Introduction to Spectral Theory of Linear Operators

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

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Subsection 1

Unbounded Operators and their Hilbert-Adjoint Operators

Unbounded Operators

- Let H be a complex Hilbert space.
- We consider linear operators $T : \mathscr{D}(T) \to H$, with $\mathscr{D}(T) \subseteq H$.
- T is bounded if and only if there is a real number c, such that

$$\|Tx\| \le c\|x\|, \quad \text{for all } x \in \mathscr{D}(T).$$

- An important unbounded linear operator is the differentiation operator.
- Note that the operator T may be unbounded.
- In the case of a bounded linear operator T on a Hilbert space H, self-adjointness of T was defined by $\langle Tx, y\rangle = \langle x, Ty\rangle$.
- The following theorem shows that an unbounded linear operator T satisfying this relationship cannot be defined on all of H.

The Hellinger-Toeplitz Boundedness Theorem

Hellinger-Toeplitz Theorem (Boundedness)

If a linear operator T is defined on all of a complex Hilbert space H and satisfies $\langle Tx, y\rangle = \langle x, Ty\rangle$, for all $x, y \in H$, then T is bounded.

- Suppose, to the contrary, that T is not bounded.

  Then H contains a sequence (y_n) such that $\|y_n\| = 1$ and $\|Ty_n\| \to \infty$.

  We consider, for $n = 1, 2, \ldots$, the functional f_n defined by

  $$f_n(x) = \langle Tx, y_n\rangle = \langle x, Ty_n\rangle.$$

  Each f_n is defined on all of H and is linear.

  For each n, f_n is bounded, since, by the Schwarz inequality,

  $$|f_n(x)| = |\langle x, Ty_n\rangle| \leq \|Ty_n\| \|x\|.$$

The Hellinger-Toeplitz Boundedness Theorem (Cont'd)

- Moreover, for every fixed $x \in H$, the sequence $(f_n(x))$ is bounded. Indeed, using the Schwarz inequality and $\|y_n\| = 1$, we have

$$|f_n(x)| = |\langle Tx, y_n \rangle| \leq \|Tx\|.$$

By the Uniform Boundedness Theorem, $(\|f_n\|)$ is bounded, say, $\|f_n\| \leq k$, for all n. Thus, for every $x \in H$, we have

$$|f_n(x)| \leq \|f_n\| \|x\| \leq k\|x\|.$$

Taking $x = Ty_n$, we get

$$\|Ty_n\|^2 = \langle Ty_n, Ty_n \rangle = |f_n(Ty_n)| \leq k\|Ty_n\|.$$

Hence, $\|Ty_n\| \leq k$. But this contradicts $\|Ty_n\| \to \infty$.

Extensions and Hilbert-Adjoints

- By the Hellinger-Toeplitz Boundedness Theorem, $\mathscr{D}(T) = H$ is impossible for unbounded linear operators satisfying $\langle Tx, y\rangle = \langle x, Ty\rangle$.
- The problem is to determine suitable domains for extensions.
- The operator T is an **extension** of the operator S, written $S \subseteq T$, if $\mathscr{D}(S) \subseteq \mathscr{D}(T)$ and $S = T\mid_{\mathscr{D}(S)}$.
- An extension T of S is a **proper extension** if $\mathscr{D}(S)$ is a proper subset of $\mathscr{D}(T)$, i.e., $\mathscr{D}(T) - \mathscr{D}(S) \neq \emptyset$.

The Role of Hilbert-Adjoint

- For bounded operators, the Hilbert-adjoint T^* of an operator T plays a basic role and we want to generalize to the unbounded case.
- In the bounded case the operator T^* is defined by

$$\langle Tx, y\rangle = \langle x, T^* y\rangle.$$

- We can write this as

$$\langle Tx, y\rangle = \langle x, y^*\rangle, \qquad y^* = T^* y.$$

- T^* exists on H and is a bounded linear operator with norm $\|T^*\| = \|T\|$.
- In the general case, T^* must be defined for those $y \in H$, for which there is a y^*, such that, for all $x \in \mathscr{D}(T)$,

$$\langle Tx, y\rangle = \langle x, y^*\rangle, \qquad y^* = T^* y.$$

Conditions for Uniqueness of $T^* y$

- The operator T^* will be defined by $y^* = T^* y$, for those $y \in H$ for which there is a y^*, such that, for all $x \in \mathscr{D}(T)$,

$$\langle Tx, y \rangle = \langle x, y^* \rangle.$$

- In order that T^* be an operator (a mapping), for each y that belongs to the domain $\mathscr{D}(T^*)$ of T^*, the value

$$y^* = T^* y$$

must be unique.

Conditions for Uniqueness of T^*y (Cont'd)

Claim: Uniqueness of y^* holds if and only if T is densely defined in H, i.e., $\mathscr{D}(T)$ is dense in H.

Suppose $\mathscr{D}(T)$ is not dense in H. Then $\overline{\mathscr{D}(T)} \neq H$.

The orthogonal complement of $\overline{\mathscr{D}(T)}$ in H contains a nonzero y_1.

So $y_1 \perp x$, for every $x \in \mathscr{D}(T)$, i.e., $\langle x, y_1 \rangle = 0$.

Then in $\langle Tx, y \rangle = \langle x, y^* \rangle$, we obtain

$$\langle x, y^* \rangle = \langle x, y^* \rangle + \langle x, y_1 \rangle = \langle x, y^* + y_1 \rangle.$$

This shows non-uniqueness.

Suppose, conversely, $\mathscr{D}(T)$ is dense in H.

Then $\mathscr{D}(T)^\perp = \{0\}$.

Hence, $\langle x, y_1 \rangle = 0$, for all $x \in \mathscr{D}(T)$, implies $y_1 = 0$.

So $y^* + y_1 = y^*$. This proves uniqueness.

Hilbert-Adjoint Operator

- We use the following terminology:
  - T is an operator **on** H if $\mathscr{D}(T)$ is all of H;
  - T is an operator **in** H if $\mathscr{D}(T)$ lies in H but may not be all of H.

Definition (Hilbert-Adjoint Operator)

Let $T : \mathscr{D}(T) \to H$ be a (possibly unbounded) densely defined linear operator in a complex Hilbert space H. Then the **Hilbert-adjoint operator** $T^* : \mathscr{D}(T^*) \to H$ of T is defined as follows. The domain $\mathscr{D}(T^*)$ of T^* consists of all $y \in H$, such that, there is a $y^* \in H$ satisfying

$$\langle Tx, y \rangle = \langle x, y^* \rangle, \quad \text{for all } x \in \mathscr{D}(T).$$

For each such $y \in \mathscr{D}(T^*)$, the **Hilbert-adjoint operator** T^* is then defined in terms of that y^* by $y^* = T^* y$.

Remarks on Hilbert-Adjoint Operators

- An element $y \in H$ is in $\mathscr{D}(T^*)$ if for that y, $\langle Tx, y\rangle$, considered as a function of x, can be represented as

$$\langle Tx, y\rangle = \langle x, y^*\rangle, \quad \text{for all } x \in \mathscr{D}(T).$$

- For that y, the formula

$$\langle Tx, y\rangle = \langle x, y^*\rangle, \quad \text{for all } x \in \mathscr{D}(T),$$

  determines y^* uniquely by density.
- Finally, T^* is a linear operator.

Sum of Operators

- Let H be a complex Hilbert space.
- Let $S : \mathscr{D}(S) \to H$ and $T : \mathscr{D}(T) \to H$ be linear operators, where $\mathscr{D}(S) \subseteq H$ and $\mathscr{D}(T) \subseteq H$.
- Then the **sum** $S + T$ of S and T is the linear operator with:
  - Domain $\mathscr{D}(S+T) = \mathscr{D}(S) \cap \mathscr{D}(T)$;
  - For every $x \in \mathscr{D}(S+T)$,

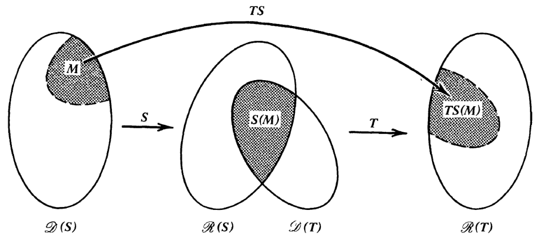
$$(S+T)x = Sx + Tx.$$

- $\mathscr{D}(S+T)$ is the largest set on which both S and T make sense.
- $\mathscr{D}(S+T)$ is a vector space.
- Always $0 \in \mathscr{D}(S+T)$, so that $\mathscr{D}(S+T)$ is never empty.
- Nontrivial results can be expected only if $\mathscr{D}(S+T)$ also contains nonzero elements.

