Introduction to Real Analysis

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Fixing Some Notation

- The following notations will be fixed:
  - $[a, b]$ is a closed interval of $\mathbb{R}$, $a < b$;
  - $f : [a, b] \to \mathbb{R}$ is a **bounded** function;
  - $M = \sup f = \sup \{f(x) : a \leq x \leq b\}$;
  - $m = \inf f = \inf \{f(x) : a \leq x \leq b\}$.

  To add emphasis to the dependence of M and m on f, we sometimes write $M = M(f)$ and $m = m(f)$.
- Further notation is introduced when needed (for subintervals of $[a, b]$, other functions, etc.).

Subsection 1

Upper and Lower Integrals

Subdivisions

Definition (Subdivision)

A **subdivision** σ of $[a, b]$ is a finite list of points, starting at a, increasing strictly, and ending at b:

$$\sigma = \{a = a_0 < a_1 < a_2 < \cdots < a_n = b\}.$$

- The a_n, $n = 0, 1, 2, \ldots, n$, are called the **points** of the subdivision.
- The **trivial subdivision** $\sigma = \{a = a_0 < a_1 = b\}$ is allowed.
- The effect of σ (when $n > 1$) is to break up the interval $[a, b]$ into n **subintervals**
$$[a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n].$$
- The length of the ν-th subinterval is denoted e_ν, $e_\nu = a_\nu - a_{\nu-1}$, $\nu = 1, \ldots, n$.
- The largest of these lengths is called the **norm** of the subdivision σ, written $N(\sigma) = \max\{e_\nu : \nu = 1, \ldots, n\}$.

Oscillations

Definition

Let $\sigma = \{a = a_0 < a_1 < a_2 < \cdots < a_n = b\}$ be a subdivision of $[a, b]$. For $\nu = 1, \ldots, n$, we write

$$\begin{aligned} M_\nu &= \sup\{f(x) : a_{\nu-1} \leq x \leq a_\nu\}, \\ m_\nu &= \inf\{f(x) : a_{\nu-1} \leq x \leq a_\nu\}. \end{aligned}$$

Obviously $m_\nu \leq M_\nu$ and the difference

$$\omega_\nu = M_\nu - m_\nu \geq 0$$

is called the **oscillation** of f over the subinterval $[a_{\nu-1}, a_\nu]$.

- To emphasize the dependence of these numbers on f, we write $M_\nu(f)$, $m_\nu(f)$, $\omega_\nu(f)$, respectively.

Upper and Lower Sums

Definition (Upper and Lower Sums)

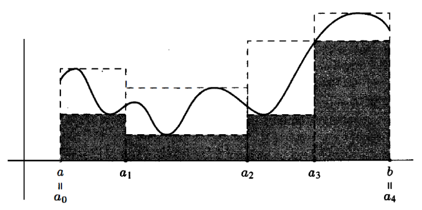
Let $\sigma = \{a = a_0 < a_1 < a_2 < \cdots < a_n = b\}$ be a subdivision of $[a, b]$. The **upper sum** of f for the subdivision σ is the number

$$S(\sigma) = \sum_{\nu=1}^{n} M_\nu e_\nu$$

and the **lower sum** of f for σ is the number

$$s(\sigma) = \sum_{\nu=1}^{n} m_\nu e_\nu.$$

- Again, we write $S_f(\sigma)$ and $s_f(\sigma)$ to express the dependence of these numbers on f and σ.
- The upper and lower sums can be interpreted as crude "rectangular" approximations to the area under the graph of f:



Boundedness of Upper and Lower Sums

Theorem

If σ is any subdivision of $[a, b]$, then

$$m(b-a) \leq s(\sigma) \leq S(\sigma) \leq M(b-a).$$

- Say $\sigma = \{a = a_0 < a_1 < \cdots < a_n = b\}$. For $\nu = 1, \ldots, n$,

  $$m \leq m_\nu \leq M_\nu \leq M.$$

  By multiplying all four sides by e_ν, we get

  $$me_\nu \leq m_\nu e_\nu \leq M_\nu e_\nu \leq Me_\nu.$$

  Finally, take the sum over $\nu = 1, \ldots, n$:

  $$m(b-a) \leq s(\sigma) \leq S(\sigma) \leq M(b-a).$$

- It follows that the sets $\{s(\sigma) : \sigma \text{ any subdivision of } [a, b]\}$ and $\{S(\sigma) : \sigma \text{ any subdivision of } [a, b]\}$ are bounded.

