 from the origin.



Definition (Distance)

For real numbers a, b the **distance** from a to b is defined to be $|a - b|$. We also write $d(a, b) = |a - b|$. The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by this formula is called the **distance function** on $\mathbb{R}$.

- Example: If $a = -2$ and $b = 5$, then $|a - b| = |-2 - 5| = 7$.