Product of Operators

- Let M be the largest subset of $\mathscr{D}(S)$ whose image $S(M)$ under S lies in $\mathscr{D}(T)$.
- Then $S(M) = \mathscr{R}(S) \cap \mathscr{D}(T)$, where $\mathscr{R}(S)$ is the range of S.
- Then the **product** TS is defined to be the operator with domain $\mathscr{D}(TS) = M$, such that for all $x \in \mathscr{D}(TS)$,

$$(TS)x = T(Sx).$$



Product of Operators (Cont'd)

- Similarly, let $\widetilde{M}$ be the largest subset of $\mathscr{D}(T)$ whose image $T(\widetilde{M})$ under T lies in $\mathscr{D}(S)$.
- Then $T(\widetilde{M}) = \mathscr{R}(T) \cap \mathscr{D}(S)$, where $\mathscr{R}(T)$ is the range of T.
- Then the **product** ST is defined to be the operator with domain $\mathscr{D}(ST) = \widetilde{M}$, such that for all $x \in \mathscr{D}(ST)$,

$$(ST)x = S(Tx).$$

- Both TS and ST are linear operators.

Subsection 2

Hilbert-Adjoint, Symmetric and Self-Adjoint Operators

Hilbert-Adjoint Operators

- By definition, $T^{**} = (T^*)^*$.

Theorem (Hilbert-Adjoint Operator)

Let $S : \mathscr{D}(S) \to H$ and $T : \mathscr{D}(T) \to H$ be linear operators which are densely defined in a complex Hilbert space H. Then:

(a) If $S \subseteq T$, then $T^* \subseteq S^*$.

(b) If $\mathscr{D}(T^*)$ is dense in H, then $T \subseteq T^{**}$.

(a) By definition, $\langle Tx, y \rangle = \langle x, T^* y \rangle$, for all $x \in \mathscr{D}(T)$ and all $y \in \mathscr{D}(T^*)$.
Since $S \subseteq T$, $\langle Sx, y \rangle = \langle x, T^* y \rangle$, for all $x \in \mathscr{D}(S)$ and y as before.
By the definition of S^*, $\langle Sx, y \rangle = \langle x, S^* y \rangle$, for all $x \in \mathscr{D}(S)$, $y \in \mathscr{D}(S^*)$.
Claim: The last two equations imply $\mathscr{D}(T^*) \subseteq \mathscr{D}(S^*)$.

Proof of the Claim

Claim: The last two equations imply $\mathscr{D}(T^*) \subseteq \mathscr{D}(S^*)$.

By the definition of the Hilbert-adjoint operator S^*, the domain $\mathscr{D}(S^*)$ includes all y for which one has a representation

$$\langle Sx, y \rangle = \langle x, S^* y \rangle, \quad \text{for all } x \text{ in } \mathscr{D}(S).$$

But $\langle Sx, y \rangle = \langle x, T^* y \rangle$ also represents $\langle Sx, y \rangle$ in the same form, for x in $\mathscr{D}(S)$.

So the set of y's for which this is valid must be a (proper or improper) subset of the set of y's for which the previous equation holds.

I.e., we must have $\mathscr{D}(T^*) \subseteq \mathscr{D}(S^*)$.

Taking into account both equations, we conclude that

$$S^* y = T^* y, \quad \text{for all } y \in \mathscr{D}(T^*).$$

So, by definition, $T^* \subseteq S^*$.

Hilbert-Adjoint Operators (Part (b))

(b) Taking complex conjugates in $\langle Tx, y\rangle = \langle x, T^* y\rangle$, we have

$$\langle T^* y, x\rangle = \langle y, Tx\rangle, \quad \text{for all } y \in \mathscr{D}(T^*),\ x \in \mathscr{D}(T).$$

Since $\mathscr{D}(T^*)$ is dense in H, the operator T^{**} exists.

By definition,

$$\langle T^* y, x\rangle = \langle y, T^{**} x\rangle, \quad \text{for all } y \in \mathscr{D}(T^*),\ x \in \mathscr{D}(T^{**}).$$

From these equations, reasoning as in Part (a), we see that:

- An $x \in \mathscr{D}(T)$ also belongs to $\mathscr{D}(T^{**})$;
- $T^{**} x = Tx$, for all $x \in \mathscr{D}(T)$.

This means that $T \subseteq T^{**}$.

Inverse of the Hilbert-Adjoint Operator

Theorem (Inverse of the Hilbert-Adjoint Operator)

Let $T:\mathscr{D}(T)\to H$ be a linear operator densely defined in a complex Hilbert space H. Suppose that T is injective and its range $\mathscr{R}(T)$ is dense in H. Then T^* is injective and

$$(T^*)^{-1}=(T^{-1})^*.$$

- T^* exists, since T is densely defined in H.
  Also T^{-1} exists, since T is injective.
  $(T^{-1})^*$ exists, since $\mathscr{D}(T^{-1})=\mathscr{R}(T)$ is dense in H.
  We must show that $(T^*)^{-1}$ exists and satisfies $(T^*)^{-1}=(T^{-1})^*$.
  Let $y\in\mathscr{D}(T^*)$. Then, for all $x\in\mathscr{D}(T^{-1})$, $T^{-1}x\in\mathscr{D}(T)$ and

$$\langle T^{-1}x,T^*y\rangle=\langle TT^{-1}x,y\rangle=\langle x,y\rangle.$$

Inverse of the Hilbert-Adjoint Operator (Cont'd)

- By the definition of the Hilbert-adjoint operator of T^{-1},

$$\langle T^{-1}x, T^*y\rangle = \langle x, (T^{-1})^* T^* y\rangle, \quad \text{for all } x \in \mathscr{D}(T^{-1}).$$

This shows that $T^*y \in \mathscr{D}((T^{-1})^*)$.

Comparing with the preceding equation, we conclude that

$$(T^{-1})^* T^* y = y, \quad y \in \mathscr{D}(T^*).$$

So $T^*y = 0$ implies $y = 0$. Hence, $(T^*)^{-1} : \mathscr{R}(T^*) \to \mathscr{D}(T^*)$ exists.

Since $(T^*)^{-1}T^*$ is the identity operator on $\mathscr{D}(T^*)$, a comparison with the preceding equation shows that $(T^*)^{-1} \subseteq (T^{-1})^*$.

It suffices now to show that $(T^*)^{-1} \supseteq (T^{-1})^*$.

Inverse of the Hilbert-Adjoint Operator (Cont'd)

- Consider any $x \in \mathscr{D}(T)$ and $y \in \mathscr{D}((T^{-1})^*)$.
  Then $Tx \in \mathscr{R}(T) = \mathscr{D}(T^{-1})$. Moreover,

  $$\langle Tx, (T^{-1})^* y \rangle = \langle T^{-1} Tx, y \rangle = \langle x, y \rangle.$$

  By the definition of the Hilbert-adjoint operator of T, we have

  $$\langle Tx, (T^{-1})^* y \rangle = \langle x, T^*(T^{-1})^* y \rangle, \quad \text{for all } x \in \mathscr{D}(T).$$

  From this and the last equation, $(T^{-1})^* y \in \mathscr{D}(T^*)$ and

  $$T^*(T^{-1})^* y = y, \quad \text{for all } y \in \mathscr{D}((T^{-1})^*).$$

  By the definition of an inverse:
  - $T^*(T^*)^{-1}$ is the identity operator on $\mathscr{D}((T^*)^{-1}) = \mathscr{R}(T^*)$;
  - $(T^*)^{-1} : \mathscr{R}(T^*) \to \mathscr{D}(T^*)$ is surjective.

  Comparing with the preceding,, we get $\mathscr{D}((T^*)^{-1}) \supseteq \mathscr{D}((T^{-1})^*)$.
  So $(T^*)^{-1} \supseteq (T^{-1})^*$.

Symmetric Linear Operators

Definition (Symmetric Linear Operator)

Let $T : \mathscr{D}(T) \to H$ be a linear operator which is densely defined in a complex Hilbert space H. T is called a **symmetric linear operator** if

$$\langle Tx, y \rangle = \langle x, Ty \rangle, \quad \text{for all } x, y \in \mathscr{D}(T).$$

Lemma (Symmetric Operator)

A densely defined linear operator T in a complex Hilbert space H is symmetric if and only if

$$T \subseteq T^*.$$

- By the definition of T^*,

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \quad \text{for all } x \in \mathscr{D}(T),\ y \in \mathscr{D}(T^*).$$

Symmetric Linear Operators (Cont'd)

- Suppose, first, that $T \subseteq T^*$.

  Then $T^* y = Ty$, for all $y \in \mathscr{D}(T)$.

  So the preceding equation, for $x, y \in \mathscr{D}(T)$, becomes

  $$\langle Tx, y \rangle = \langle x, Ty \rangle.$$

  Hence, T is symmetric.

  Suppose, next, that

  $$\langle Tx, y \rangle = \langle x, Ty \rangle, \quad \text{for all } x, y \in \mathscr{D}(T).$$

  Then a comparison with $\langle Tx, y \rangle = \langle x, T^* y \rangle$ shows that:
  - $\mathscr{D}(T) \subseteq \mathscr{D}(T^*)$;
  - $T = T^* |_{\mathscr{D}(T)}$.

  By definition, T^* is an extension of T.

Self-Adjoint Linear Operators

Definition (Self-Adjoint Linear Operator)

Let $T : \mathscr{D}(T) \to H$ be a linear operator which is densely defined in a complex Hilbert space H. T is called a **self-adjoint linear operator** if

$$T = T^*.$$

- Every self-adjoint linear operator is symmetric.
- But a symmetric linear operator need not be self-adjoint.
- In fact, T^* may be a proper extension of T, i.e., $\mathscr{D}(T) \neq \mathscr{D}(T^*)$.