Lower and Upper Integrals

Definition (Lower and Upper Integrals)

The **lower integral of f over** $[a,b]$ is defined to be the supremum of the lower sums, written

$$\underline{\int_a^b} f = \sup\{s(\sigma) : \sigma \text{ any subdivision of } [a,b]\},$$

and the **upper integral** is defined to be the infimum of all the upper sums, written

$$\overline{\int_a^b} f = \inf\{S(\sigma) : \sigma \text{ any subdivision of } [a,b]\}.$$

- Example: Consider

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational in } [a,b] \\ 0, & \text{if } x \text{ is irrational in } [a,b] \end{cases}$$

For this function, every lower sum is 0 and every upper sum is $b-a$. Thus, $\underline{\int_a^b} f = 0$ and $\overline{\int_a^b} f = b-a$.

Convergence and Divergence

- For the upper integral:
  - For each subdivision σ, we take a supremum (actually, one for each term of $S(\sigma)$),
  - then we take the infimum of the $S(\sigma)$ over all possible subdivisions σ,

  a process analogous to the limit superior of a bounded sequence.
- Similarly, the definition of lower integral is analogous to the limit inferior of a bounded sequence (inf followed by sup).
- The preceding example represents a sort of "divergence".
- Just as the "nice" bounded sequences are the convergent ones (those for which $\liminf = \limsup$), the "nice"' bounded functions should, by analogy, be those for which the lower integral is equal to the upper integral.

Bounds

- Necessarily, for every subdivision σ, we have

$$s(\sigma) \leq \underline{\int_a^b} f \quad \text{and} \quad \overline{\int_a^b} f \leq S(\sigma).$$

Theorem

For every bounded function $f : [a, b] \to \mathbb{R}$,

$$m(b-a) \leq \underline{\int_a^b} f \leq M(b-a) \quad \text{and} \quad m(b-a) \leq \overline{\int_a^b} f \leq M(b-a),$$

where $m = \inf f$ and $M = \sup f$.

Refinements

- Upper and lower sums are in a sense approximations to the upper and lower integrals. The way to improve the approximation is to make the subdivision "finer":

Definition (Refinement)

Let σ and τ be subdivisions of $[a, b]$. We say that τ **refines** σ (or that τ is a **refinement of** σ), written $\tau \succ \sigma$ or $\sigma \prec \tau$, if every point of σ is also a point of τ. Thus, if

$$\begin{array}{rcl} \sigma & = & \{a = a_0 < a_1 < \cdots < a_n = b\} \\ \tau & = & \{a = b_0 < b_1 < \cdots < b_m = b\}, \end{array}$$

then $\tau \succ \sigma$ means that each a_ν is equal to some b_μ, i.e., as sets, $\{a_0, a_1, \ldots, a_n\} \subseteq \{b_0, b_1, \ldots, b_m\}$.

- Remarks: Note $\sigma \succ \sigma$; if $\rho \succ \tau$ and $\tau \succ \sigma$ then $\rho \succ \sigma$. If $\tau \succ \sigma$ and $\sigma \succ \tau$, then σ and τ are the same subdivision and we write $\sigma = \tau$.
- Also note that if $\tau \succ \sigma$, then, obviously, $N(\tau) \leq N(\sigma)$.

Effect of Refinements on Sums

- The effect of refinement on upper and lower sums is described in the following:

Lemma

If $\tau \succ \sigma$, then $S(\tau) \leq S(\sigma)$ and $s(\tau) \geq s(\sigma)$.