On Symmetry and Self-Adjointness

- Of course, $\mathscr{D}(T) \subsetneq \mathscr{D}(T^*)$ cannot happen if $\mathscr{D}(T)$ is all of H.

  For a linear operator $T : H \to H$ on a complex Hilbert space H, the concepts of symmetry and self-adjointness are identical.
- Note that in this case, T is bounded, and this explains why the concept of symmetry did not occur earlier.
- A densely defined linear operator T in a complex Hilbert space H is symmetric if and only if

$$\langle Tx, x \rangle \text{ is real, for all } x \in \mathscr{D}(T).$$

Subsection 3

Closed Linear Operators and Closures

Closed Linear Operators

Definition (Closed Linear Operator)

Let $T:\mathscr{D}(T)\to H$ be a linear operator, where $\mathscr{D}(T)\subseteq H$ and H is a complex Hilbert space. T is called a **closed linear operator** if its graph

$$\mathscr{G}(T)=\{(x,y):x\in\mathscr{D}(T),y=Tx\}$$

is closed in $H\times H$, where the norm on $H\times H$ is defined by

$$\|(x,y)\|=(\|x\|^2+\|y\|^2)^{1/2}.$$

This norm results from the inner product defined by

$$\langle(x_1,y_1),(x_2,y_2)\rangle=\langle x_1,x_2\rangle+\langle y_1,y_2\rangle.$$

The Closed Linear Operator Theorem

- From the theory of closed linear operators, we get the following facts.

Theorem (Closed Linear Operator)

Let $T : \mathscr{D}(T) \to H$ be a linear operator, where $\mathscr{D}(T) \subseteq H$ and H is a complex Hilbert space. Then:

(a) T is closed if and only if $x_n \to x$, $x_n \in \mathscr{D}(T)$ and $Tx_n \to y$ together imply that $x \in \mathscr{D}(T)$ and $Tx = y$.

(b) If T is closed and $\mathscr{D}(T)$ is closed, then T is bounded.

(c) For T be bounded, T is closed if and only if $\mathscr{D}(T)$ is closed.

The Hilbert-Adjoint Operator Theorem

Theorem (Hilbert-Adjoint Operator)

Let $T:\mathscr{D}(T)\to H$ be a linear operator, where $\mathscr{D}(T)\subseteq H$ and H is a complex Hilbert space. The Hilbert-adjoint operator T^* is closed.

- Consider any sequence (y_n) in $\mathscr{D}(T^*)$, such that:
  - $y_n\to y_0$;
  - $T^*y_n\to z_0$.

  We show that $y_0\in\mathscr{D}(T^*)$ and $z_0=T^*y_0$.
  By the definition of T^*, for every $y\in\mathscr{D}(T)$,

  $$\langle Ty,y_n\rangle=\langle y,T^*y_n\rangle.$$

  By continuity of the inner product,

  $$\langle Ty,y_0\rangle=\langle y,z_0\rangle,\quad \text{for every } y\in\mathscr{D}(T).$$

  By the definition of T^*, we get $y_0\in\mathscr{D}(T^*)$ and $z_0=T^*y_0$.
  Applying the preceding theorem, we conclude that T^* is closed.

Closable Operator and Closure

Definition (Closable Operator, Closure)

Let $T : \mathscr{D}(T) \to H$ be a linear operator, where $\mathscr{D}(T) \subseteq H$ and H is a complex Hilbert space.

- If T has an extension T_1 which is a closed linear operator, then T is said to be **closable**, and T_1 is called a **closed linear extension** of T.
- A closed linear extension $\overline{T}$ of a closable linear operator T is said to be **minimal** if every closed linear extension T_1 of T is a closed linear extension of $\overline{T}$. This minimal extension $\overline{T}$ of T - if it exists - is called the **closure** of T.

- If $\overline{T}$ exists, it is unique.
- If T is not closed, the problem arises whether T has closed extensions.

The Closure Theorem

Theorem (Closure)

Let $T : \mathscr{D}(T) \to H$ be a linear operator, where H is a complex Hilbert space and $\mathscr{D}(T)$ is dense in H. Then, if T is symmetric, its closure $\overline{T}$ exists and is unique.

- We define $\overline{T}$ by:
  - First defining the domain $M = \mathscr{D}(\overline{T})$;
  - Then defining $\overline{T}$ itself.

  Then we show that $\overline{T}$ is indeed the closure of T.

  Let M be the set of all $x \in H$ for which there is a sequence (x_n) in $\mathscr{D}(T)$ and a $y \in H$, such that

$$x_n \to x \quad \text{and} \quad Tx_n \to y.$$

  We can show that M is a vector space. Clearly, $\mathscr{D}(T) \subseteq M$.

The Closure Theorem (Cont'd)

- On M we define $\overline{T}$ by setting

$$y = \overline{T}x, \quad x \in M,$$

with y given by

$$x_n \to x, \quad Tx_n \to y.$$

To show that $\overline{T}$ is the closure of T, we have to prove that $\overline{T}$ has all the properties by which the closure is defined.

Obviously, T has the domain $\mathscr{D}(\overline{T}) = M$.

We shall prove:

(a) To each $x \in \mathscr{D}(\overline{T})$, there corresponds a unique y.

(b) $\overline{T}$ is a symmetric linear extension of T.

(c) $\overline{T}$ is closed and is the closure of T.

The Closure Theorem Property (a)

(a) **Uniqueness of** y, for every $x \in \mathscr{D}(\overline{T})$.

In addition to (x_n), let $(\widetilde{x}_n)$ be another sequence in $\mathscr{D}(T)$, such that

$$\widetilde{x}_n \to x \quad \text{and} \quad T\widetilde{x}_n \to \widetilde{y}.$$

Since T is linear, $Tx_n - T\widetilde{x}_n = T(x_n - \widetilde{x}_n)$.

Since T is symmetric, for every $v \in \mathscr{D}(T)$,

$$\langle v, Tx_n - T\widetilde{x}_n \rangle = \langle v, T(x_n - \widetilde{x}_n) \rangle = \langle Tv, x_n - \widetilde{x}_n \rangle.$$

Letting $n \to \infty$ and using the continuity of the inner product,

$$\langle v, y - \widetilde{y} \rangle = \langle Tv, x - x \rangle = 0.$$

Therefore, $y - \widetilde{y} \perp \mathscr{D}(T)$. Since $\mathscr{D}(T)$ is dense in H, $\mathscr{D}(T)^{\perp} = \{0\}$. Hence, $y - \widetilde{y} = 0$. Thus, $y = \widetilde{y}$.

The Closure Theorem Property (b)

(b) **$\overline{T}$ is a symmetric linear extension of T:**

Since T is linear, so is $\overline{T}$.

This also shows that $\overline{T}$ is an extension of T.

We show that the symmetry of T implies that of $\overline{T}$.

For all $x, z \in \mathscr{D}(\overline{T})$, there are sequences $(x_n), (z_n)$ in $\mathscr{D}(T)$, such that

$$\begin{aligned} x_n \to x, \quad Tx_n \to \overline{T}x \\ z_n \to z, \quad Tz_n \to \overline{T}z. \end{aligned}$$

Since T is symmetric, $\langle z_n, Tx_n \rangle = \langle Tz_n, x_n \rangle$.

Letting $n \to \infty$ and using the continuity of the inner product,

$$\langle z, \overline{T}x \rangle = \langle \overline{T}z, x \rangle.$$

Since $x, z \in \mathscr{D}(\overline{T})$ were arbitrary, this shows that $\overline{T}$ is symmetric.

The Closure Theorem Property (c)

(c) **$\overline{T}$ is closed and is the closure of T:**
We prove closedness of $\overline{T}$ by considering any sequence (w_m) in $\mathscr{D}(\overline{T})$, such that $w_m \to x$ and $\overline{T}w_m \to y$ and proving $x \in \mathscr{D}(\overline{T})$ and $\overline{T}x = y$.
Every w_m (m fixed) is in $\mathscr{D}(\overline{T})$.
By the definition of $\mathscr{D}(\overline{T})$, there is a sequence in $\mathscr{D}(T)$ which converges to w_m and whose image under T converges to $\overline{T}w_m$.
Hence, for every fixed m, there is a $v_m \in \mathscr{D}(T)$, such that

$$\|w_m - v_m\| < \frac{1}{m} \quad \text{and} \quad \|\overline{T}w_m - Tv_m\| < \frac{1}{m}.$$

From this, we conclude that $v_m \to x$ and $Tv_m \to y$.
By the definitions of $\mathscr{D}(\overline{T})$ and $\overline{T}$, we get $x \in \mathscr{D}(\overline{T})$ and $y = \overline{T}x$.
Hence, $\overline{T}$ is closed.
By the Closed Linear Operator Theorem, every point of $\mathscr{D}(\overline{T})$ must also belong to the domain of every closed linear extension of T.
So $\overline{T}$ is the closure of T. We also get that the closure is unique.