- The lemma asserts that refinement can only decrease (or leave fixed) an upper sum and can only increase (or leave fixed) a lower sum.

  If $\tau = \sigma$, there is nothing to prove. Otherwise, if τ has $r \geq 1$ points not in σ, we can start at σ and arrive at τ in r steps by inserting one of these points at a time, say $\sigma = \sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_r = \tau$, where σ_k is obtained from σ_{k-1} by inserting one new point. We need only show that $S(\sigma_k) \leq S(\sigma_{k-1})$ and $s(\sigma_k) \geq s(\sigma_{k-1})$, i.e., it suffices to consider the case that τ is obtained from σ by adding only one new point c.

Effect of Refinements on Sums (Cont'd)

- Suppose $\sigma = \{a = a_0 < a_1 < \cdots < a_n = b\}$. Say c belongs to the μ-th subinterval, $a_{\mu-1} < c < a_\mu$. Then, $\tau = \{a = a_0 < a_1 < \cdots < a_{\mu-1} < c < a_\mu < a_{\mu+1} < \cdots < a_n = b\}$. The terms of $S(\tau)$ are the same as those of $S(\sigma)$ except that the μ-th term of $S(\sigma)$ is replaced by two terms of $S(\tau)$. Thus, in calculating $S(\sigma) - S(\tau)$ all of the action is in the μ-th term of $S(\sigma)$. By replacing f by its restriction to $[a_{\mu-1}, a_\mu]$, we are reduced to the case where $\sigma = \{a < b\}$, $\tau = \{a < c < b\}$. Writing $M = \sup f$ as before, and

$$M' = \sup\{f(x) : a \leq x \leq c\}, \quad M'' = \sup\{f(x) : c \leq x \leq b\},$$

we obtain $S(\sigma) = M(b-a)$ and $S(\tau) = M'(c-a) + M''(b-c)$. Obviously $M' \leq M$ and $M'' \leq M$. Therefore, $S(\tau) \leq M(c-a) + M(b-c) = M(b-a) = S(\sigma)$, whence $S(\tau) \leq S(\sigma)$.
A similar argument shows that $s(\tau) \geq s(\sigma)$.

Any Lower Sum Dominated by Any Upper Sum

- We have already seen that, for any subdivision σ of $[a, b]$

$$m(b-a) \leq s(\sigma) \leq S(\sigma) \leq M(b-a).$$

In fact, even more is true:

Lemma

If σ and τ are any two subdivisions of $[a, b]$, then $s(\sigma) < S(\tau)$.

- Let ρ be a subdivision, such that $\rho \succ \sigma$ and $\rho \succ \tau$. Such a ρ is called a **common refinement** of σ and τ and may be constructed, e.g., by taking together all of the points of σ and τ. By previous results,

$$s(\sigma) \leq s(\rho) \leq S(\rho) \leq S(\tau).$$

Lower Integral Dominated by Upper Integral

Theorem ($\liminf \leq \limsup$)

For every bounded function $f : [a, b] \to \mathbb{R}$,

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

- Fix a subdivision τ. By the lemma, for every subdivision σ, $s(\sigma) \leq S(\tau)$. Thus, by the definition of lower integral (as the least upper bound of the set of all lower sums), $\underline{\int_a^b} f \leq S(\tau)$. Letting τ vary, the previous inequality holding for all τ implies $\underline{\int_a^b} f \leq \overline{\int_a^b} f$, by the definition of the upper integral (as the greatest lower bound of the set of all upper sums).