The Hilbert-Adjoint of the Closure

Theorem (Hilbert-Adjoint of the Closure)

For a symmetric linear operator T, we have $(\overline{T})^* = T^*$.

- Since $T \subseteq \overline{T}$, by a preceding theorem, $(\overline{T})^* \subseteq T^*$. Hence $\mathscr{D}((\overline{T})^*) \subseteq \mathscr{D}(T^*)$. We show $y \in \mathscr{D}(T^*)$ implies $y \in \mathscr{D}((\overline{T})^*)$.
  Let $y \in \mathscr{D}(T^*)$. By the definition of the Hilbert-adjoint operator, it suffices to prove that, for every $x \in \mathscr{D}(\overline{T})$,
  $$\langle \overline{T}x, y\rangle = \langle x, (\overline{T})^* y\rangle = \langle x, T^* y\rangle,$$
  where the second equality follows from $(\overline{T})^* \subseteq T^*$.
  By the definitions of $\mathscr{D}(\overline{T})$ and $\overline{T}$, for each $x \in \mathscr{D}(\overline{T})$, there is a sequence (x_n) in $\mathscr{D}(T)$, such that $x_n \to x$ and $Tx_n \to y_0 = \overline{T}x$.
  Since $y \in \mathscr{D}(T^*)$ and $x_n \in \mathscr{D}(T)$, by definition, $\langle Tx_n, y\rangle = \langle x_n, T^* y\rangle$.
  By continuity of the inner product, $\langle \overline{T}x, y\rangle = \langle x, T^* y\rangle$, $x \in \mathscr{D}(\overline{T})$.

Subsection 4

Spectral Properties of Self-Adjoint Operators

Regular Values

Theorem (Regular Values)

Let $T:\mathscr{D}(T)\to H$ be a self-adjoint linear operator which is densely defined in a complex Hilbert space H. Then a number λ belongs to the resolvent set $\rho(T)$ of T if and only if, there exists a $c>0$, such that, for every $x\in\mathscr{D}(T)$,

$$\|T_\lambda x\|\geq c\|x\|,$$

where $T_\lambda = T-\lambda I$.

(a) Let $\lambda\in\rho(T)$. Then, the resolvent $R_\lambda=(T-\lambda I)^{-1}$ exists and is bounded, say, $\|R_\lambda\|=k>0$. Since $R_\lambda T_\lambda x=x$, for $x\in\mathscr{D}(T)$, we get

$$\|x\|=\|R_\lambda T_\lambda x\|\leq\|R_\lambda\|\|T_\lambda x\|=k\|T_\lambda x\|.$$

Division by k yields

$$\|T_\lambda x\|\geq c\|x\|,$$

where $c=\frac{1}{k}$.

Regular Values (The Converse)

(b) Conversely, suppose $\|T_\lambda x\| \geq c\|x\|$, $x \in \mathscr{D}(T)$, holds for some $c > 0$.

We consider the vector space

$$Y = \{y : y = T_\lambda x, x \in \mathscr{D}(T)\},$$

i.e., the range of T_λ. We show that:

(i) $T_\lambda : \mathscr{D}(T) \to Y$ is bijective;
(ii) Y is dense in H;
(iii) Y is closed.

These imply that the resolvent $R_\lambda = T_\lambda^{-1}$ is defined on all of H.

Boundedness of R_λ will then follow from hypothesis.

So we will have $\lambda \in \rho(T)$.

Regular Values (The Converse Part (i))

(i) Consider any $x_1, x_2 \in \mathscr{D}(T)$, such that $T_\lambda x_1 = T_\lambda x_2$.
Since T_λ is linear, the hypothesis yields

$$0 = \| T_\lambda x_1 - T_\lambda x_2 \| = \| T_\lambda (x_1 - x_2) \| \geq c \| x_1 - x_2 \|.$$

Since $c > 0$, this implies $\|x_1 - x_2\| = 0$.
Hence, $x_1 = x_2$.
So the operator $T_\lambda : \mathscr{D}(T) \to Y$ is bijective.

Regular Values (The Converse Part (ii))

(ii) We prove that $\overline{Y} = H$ by showing that $x_0 \perp Y$ implies $x_0 = 0$.
Let $x_0 \perp Y$. Then, for every $y = T_\lambda x \in Y$,

$$0 = \langle T_\lambda x, x_0 \rangle = \langle Tx, x_0 \rangle - \lambda \langle x, x_0 \rangle.$$

Hence, for all $x \in \mathscr{D}(T)$,

$$\langle Tx, x_0 \rangle = \langle x, \overline{\lambda} x_0 \rangle.$$

By definition of the Hilbert-adjoint, $x_0 \in \mathscr{D}(T^*)$ and $T^* x_0 = \overline{\lambda} x_0$.
Since T is self-adjoint, $\mathscr{D}(T^*) = \mathscr{D}(T)$ and $T^* = T$. So $Tx_0 = \overline{\lambda} x_0$.
Suppose $x_0 \neq 0$. This implies that $\overline{\lambda}$ is an eigenvalue of T.
Hence, $\overline{\lambda} = \lambda$ must be real. So $Tx_0 = \lambda x_0$. I.e., $T_\lambda x_0 = 0$.
But now, the hypothesis yields a contradiction:

$$0 = \| T_\lambda x_0 \| \geq c \| x_0 \| \quad \text{implies} \quad \| x_0 \| = 0.$$

It follows that $\overline{Y}^\perp = \{0\}$. So $\overline{Y} = H$.

Regular Values (The Converse Part (iii))

(iii) We prove that Y is closed. Let $y_0 \in \overline{Y}$.

Then there is a sequence (y_n) in Y, such that $y_n \to y_0$.

Since $y_n \in Y$, we have $y_n = T_\lambda x_n$, for some $x_n \in \mathscr{D}(T_\lambda) = \mathscr{D}(T)$.

By the hypothesis,

$$\|x_n - x_m\| \leq \frac{1}{c} \| T_\lambda (x_n - x_m)\| = \frac{1}{c} \|y_n - y_m\|.$$

Since (y_n) converges, this shows that (x_n) is Cauchy.

Since H is complete, (x_n) converges, say, $x_n \to x_0$.

Since T is self-adjoint, by a previous theorem, it is closed.

Thus, we have $x_0 \in \mathscr{D}(T)$ and $T_\lambda x_0 = y_0$.

This shows that $y_0 \in Y$. Since $y_0 \in Y$ was arbitrary, Y is closed.

Regular Values (The Converse Part (iii) Cont'd)

- Parts (ii) and (iii) imply that $Y = H$.

  From this and Part (i), the resolvent R_λ exists and is defined on H,

$$R_\lambda = T_\lambda^{-1} : H \to \mathscr{D}(T).$$

  By a previous result, R_λ is linear.

  For all $y \in H$ and corresponding $x = R_\lambda y$, we have $y = T_\lambda x$.

  Moreover, by hypothesis,

$$\|R_\lambda y\| = \|x\| \le \frac{1}{c}\|T_\lambda x\| = \frac{1}{c}\|y\|.$$

  So $\|R_\lambda\| \le \frac{1}{c}$ and R_λ is bounded.

  By definition this proves that $\lambda \in \rho(T)$.

The Spectrum Theorem

Theorem (Spectrum)

Let H be a complex Hilbert space. Let $T : \mathscr{D}(T) \to H$ be a self-adjoint linear operator, with $\mathscr{D}(T)$ dense in H. The spectrum $\sigma(T)$ of T is real and closed.

(a) We first show that $\sigma(T)$ is real.

For every $x \neq 0$ in $\mathscr{D}(T)$ we have

$$\langle T_\lambda x, x\rangle = \langle Tx, x\rangle - \lambda\langle x, x\rangle.$$

Since $\langle x, x\rangle$ and $\langle Tx, x\rangle$ are real,

$$\overline{\langle T_\lambda x, x\rangle} = \langle Tx, x\rangle - \overline{\lambda}\langle x, x\rangle.$$

We write $\lambda = \alpha + i\beta$, with real α and β.

Then $\overline{\lambda} = \alpha - i\beta$.

The Spectrum Theorem (Cont'd)

- Subtraction yields

$$\overline{\langle T_\lambda x, x\rangle} - \langle T_\lambda x, x\rangle = (\lambda - \overline{\lambda})\langle x, x\rangle = 2i\beta\|x\|^2.$$

The left side equals $-2i\mathrm{Im}\langle T_\lambda x, x\rangle$.

Since the imaginary part of a complex number cannot exceed the absolute value, we have by the Schwarz inequality

$$|\beta|\|x\|^2 \le |\langle T_\lambda x, x\rangle| \le \|T_\lambda x\|\|x\|.$$

Division by $\|x\| \neq 0$ gives $|\beta|\|x\| \le \|T_\lambda x\|$.

Note that this inequality holds for all $x \in \mathscr{D}(T)$.

If λ is not real, $\beta \neq 0$. So, by the previous theorem, $\lambda \in \rho(T)$.

Hence, $\sigma(T)$ must be real.

The Spectrum Theorem Part (b)

(b) We now show that $\sigma(T)$ is closed.