Subsection 2

First Properties of Upper and Lower Integrals

Lower in Terms of Upper Integrals

- The following theorem reduces the study of lower integrals to that of upper integrals:

Theorem

For every bounded function $f : [a, b] \to \mathbb{R}$,

$$\underline{\int_a^b} f = -\overline{\int_a^b} (-f).$$

- Let σ be any subdivision of $[a, b]$. With $A_\nu = \{f(x) : a_{\nu-1} \leq x \leq a_\nu\}$, we have $\sup(-A_\nu) = -(\inf A_\nu)$. Therefore, $M_\nu(-f) = -m_\nu(f)$, for $\nu = 1, \ldots, n$, whence $S_{-f}(\sigma) = -s_f(\sigma)$. Writing

$$B = \{s_f(\sigma) : \sigma \text{ any subdivision of } [a, b]\},$$

we have

$$-B = \{S_{-f}(\sigma) : \sigma \text{ any subdivision of } [a, b]\}.$$

Thus, $\underline{\int_a^b} f = \sup B = -\inf(-B) = -\overline{\int_a^b}(-f)$.

Notation for Restrctions

Definition (Notation for Restrictions)

If $a \leq c < d \leq b$, the definitions for f can be applied to the restriction $f \restriction_{[c,d]}$ of f to $[c, d]$, i.e., to the function $x \mapsto f(x), c \leq x \leq d$. Instead of the cumbersome notations

$$\underline{\int_c^d} f \restriction_{[c,d]} \quad \text{and} \quad \overline{\int_c^d} f \restriction_{[c,d]},$$

we write simply

$$\underline{\int_c^d} f \quad \text{and} \quad \overline{\int_c^d} f.$$

It is also convenient to define

$$\underline{\int_c^c} f = \overline{\int_c^c} f = 0,$$

for any $c \in [a, b]$.

Additivity of Upper and Lower Integrals

- We show that the upper and lower integral is (for a fixed function f) an additive function of the endpoints of integration:

Theorem

If $a \leq c \leq b$, then

$$\text{(i)} \ \overline{\int_a^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f; \qquad \text{(ii)} \ \underline{\int_a^b} f = \underline{\int_a^c} f + \underline{\int_c^b} f.$$

- Both equations are trivial when $c = a$ or $c = b$. Suppose $a < c < b$. It suffices to prove (i). Writing L for the left side and R for the right side, we show that $L \leq R$ and $L \geq R$.
  - $L \leq R$: Let σ_1 be any subdivision of $[a, c]$, σ_2 any subdivision of $[c, b]$, and write $\sigma = \sigma_1 \oplus \sigma_2$ for the subdivision of $[a, b]$ obtained by joining σ_1 and σ_2 at their common point c. Then $S(\sigma) = S(\sigma_1) + S(\sigma_2)$. (the upper sum on the left pertains to f, those on the right pertain to the restrictions of f to $[a, c]$ and $[c, b]$).

Additivity of Upper and Lower Integrals (Cont'd)

- We continue with the proof of (i):
  - Showing that $L \leq R$, we have $S(\sigma) = S(\sigma_1) + S(\sigma_2)$. Thus, $\overline{\int_a^b} f \leq S(\sigma) = S(\sigma_1) + S(\sigma_2)$. So $\overline{\int_a^b} f - S(\sigma_1) \leq S(\sigma_2)$. Varying σ_2 over all possible subdivisions of $[c, b]$, it follows that $\overline{\int_a^b} f - S(\sigma_1) \leq \overline{\int_c^b} f$. Thus, $\overline{\int_a^b} f - \overline{\int_c^b} f \leq S(\sigma_1)$. Since this holds for all σ_1, we get $\overline{\int_a^b} f - \overline{\int_c^b} f \leq \overline{\int_a^c} f$.
  - $L \geq R$: Let σ be any subdivision of $[a, b]$. Let τ be a subdivision of $[a, b]$, such that $\tau \succ \sigma$ and τ includes the point c (for example, let τ be the result of inserting c into σ if it is not already there). Since c is a point of τ, as in the first part of the proof we can write $\tau = \tau_1 \oplus \tau_2$, with τ_1 a subdivision of $[a, c]$ and τ_2 a subdivision of $[c, b]$. Then $S(\sigma) \geq S(\tau) = S(\tau_1) + S(\tau_2) \geq \overline{\int_a^c} f + \overline{\int_c^b} f$. Thus, $S(\sigma) \geq R$, for every subdivision σ of $[a, b]$, whence $L \geq R$.

Subsection 3

Indefinite Upper and Lower Integrals

Indefinite Integrals

Definition (Indefinite Integrals)

For the given bounded function $f : [a, b] \to \mathbb{R}$, we define functions $F : [a, b] \to \mathbb{R}$ and $H : [a, b] \to \mathbb{R}$ by the formulas

$$F(x) = \overline{\int_a^x} f, \quad H(x) = \underline{\int_a^x} f, \quad a \leq x \leq b.$$

We may also consider variable lower endpoints of integration, leading to a function G complementary to F, and a function K complementary to H. The function F is called the **indefinite upper integral** of f. H is called the **indefinite lower integral** of f.

- By a previously adopted convention, $F(a) = H(a) = 0$.
- Moreover, we know that $H(x) \leq F(x)$, for all $x \in [a, b]$.
- We show that the functions F and H have nice properties even if nothing is assumed about the given bounded function f.
  Moreover, every nice property of f (like continuity) yields an even nicer property of F (like differentiability).

Lipschitz Continuity of the Indefinite Integrals

Theorem

Let $k = \max\{|m|, |M|\}$, where $m = \inf f$ and $M = \sup f$. Then

$$|F(x) - F(y)| \leq k|x-y|, \quad |H(x) - H(y)| \leq k|x-y|,$$

for all $x, y \in [a,b]$. In particular, F and H are continuous on $[a,b]$.

- We can suppose $x < y$. By the additivity property, $\underline{\int}_a^y f = \underline{\int}_a^x f + \underline{\int}_x^y f$. Thus, $\underline{\int}_x^y f = F(y) - F(x)$. If m' and M' are the infimum and supremum of f on the interval $[x,y]$, we have $m \leq m' \leq M' \leq M$. This yields $m(y-x) \leq m'(y-x) \leq \underline{\int}_x^y f \leq M'(y-x) \leq M(y-x)$. Therefore, $m(y-x) \leq F(y) - F(x) \leq M(y-x)$. Since $|m| \leq k$ and $|M| \leq k$, $-k(y-x) \leq F(y) - F(x) \leq k(y-x)$, whence $|F(y) - F(x)| \leq k(y-x) = k|y-x|$.

  The proof for H is similar.

Monotonicity of Indefinite Integrals

Theorem (Monotonicity of Indefinite Integrals)

If $f \geq 0$, then F and H are increasing functions.

- If $f \geq 0$, then $m \geq 0$, whence the upper and lower integrals of a nonnegative function are nonnegative. If $a \leq c < d \leq b$, then $F(d) = F(c) + \overline{\int_c^d} f \geq F(c)$. Hence F is increasing.

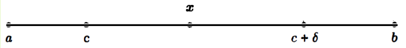
  A similar reasoning applies to H.

Right Differentiability of Indefinite Integrals

Theorem (Right Differentiability of Indefinite Integrals)

If $a \leq c < b$ and f is right continuous at c, then F and H are right differentiable at c and $F_r'(c) = H_r'(c) = f(c)$.

- We give the proof for F; the proof for H is similar. Let $\epsilon > 0$. We seek $\delta > 0$, $c + \delta < b$, with $c < x < c + \delta \Rightarrow \left|\frac{F(x)-F(c)}{x-c} - f(c)\right| \leq \epsilon$. Since f is right continuous at c, there exists a $\delta > 0$, with $c + \delta < b$, such that $c \leq t \leq c + \delta \Rightarrow |f(t) - f(c)| \leq \epsilon$. Consider $c < x < c + \delta$:



For $t \in [c, x]$, $|f(t) - f(c)| \leq \epsilon$, whence $f(c) - \epsilon \leq f(t) \leq f(c) + \epsilon$. If m_x and M_x are the infimum and supremum of f on $[c, x]$, then $f(c) - \epsilon \leq m_x \leq M_x \leq f(c) + \epsilon$. Therefore,
$[f(c) - \epsilon](x - c) \leq m_x(x - c) \leq \overline{\int_c^x} f \leq M_x(x - c) \leq [f(c) + \epsilon](x - c)$.
Finally, we get $[f(c) - \epsilon](x - c) \leq F(x) - F(c) \leq [f(c) + \epsilon](x - c)$.