We do this by proving that the resolvent set $\rho(T)$ is open.

We consider an arbitrary $\lambda_0 \in \rho(T)$.

We show that every λ sufficiently close to λ_0 also belongs to $\rho(T)$.

By the triangle inequality,

$$\|Tx - \lambda_0 x\| = \|Tx - \lambda x + (\lambda - \lambda_0)x\| \leq \|Tx - \lambda x\| + |\lambda - \lambda_0| \|x\|.$$

So

$$\|Tx - \lambda x\| \geq \|Tx - \lambda_0 x\| - |\lambda - \lambda_0| \|x\|.$$

Since $\lambda_0 \in \rho(T)$, there is a $c > 0$, such that for all $x \in \mathscr{D}(T)$,

$$\|Tx - \lambda_0 x\| \geq c\|x\|.$$

The Spectrum Theorem Part (b) (Cont'd)

- Assume that λ is close to λ_0, say, $|\lambda - \lambda_0| \leq \frac{c}{2}$.

  Then previous inequalities imply, for all $x \in \mathscr{D}(T)$,

  $$\|Tx - \lambda x\| \geq c\|x\| - \frac{1}{2}c\|x\| = \frac{1}{2}c\|x\|.$$

  By a previous theorem, $\lambda \in \rho(T)$.

  So λ_0 has a neighborhood lying entirely in $\rho(T)$.

  Since $\lambda_0 \in \rho(T)$ was arbitrary, we conclude that $\rho(T)$ is open.

  Hence, $\sigma(T) = \mathbb{C} - \rho(T)$ is closed.

Subsection 5

Spectral Representation of Unitary Operators

The Spectrum Theorem

Theorem (Spectrum)

If $U : H \to H$ is a unitary linear operator on a complex Hilbert space $H \neq \{0\}$, then the spectrum $\sigma(U)$ is a closed subset of the unit circle. Thus, $|\lambda| = 1$, for every $\lambda \in \sigma(U)$.

- We have $\|U\| = 1$, by a preceding theorem.

  Hence, $|\lambda| \leq 1$, for all $\lambda \in \sigma(U)$, also by a previous theorem.

  Also $0 \in \rho(U)$, since for $\lambda = 0$ the resolvent operator of U is $U^{-1} = U^*$.

  The operator U^{-1} is unitary by a preceding theorem.

  Hence, $\|U^{-1}\| = 1$.

  Also, a preceding theorem, with $T = U$ and $\lambda_0 = 0$, now implies that every λ satisfying $|\lambda| < \frac{1}{\|U^{-1}\|} = 1$ belongs to $\rho(U)$.

  Hence, the spectrum of U must lie on the unit circle.

  It is closed, by another theorem.

The Power Series Lemma

Lemma (Power Series)

Let

$$h(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n, \quad \alpha_n \text{ real},$$

be absolutely convergent, for all λ, such that $|\lambda| \leq k$. Suppose that $S \in B(H,H)$ is self-adjoint and has norm $\|S\| \leq k$, where H is a complex Hilbert space. Then

$$h(S) = \sum_{n=0}^{\infty} \alpha_n S^n$$

is a bounded self-adjoint linear operator and

$$\|h(S)\| \leq \sum_{n=0}^{\infty} |\alpha_n| k^n.$$

If a bounded linear operator commutes with S, it does so with $h(S)$.

The Power Series Lemma

- Let $h_n(\lambda)$ denote the n-th partial sum of the λ-series.

  For $|\lambda| \leq k$, the series converges absolutely (hence also uniformly).

  Since H is complete, absolute convergence implies convergence.

  Hence, convergence of the S-series follows from $\|S\| \leq k$ and

  $$\left\|\sum \alpha_n S^n\right\| \leq \sum |\alpha_n| \|S\|^n \leq |\alpha_n| k^n.$$

  We denote the sum of the series by $h(S)$.

  This is in agreement with a preceding section, because $h(\lambda)$ is continuous and $h_n(\lambda) \to h(\lambda)$, uniformly for $|\lambda| \leq k$.

The Power Series Lemma (Cont'd)

- We show, next, that the operator $h(S)$ is self-adjoint.

  Since the $h_n(S)$ are self-adjoint, $\langle h_n(S)x, x\rangle$ is real.

  Hence, $\langle h(S)x, x\rangle$ is real by the continuity of the inner product.

  So that $h(S)$ is self-adjoint, since H is complex.

  Finally, we prove the last inequality.

  Since $\|S\| \leq k$, a preceding theorem gives $[m, M] \subseteq [-k, k]$.

  Another theorem yields, for $J = [m, M]$,

$$\|h_n(S)\| \leq \max_{\lambda \in J} |h_n(\lambda)| \leq \sum_{j=0}^{n} |\alpha_j| k^j.$$

  Letting $n \to \infty$, the conclusion follows.

Wecken's Lemma

Wecken's Lemma

Let W and A be bounded self-adjoint linear operators on a complex Hilbert space H. Suppose that $WA = AW$ and $W^2 = A^2$. Let P be the projection of H onto the null space $\mathcal{N}(W - A)$. Then:

(a) If a bounded linear operator commutes with $W - A$, it also commutes with P.

(b) $Wx = 0$ implies $Px = x$.

(c) We have $W = (2P - I)A$.

(a) Suppose that B commutes with $W - A$.

By hypothesis, $Px \in \mathcal{N}(W - A)$, for every $x \in H$.

Thus, $(W - A)BPx = B(W - A)Px = 0$. So $BPx \in \mathcal{N}(W - A)$.

This implies $P(BPx) = BPx$. I.e., $PBP = BP$.

It now suffices to show that $PBP = PB$.

Wecken's Lemma Parts (a) and (b)

- We must show $PBP = PB$.
  Since $W - A$ is self-adjoint,
  $$(W-A)B^* = [B(W-A)]^* = [(W-A)B]^* = B^*(W-A).$$
  This shows that $W - A$ and B^* also commute.
  Hence, reasoning as before, we obtain $PB^*P = B^*P$.
  Since projections are self-adjoint,
  $$PBP = (PB^*P)^* = (B^*P)^* = PB.$$
  Together with $PBP = BP$, we have $BP = PB$.

(b) Let $Wx = 0$.
Since A and W are self-adjoint and $A^2 = W^2$,
$$\|Ax\|^2 = \langle Ax, Ax\rangle = \langle A^2x, x\rangle = \langle W^2x, x\rangle = \|Wx\|^2 = 0.$$
So $Ax = 0$. Hence, $(W-A)x = 0$. This shows that $x \in \mathcal{N}(W-A)$.
But P is the projection of H onto $\mathcal{N}(W-A)$. So $Px = x$.

Wecken's Lemma Part (c)

(c) From the assumptions $W^2 = A^2$ and $WA = AW$, we have
$$(W-A)(W+A) = W^2 - A^2 = 0.$$
Hence, $(W+A)x \in \mathcal{N}(W-A)$, for every $x \in H$.
Since P projects H onto $\mathcal{N}(W-A)$, we get $P(W+A)x = (W+A)x$, for every $x \in H$. Thus,
$$P(W+A) = W+A.$$
But note that:
- $P(W-A) = (W-A)P$, by Part (a);
- $(W-A)P = 0$, since P projects H onto $\mathcal{N}(W-A)$.

So
$$P(W-A) = 0.$$
Hence,
$$2PA = P(W+A) - P(W-A) = W+A.$$
Therefore, $2PA - A = W$.

Spectral Theorem for Unitary Operators

Spectral Theorem for Unitary Operators

Let $U : H \to H$ be a unitary operator on a complex Hilbert space $H \neq \{0\}$. Then, there exists a spectral family $\mathscr{E} = (E_\theta)$ on $[-\pi, \pi]$, such that

$$U = \int_{-\pi}^{\pi} e^{i\theta} dE_\theta = \int_{-\pi}^{\pi} (\cos\theta + i\sin\theta) dE_\theta.$$

More generally, for every continuous function f defined on the unit circle,

$$f(U) = \int_{\pi}^{\pi} f(e^{i\theta}) dE_\theta,$$

where the integral is to be understood in the sense of uniform operator convergence. Moreover, for all $x, y \in H$,

$$\langle f(U)x, y\rangle = \int_{-\pi}^{\pi} f(e^{i\theta}) dw(\theta), \quad w(\theta) = \langle E_\theta x, y\rangle,$$

where the integral is an ordinary Riemann-Stieltjes integral.

Proof of the Spectral Theorem Plan

- We prove that, for a given unitary operator U, there is a bounded self-adjoint linear operator S, with $\sigma(S) \subseteq [-\pi,\pi]$, such that

$$U = e^{iS} = \cos S + i \sin S.$$

Then we use the spectral theorems of the preceding chapter.
We proceed stepwise as follows:

(a) We prove that U is unitary, provided S exists.
(b) We write $U = V + iW$, where

$$V = \frac{1}{2}(U + U^*), \quad W = \frac{1}{2i}(U + U^*),$$

and prove that V and W are self-adjoint and $-I \leq V \leq I, -I \leq W \leq I$.
(c) We investigate some properties of $g(V) = \arccos V$ and $A = \sin g(V)$.
(d) We prove that the desired operator S is

$$S = (2P - I)(\arccos V),$$

where P is the projection of H onto $\mathcal{N}(W - A)$.