Differentiability of Indefinite Integrals

Theorem (Left Differentiability of Indefinite Integrals)

If $a < c \leq b$ and f is left continuous at c, then F and H are left differentiable at c and $F'_\ell(c) = H'_\ell(c) = f(c)$.

- The easiest strategy is to modify the preceding proof: Replace $c < x < c + \delta$ by $c - \delta < x < c$, $[c, x]$ by $[x, c]$, etc.
  An alternative strategy is to apply the "right" version to the function $g : [-b, -a] \to \mathbb{R}$ defined by $g(y) = f(-y)$, which is right continuous at $-c$ when f is left continuous at c. The relations among the indefinite integrals of f and g are easy to verify, but cumbersome.

Corollary

If $a < c < b$ and f is continuous at c, then F and H are differentiable at c and $F'(c) = H'(c) = f(c)$.

- By assumption, f is both left and right continuous at c, whence $F'_\ell(c) = f(c) = F'_r(c)$ and $H'_\ell(c) = f(c) = H'_r(c)$. F and H are differentiable at c, with $F'(c) = f(c)$ and $H'(c) = f(c)$.

Indefinite Integrals in Terms of Lower Points

- We look at the upper and lower integrals as functions of the lower endpoint of integration:

Definition (Indefinite Integrals Revisited)

For the given bounded function $f : [a, b] \to \mathbb{R}$, we define functions $G : [a, b] \to \mathbb{R}$ and $K : [a, b] \to \mathbb{R}$ by the formulas

$$G(x) = \overline{\int_x^b} f, \quad K(x) = \underline{\int_x^b} f, \quad a \le x \le b.$$

- Remarks: We have $F(x) + G(x) = \overline{\int_a^b} f$ and $H(x) + K(x) = \underline{\int_a^b} f$, for $a \le x \le b$. Thus, G is in a sense complementary to F, and K to H. This is the key to deducing the properties of G from those of F, and the properties of K from those of H: E.g., since F and H are continuous, so are G and K.

Differentiability of G and K

Theorem (Right Differentiability of G and K)

If $a \leq c < b$ and f is right continuous at c, then G and K are right differentiable at c and $G_r'(c) = K_r'(c) = -f(c)$.

- This is immediate from right differentiability of F and H and the preceding complementarity formulas.

Theorem (Left Differentiability of G and K)

If $a < c \leq b$ and f is left continuous at c, then G and K are left differentiable at c and $G_\ell'(c) = K_\ell'(c) = -f(c)$.

Corollary (Differentiability of G and K)

If $a < c < b$ and f is continuous at c, then G and K are differentiable at c and $G'(c) = K'(c) = -f(c)$.

Subsection 4

Riemann Integrable Functions

Riemann Integrability

Definition (Riemann Integral)

A bounded function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann-integrable** (briefly, **integrable**) if

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

(The analogous concept for bounded sequences ($\liminf = \limsup$) is convergence!) We write simply $\int_a^b f$ or (especially when $f(x)$ is replaced by a formula for it) $\int_a^b f(x)dx$ for the common value of the lower and upper integral, and call it the **integral** (or **Riemann integral**) of f.

- Remark: If f is Riemann-integrable, then so is $-f$, and $\int_a^b (-f) = -\int_a^b f$.

Monotonicity and Riemann Integrability

- If $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$, then f is not Riemann-integrable.

Theorem

If f is monotone, then it is Riemann-integrable.