Proof of the Spectral Theorem Part (a)

(a) Suppose S is bounded and self-adjoint.

By the Power Series Lemma, so are $\cos S$ and $\sin S$.

These operators commute by the same lemma.

This implies that U is unitary since

$$\begin{array}{rcl} UU^* & = & (\cos S + i \sin S)(\cos S - i \sin S) \\ & = & (\cos S)^2 + (\sin S)^2 \\ & = & (\cos^2 + \sin^2)(S) \\ & = & I. \end{array}$$

Similarly, $U^* U = I$.

Proof of the Spectral Theorem Part (b)

(b) Self-adjointness of $V = \frac{1}{2}(U + U^*)$ and $W = \frac{1}{2i}(U - U^*)$ follows by a direct calculation using a previous result.

Since $UU^* = U^*U$ $(= I)$, we have $VW = WV$.

Also $\|U\| = \|U^*\| = 1$ imply $\|V\| \leq 1$, $\|W\| \leq 1$.

Hence, the Schwarz inequality yields

$$|\langle Vx, x\rangle| \leq \|Vx\|\|x\| \leq \|V\|\|x\|^2 \leq \langle x, x\rangle.$$

So we have

$$-\langle x, x\rangle \leq \langle Vx, x\rangle \leq \langle x, x\rangle.$$

This proves the first formula.

The second follows by the same argument.

Furthermore, by direct calculation,

$$V^2 + W^2 = \frac{1}{4}(U^2 + 2UU^* + (U^*)^2) - \frac{1}{4}(U^2 - 2UU^* + (U^*)^2) = UU^* = I.$$

Proof of the Spectral Theorem Part (c)

(c) We consider

$$g(\lambda) = \arccos\lambda = \frac{\pi}{2} - \arcsin\lambda = \frac{\pi}{2} - \lambda - \frac{1}{6}\lambda^3 - \cdots.$$

The Maclaurin series on the right converges for $|\lambda| \leq 1$.

- At $\lambda = 1$ the series of $\arcsin\lambda$ has positive coefficients.
  So it has a monotone sequence of partial sums s_n, when $\lambda > 0$.
  This sequence is bounded on $(0,1)$, since $s_n(\lambda) < \arcsin\lambda < \frac{\pi}{2}$.
  So, for every fixed n, we have $s_n(\lambda) \to s_n(1) \leq \frac{\pi}{2}$, as $\lambda \to 1$.
  It follows that the series converges at $\lambda = 1$.
- Convergence at $\lambda = -1$ follows readily from that at $\lambda = 1$.

Note that $\|V\| \leq 1$.

So, by a previous lemma, the operator

$$g(V) = \arccos V = \frac{\pi}{2}I - V - \frac{1}{6}V^3 - \cdots$$

exists and is self-adjoint.

Proof of the Spectral Theorem Part (c) (Cont'd)

- Now define
$$A = \sin g(V).$$
This is a power series in V.
By a previous lemma, A is self-adjoint and commutes with V.
Moreover, it also commutes with W.
By the power-series expression $\cos g(V) = V$.
So we have
$$V^2 + A^2 = (\cos^2 + \sin^2)(g(V)) = I.$$
A comparison with $V^2 + W^2 = I$ yields $W^2 = A^2$.
Hence, we can apply Wecken's lemma to conclude that:
    - $W = (2P - I)A$;
    - $Wx = 0$ implies $Px = x$;
    - P commutes with V and with $g(V)$, since these operators commute with $W - A$.

Proof of the Spectral Theorem Part (d)

(d) Define

$$S=(2P-I)g(V)=g(V)(2P-I).$$

Obviously, S is self-adjoint.

Claim: S satisfies $U=e^{iS}=\cos S+i\sin S$.

Set $\kappa=\lambda^2$. Define h_1 and h_2 by

$$\begin{aligned} h_1(\kappa) &= \cos\lambda=1-\frac{1}{2!}\lambda^2+-\cdots; \\ \lambda h_2(\kappa) &= \sin\lambda=\lambda-\frac{1}{3!}\lambda^3+-\cdots. \end{aligned}$$

These functions exist for all κ.

Since P is a projection, $(2P-I)^2=4P^2-4P+I=4P-4P+I=I$.

So we get

$$S^2=(2P-I)^2g(V)^2=g(V)^2.$$

Hence,

$$\cos S=h_1(S^2)=h_1(g(V)^2)=\cos g(V)=V.$$

Proof of the Spectral Theorem Part (d) (Cont'd)

- Next we show that $\sin S = W$.

  Indeed, we have

$$\begin{aligned} \sin S &= Sh_2(S^2) \\ &= (2P-I)g(V)h_2(g(V)^2) \\ &= (2P-I)\sin g(V) \\ &= (2P-I)A \\ &= W. \end{aligned}$$

  We conclude that $e^{iS} = V + iW = U$.

  Claim: $\sigma(S) \subseteq [-\pi, \pi]$.

  Since $|\arccos \lambda| \le \pi$, we get that $\|S\| \le \pi$.

  Since S is self-adjoint and bounded, $\sigma(S)$ is real.

  A preceding theorem yields the result.

Proof of the Spectral Theorem (Conclusion)

- Let (E_θ) be the spectral family of S.

  Then the equations for U and $f(U)$ follow from $U = e^{iS}$ and the spectral theorem for bounded self-adjoint linear operators.

  Claim: We can take $-\pi$ (instead of $-\pi^-$) as the lower limit of integration without restricting generality.

  If we had a spectral family, call it $(\widetilde{E}_\theta)$, such that $\widetilde{E}_{-\pi} \neq 0$, we would have to take $-\pi^-$ as the lower limit of integration in those integrals.

  However, instead of $\widetilde{E}_\theta$ we could then equally well use E_θ defined by

$$E_\theta = \begin{cases} 0, & \text{if } \theta = -\pi \\ \widetilde{E}_\theta - \widetilde{E}_{-\pi}, & \text{if } -\pi < \theta < \pi \\ I, & \text{if } \theta = \pi \end{cases}.$$

  E_θ is continuous at $\theta = -\pi$.

  So the lower limit of integration $-\pi$ is in order.

Subsection 6

Spectral Representation of Self-Adjoint Linear Operators

The Cayley Transform

- Let H be a complex Hilbert space.
- Consider a self-adjoint linear operator $T : \mathscr{D}(T) \to H$ on H, where $\mathscr{D}(T)$ is dense in H and T may be unbounded.
- We associate with T the operator

$$U = (T - iI)(T + iI)^{-1},$$

  called the **Cayley transform** of T.
- We show that the operator U is unitary.

Cayley Transform and Spectra

- We defined the Cayley transform $U=(T-iI)(T+iI)^{-1}$ of T, which is unitary.
- We obtain the spectral theorem for the (possibly unbounded) T from that for the bounded operator U.
- T has its spectrum $\sigma(T)$ on the real axis of the complex plane $\mathbb{C}$.
- On the other hand, the spectrum of a unitary operator lies on the unit circle of $\mathbb{C}$.
- A mapping $\mathbb{C}\to\mathbb{C}$ which transforms the real axis into the unit circle is
$$u=\frac{t-i}{t+i}.$$
- This mapping suggests the Cayley transform.

First Cayley Transform Lemma

Lemma (Cayley Transform)

The Cayley transform of a self-adjoint linear operator $T : \mathscr{D}(T) \to H$ exists on H and is a unitary operator, where $H \neq \{0\}$ is a complex Hilbert space.

- Since T is self-adjoint, $\sigma(T)$ is real.
  Hence, i and $-i$ belong to the resolvent set $\rho(T)$.
  Consequently, by the definition of $\rho(T)$, the inverses $(T+iI)^{-1}$ and $(T-iI)^{-1}$ exist on a dense subset of H and are bounded operators.
  A preceding theorem implies that T is closed because $T = T^*$.
  By a previous lemma, those inverses are defined on all of H.
  That is, $\mathscr{R}(T+iI) = H$ and $\mathscr{R}(T-iI) = H$.
  We thus have, since I is defined on all of H,
  $$(T+iI)^{-1}(H) = \mathscr{D}(T+iI) = \mathscr{D}(T) = \mathscr{D}(T-iI).$$
  We also have $(T-iI)(\mathscr{D}(T)) = H$.
  This shows that U is a bijection of H onto itself.

First Cayley Transform Lemma (Cont'd)

- By a previous theorem, it remains to prove that U is isometric.

  Take any $x \in H$, set $y = (T + iI)^{-1}x$ and use $\langle y, Ty \rangle = \langle Ty, y \rangle$.

  We calculate

$$\begin{array}{rcl} \|Ux\|^2 & = & \|(T - iI)y\|^2 \\ & = & \langle Ty - iy, Ty - iy \rangle \\ & = & \langle Ty, Ty \rangle + i\langle Ty, y \rangle - i\langle y, Ty \rangle + \langle iy, iy \rangle \\ & = & \langle Ty + iy, Ty + iy \rangle \\ & = & \|(T + iI)y\|^2 \\ & = & \|(T + iI)(T + iI)^{-1}x\|^2 \\ & = & \|x\|^2. \end{array}$$

  A previous theorem now implies that U is unitary.

Second Cayley Transform Lemma

Lemma (Cayley Transform)

Let $T : \mathscr{D}(T) \to H$ be a self-adjoint linear operator, where, $H \neq \{0\}$ is a complex Hilbert space, and let U be defined by $U = (T - iI)(T + iI)^{-1}$. Then

$$T = i(I + U)(I - U)^{-1}.$$

Furthermore, 1 is not an eigenvalue of U.

- Let $x \in \mathscr{D}(T)$ and $y = (T + iI)x$.
  Then $Uy = (T - iI)x$, since $(T + iI)^{-1}(T + iI) = I$.
  By addition and subtraction, we get

  $$(I + U)y = 2Tx \quad \text{and} \quad (I - U)y = 2ix.$$

  We know $y \in \mathscr{R}(T + iI) = H$. Hence, $I - U$ maps H onto $\mathscr{D}(T)$.
  We also see that, if $(I - U)y = 0$, then $x = 0$.
  So, by $y = (T + iI)x$, $y = 0$.

Second Cayley Transform Lemma (Cont'd)

- Hence, $(I-U)^{-1}$ exists by a previous theorem.

  Moreover, it is defined on the range of $I-U$, which is $\mathscr{D}(T)$.

  Hence, since $(I-U)y=2ix$,

  $$y=2i(I-U)^{-1}x, \quad \text{for all } x\in\mathscr{D}(T).$$

  By substitution into $(I+U)y=2Tx$, for all $x\in\mathscr{D}(T)$,

  $$Tx=\frac{1}{2}(I+U)y=i(I+U)(I-U)^{-1}x.$$

  Since $(I-U)^{-1}$ exists, 1 cannot be an eigenvalue of the Cayley transform U.

Spectral Theorem for Self-Adjoint Linear Operators

Spectral Theorem for Self-Adjoint Linear Operators

Let $T:\mathscr{D}(T)\to H$ be a self-adjoint linear operator, where $H\neq\{0\}$ is a complex Hilbert space and $\mathscr{D}(T)$ is dense in H. Let U be the Cayley transform of T and (E_θ) the spectral family in the spectral representation

$$-U=\int_{-\pi}^{\pi}e^{i\theta}dE_\theta=\int_{-\pi}^{\pi}(\cos\theta+i\sin\theta)dE_\theta$$

of $-U$. Then, for all $x\in\mathscr{D}(T)$,

$$\begin{aligned}\langle Tx,x\rangle &= \int_{-\pi}^{\pi}\tan\frac{\theta}{2}dw(\theta) & w(\theta)=\langle E_\theta x,x\rangle\\ &= \int_{-\infty}^{\infty}\lambda dv(\lambda), & v(\lambda)=\langle F_\lambda x,x\rangle\end{aligned}$$

where $F_\lambda=E_{2\arctan\lambda}$.

Spectral Theorem for Self-Adjoint Operators (Plan)

- From a previous spectral theorem,we have

$$-U = \int_{-\pi}^{\pi} e^{i\theta} dE_\theta = \int_{-\pi}^{\pi} (\cos\theta + i\sin\theta) dE_\theta.$$

We prove the statement in two steps:

(a) We show that (E_θ) is continuous at $-\pi$ and π.

(b) We use Property (a) to establish the claimed equations.

Spectral Theorem for Self-Adjoint Operators Part (a)

(a) (E_θ) is the spectral family of a bounded self-adjoint linear operator which we call S. Then $-U = \cos S + i \sin S$.

From a previous theorem, we know that a θ_0 at which (E_θ) is discontinuous is an eigenvalue of S.

Then, there is an $x \neq 0$, such that $Sx = \theta_0 x$.

Hence, for any polynomial q, $q(S)x = q(\theta_0)x$.

Also, for any continuous function g on $[-\pi, \pi]$, $g(S)x = g(\theta_0)x$.

Since $\sigma(S) \subseteq [-\pi, \pi]$, we have $E_{-\pi^-} = 0$.

Hence, if $E_{-\pi} \neq 0$, then $-\pi$ would be an eigenvalue of S.

By the preceding relations, the operator U would have the eigenvalue $-\cos(-\pi) - i \sin(-\pi) = 1$.

This contradicts a preceding lemma.

Similarly, $E_\pi = I$ and, if $E_{\pi^-} \neq I$, U would have an eigenvalue 1.

Spectral Theorem for Self-Adjoint Operators Part (b)

(b) Let $x \in H$ and $y = (I - U)x$.
In the proof of a previous lemma, it was shown that $I - U : H \to \mathscr{D}(T)$.
Hence, $y \in \mathscr{D}(T)$.
Now, we have $T = i(I + U)(I - U)^{-1}$. So we get

$$Ty = i(I + U)(I - U)^{-1}y = i(1 + U)x.$$

Since $\|Ux\| = \|x\|$, we obtain

$$\begin{array}{rcl} \langle Ty, y\rangle & = & \langle i(I + U)x, (I - U)x\rangle \\ & = & i(\langle Ux, x\rangle - \langle x, Ux\rangle) \\ & = & i(\langle Ux, x\rangle - \overline{\langle Ux, x\rangle}) \\ & = & -2\mathrm{Im}\langle Ux, x\rangle \\ & = & 2\int_{-\pi}^{\pi} \sin\theta d\langle E_\theta x, x\rangle. \end{array}$$

Hence

$$\langle Ty, y\rangle = 4\int_{-\pi}^{\pi} \sin\frac{\theta}{2}\cos\frac{\theta}{2} d\langle E_\theta x, x\rangle.$$

Spectral Theorem Part (b) (Cont'd)

- Recall that (E_θ) is the spectral family of the bounded self-adjoint linear operator S in $-U = \cos S + i \sin S$.

  Hence E_θ and S commute. So E_θ and U commute.

  Now, we obtain

$$\begin{array}{rcl} \langle E_\theta y, y\rangle & = & \langle E_\theta (I-U)x, (I-U)x\rangle \\ & = & \langle (I-U)^*(I-U)E_\theta x, x\rangle \\ & = & \int_{-\pi}^{\pi}(1+e^{-i\varphi})(1+e^{i\varphi})d\langle E_\varphi z, x\rangle, \quad \text{where } z = E_\theta x. \end{array}$$

  We also have:
  - $E_\varphi E_\theta = E_\varphi$, when $\varphi \le \theta$;
  - $(1+e^{-i\varphi})(1+e^{i\varphi}) = (e^{i\varphi/2}+e^{-i\varphi/2})^2 = 4\cos^2\frac{\varphi}{2}$.

  So we obtain

$$\langle E_\theta y, y\rangle = 4\int_{-\pi}^{\theta}\cos^2\frac{\varphi}{2}d\langle E_\varphi x, x\rangle.$$

Spectral Theorem Part (b) (Cont'd)

- We obtained
$$\langle E_\theta y, y\rangle = 4\int_{-\pi}^{\theta} \cos^2\frac{\varphi}{2} d\langle E_\varphi x, x\rangle.$$
Using this, the continuity of E_θ at $\pm\pi$ and the rule for transforming a Stieltjes integral, we finally have
$$\begin{array}{rcl} \int_{-\pi}^{\pi}\tan\frac{\theta}{2}d\langle E_\theta y, y\rangle & = & \int_{-\pi}^{\pi}\tan\frac{\theta}{2}(4\cos^2\frac{\theta}{2})d\langle E_\theta x, x\rangle \\ & = & 4\int_{-\pi}^{\pi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\langle E_\theta x, x\rangle. \end{array}$$
We now have the first formula with y instead of x.
The second follows by the indicated transformation $\theta = 2\arctan\lambda$.
Note that (F_λ) is indeed a spectral family. In particular:
  - $F_\lambda \stackrel{\lambda\to-\infty}{\longrightarrow} 0$;
  - $F_\lambda \stackrel{\lambda\to+\infty}{\longrightarrow} I$.

Subsection 7

Multiplication Operator and Differentiation Operator

The Multiplication Operator

- Consider the operator

$$\begin{array}{rccl} T: & \mathscr{D}(T) & \to & L^2(-\infty,+\infty); \\ & x & \mapsto & tx \end{array}$$

where $\mathscr{D}(T) \subseteq L^2(-\infty,+\infty)$.
- $\mathscr{D}(T)$ consists of all $x \in L^2(-\infty,+\infty)$, such that $Tx \in L^2(-\infty,+\infty)$.
- So $x \in \mathscr{D}(T)$ if and only if $x \in L^2(-\infty,+\infty)$ and

$$\int_{-\infty}^{+\infty} t^2 |x(t)|^2 dt < \infty.$$

The Domain of the Multiplication Operator

- The definition implies that $\mathscr{D}(T) \neq L^2(-\infty, +\infty)$.