- We can suppose that f is increasing. For every subdivision σ of $[a,b]$, we have $s(\sigma) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq S(\sigma)$. To show that the lower integral is equal to the upper integral, we need only show that $S(\sigma) - s(\sigma)$ can be made as small as we like (by choosing σ appropriately). Say $\sigma = \{a = a_0 < a_1 < \cdots < a_n = b\}$. Since f is increasing, we have $m_\nu = f(a_{\nu-1})$, $M_\nu = f(a_\nu)$. Thus, $s(\sigma) = \sum_{\nu=1}^n f(a_{\nu-1})e_\nu$ and $S(\sigma) = \sum_{\nu=1}^n f(a_\nu)e_\nu$. So $S(\sigma) - s(\sigma) = \sum_{\nu=1}^n [f(a_\nu) - f(a_{\nu-1})]e_\nu$. Now assume that the points of σ are equally spaced, so that $e_\nu = \frac{1}{n}(b-a)$. The sum, then, "telescopes":
$S(\sigma) - s(\sigma) = \frac{1}{n}(b-a)\sum_{\nu=1}^n [f(a_\nu) - f(a_{\nu-1})] = \frac{1}{n}(b-a)[f(b) - f(a)]$,
which can be made arbitrarily small by taking n sufficiently large.

Continuity and Riemann Integrability

Theorem

If f is continuous on $[a,b]$ then f is Riemann integrable.

- Let $F = \overline{\int_a^x} f$ and $H = \underline{\int_a^x} f$ be the indefinite upper integral and indefinite lower integral. We know that $F(a) = H(a) = 0$. We must show that $F(b) = H(b)$.

  We know F and H are continuous on $[a,b]$. Also, F and H are differentiable on (a,b) with $F'(x) = f(x) = H'(x)$, for all $x \in (a,b)$. Thus, $F - H$ is continuous on $[a,b]$, differentiable on (a,b), and $(F-H)'(x) = 0$, for all $x \in (a,b)$. Therefore, $F - H$ is constant by a corollary of the Mean Value Theorem. Since $(F-H)(a) = 0$, also $(F-H)(b) = 0$. Thus, $F(b) = H(b)$, as we wished to show.

The Fundamental Theorem of Calculus

Theorem (The Fundamental Theorem of Calculus)

If $f : [a, b] \to \mathbb{R}$ is continuous, then:

(1) f is Riemann-integrable on $[a, b]$;

(2) There exists a continuous function $F : [a, b] \to \mathbb{R}$, differentiable on (a, b), such that $F'(x) = f(x)$, for all $x \in (a, b)$;

(3) For any F satisfying (2), $F(x) = F(a) + \int_a^x f$, for all $x \in [a, b]$. Moreover, F is right differentiable at a, left differentiable at b, and $F_r'(a) = f(a)$, $F_\ell'(b) = f(b)$.

- Part (1) is the conclusion of the preceding theorem. $F(x) = \overline{\int_a^x} f$ has the properties in (2) and (3). Suppose that $J : [a, b] \to \mathbb{R}$ is also a continuous function having derivative $f(x)$ at every $x \in (a, b)$. By the argument used in the preceding theorem, $J - F$ is constant, say $J(x) = F(x) + C$, for all $x \in [a, b]$. Then $J(x) - J(a) = F(x) - F(a)$ $= \int_a^x f$, for all $x \in [a, b]$. Finally, J has the one-sided derivatives $f(a)$ and $f(b)$ at the endpoints since F does.

Consequences of the Fundamental Theorem

Corollary

If $f : [a, b] \to \mathbb{R}$ is continuous and $F : [a, b] \to \mathbb{R}$ is a continuous function, differentiable on (a, b), such that $F'(x) = f(x)$, for all $x \in (a, b)$, then

$$\int_a^b f = F(b) - F(a).$$

Corollary

If $f : [a, b] \to \mathbb{R}$ is continuous, $f \geq 0$ on $[a, b]$, and $\int_a^b f = 0$, then $f \equiv 0$.

- If $F = \int_a^x f$, then F is increasing and $F(b) - F(a) = \int_a^b f = 0$. Therefore, F is constant. Then $f = F' = 0$ on (a, b), whence $f = 0$ on $[a, b]$ by continuity.