  An $x \in L^2(-\infty, +\infty)$ not satisfying finiteness is

  $$x(t) = \begin{cases} \frac{1}{t}, & \text{if } t \geq 1 \\ 0, & \text{if } t < 1 \end{cases}$$

  Hence $x \notin \mathscr{D}(T)$.
- $\mathscr{D}(T)$ contains all functions $x \in L^2(-\infty, +\infty)$ which are zero outside a compact interval.
- It can be shown that this set of functions is dense in $L^2(-\infty, +\infty)$.
- Hence $\mathscr{D}(T)$ is dense in $L^2(-\infty, +\infty)$.

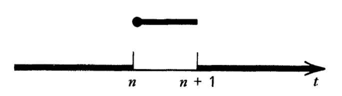
Unboundedness of the Multiplication Operator

Lemma (Multiplication Operator)

The multiplication operator T defined by $U = (T - iI)(T + iI)^{-1}$ is not bounded.

- Consider

$$x_n(t) = \begin{cases} 1, & \text{if } n \leq t < n+1 \\ 0, & \text{elsewhere} \end{cases}.$$



We have
  - $\|x_n\| = 1$;
  - $\|Tx_n\|^2 = \int_n^{n+1} t^2 dt > n^2$.

So $\frac{\|Tx_n\|}{\|x_n\|} > n$, where $n \in \mathbb{N}$ can be chosen as large as desired.

Comparison with Finite Domains

- The unboundedness results from the fact that we are dealing with functions on an infinite interval.
- For comparison, in the case of a finite interval $[a,b]$ the operator

$$\begin{array}{rccl} \widetilde{T}: & \mathscr{D}(\widetilde{T}) & \to & L^2[a,b]; \\ & x & \mapsto & tx, \end{array}$$

is bounded.

  - If $|b| \geq |a|$, then

$$\|\widetilde{T}x\|^2 = \int_a^b t^2 |x(t)|^2 dt \leq b^2 \|x\|^2;$$

  - If $|b| < |a|$, the proof is similar.

This also shows that $x \in L^2[a,b]$ implies $\widetilde{T}x \in L^2[a,b]$. Hence $\mathscr{D}(\widetilde{T}) = L^2[a,b]$, i.e., $\widetilde{T}$ is defined on all of $L^2[a,b]$.

Self-Adjointness

Theorem (Self-Adjointness)

The multiplication operator T defined by $U = (T - iI)(T + iI)^{-1}$ is self-adjoint.

- T is densely defined in $L^2(-\infty, +\infty)$, as was mentioned before. T is symmetric because, using $t = \overline{t}$, we have
$$\langle Tx, y\rangle = \int_{-\infty}^{+\infty} tx(t)\overline{y(t)}dt = \int_{-\infty}^{+\infty} x(t)\overline{ty(t)}dt = \langle x, Ty\rangle.$$
Hence, $T \subseteq T^*$, by a preceding theorem.
Thus, it suffices to show that $\mathscr{D}(T) \supseteq \mathscr{D}(T^*)$.
This we do by proving that $y \in \mathscr{D}(T^*)$ implies $y \in \mathscr{D}(T)$.
Let $y \in \mathscr{D}(T^*)$. Then, for all $x \in \mathscr{D}(T)$,
$$\langle Tx, y\rangle = \langle x, y^*\rangle, \quad y^* = T^* y.$$
Written out $\int_{-\infty}^{+\infty} tx(t)\overline{y(t)}dt = \int_{-\infty}^{+\infty} x(t)\overline{y^*(t)}dt$.

Self-Adjointness

- Now we have
$$\int_{-\infty}^{+\infty} x(t)[\overline{ty(t)} - \overline{y^*(t)}]dt = 0.$$
In particular, this holds for every $x \in L^2(-\infty,+\infty)$ which is zero outside an arbitrary given bounded interval (a,b).
Clearly, such an x is in $\mathscr{D}(T)$. Choose
$$x(t) = \begin{cases} ty(t) - y^*(t), & \text{if } t \in (a,b) \\ 0, & \text{elsewhere} \end{cases}.$$
Then we have $\int_a^b |ty(t) - y^*(t)|^2 dt = 0$.
It follows that $ty(t) - y^*(t) = 0$ almost everywhere on (a,b).
Hence, $ty(t) = y^*(t)$ almost everywhere on (a,b).
Since (a,b) was arbitrary, we have $ty = y^* \in L^2(-\infty,+\infty)$. So $y \in \mathscr{D}(T)$. We also have $T^*y = y^* = ty = Ty$.
- Note that the theorem implies that T is closed, because $T = T^*$.

Spectral Properties

Theorem (Spectrum)

Let T be the multiplication operator and $\sigma(T)$ its spectrum. Then:

(a) T has no eigenvalues.

(b) $\sigma(T)$ is all of $\mathbb{R}$.

(a) For any λ, let $x \in \mathscr{D}(T)$ be such that $Tx = \lambda x$. Then $(T - \lambda I)x = 0$. Hence, by the definition of T,

$$0 = \|(T - \lambda I)x\|^2 = \int_{-\infty}^{+\infty} |t - \lambda|^2 |x(t)|^2 dt.$$

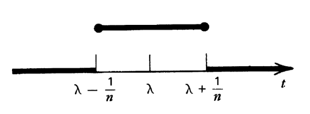
Since $|t - \lambda| > 0$, for all $t \neq \lambda$, we have $x(t) = 0$, for almost all $t \in \mathbb{R}$. Hence, $x = 0$. So x is not an eigenvector and λ not an eigenvalue of T. Since λ was arbitrary, T has no eigenvalues.

Spectral Properties Part (b)

(b) We have $\sigma(T) \subseteq \mathbb{R}$, by previous theorems.

Let $\lambda \in \mathbb{R}$. We define

$$v_n(t) = \begin{cases} 1, & \text{if } \lambda - \frac{1}{n} \le t \le \lambda + \frac{1}{n} \\ 0, & \text{elsewhere} \end{cases}.$$



Consider $x_n = \frac{1}{\|v_n\|} v_n$. Then $\|x_n\| = 1$.

Write $T_\lambda = T - \lambda I$, as usual.

Note that $(t - \lambda)^2 \le \frac{1}{n^2}$ on the interval on which v_n is not zero.

So, by the definition of T,

$$\|T_\lambda x_n\|^2 = \int_{-\infty}^{+\infty} (t - \lambda)^2 |x_n(t)|^2 dt \le \frac{1}{n^2} \int_{-\infty}^{+\infty} |x_n(t)|^2 dt = \frac{1}{n^2}.$$

Spectral Properties Part (b) (Cont'd)

- Taking square roots, we have $\| T_\lambda x_n \| \leq \frac{1}{n}$.

  Since T has no eigenvalues, the resolvent $R_\lambda = T_\lambda^{-1}$ exists.

  Moreover, $T_\lambda x_n \neq 0$ because $x_n \neq 0$, by a preceding result.

  Consider the vectors

  $$y_n = \frac{1}{\| T_\lambda x_n \|} T_\lambda x_n.$$

  - They are in the range of T_λ, which is the domain of R_λ;
  - They have norm 1.

  Applying R_λ, we get

  $$\| R_\lambda y_n \| = \frac{1}{\| T_\lambda x_n \|} \| x_n \| \geq n.$$

  This shows that the resolvent R_λ is unbounded. Hence, $\lambda \in \sigma(T)$.

  Since $\lambda \in \mathbb{R}$ was arbitrary, $\sigma(T) = \mathbb{R}$.

The Spectral Family of T

- The spectral family of T is (E_λ), where $\lambda \in \mathbb{R}$ and

$$E_\lambda : L^2(-\infty, +\infty) \to L^2(-\infty, \lambda)$$

  is the projection of $L^2(-\infty, +\infty)$ onto $L^2(-\infty, \lambda)$, considered as a subspace of $L^2(-\infty, +\infty)$.

- Thus,

$$E_\lambda x(t) = \begin{cases} x(t), & \text{if } t < \lambda \\ 0, & \text{if } t \geq \lambda \end{cases}.$$

Absolute Continuity

- Let $x(t)$ be a function in $L^2(-\infty,\infty)$.
- Recall that x is said to be **absolutely continuous on** an interval $[a,b]$ if, given $\varepsilon > 0$, there is a $\delta > 0$, such that:

  For every finite set of disjoint open subintervals $(a_1,b_1),\ldots,(a_n,b_n)$ of $[a,b]$ of total length less than δ, we have

$$\sum_{j=1}^{n} |x(b_j) - x(a_j)| < \varepsilon.$$

- Recall, also, that, if x is absolutely continuous on $[a,b]$, then:
  - It is differentiable almost everywhere on $[a,b]$;
  - $x' \in L[a,b]$.

The Differentiation Operator

- Consider the **differentiation operator**

$$\begin{array}{rcl} D: \ \mathscr{D}(D) & \to & L^2(-\infty,+\infty); \\ x & \mapsto & ix', \end{array}$$

  where $x' = \frac{dx}{dt}$ and i helps to make D self-adjoint.
- By definition, the domain $\mathscr{D}(D)$ of D consists of all $x \in L^2(-\infty,+\infty)$ which are:
  - Absolutely continuous on every compact interval on $\mathbb{R}$;
  - Such that $x' \in L^2(-\infty,+\infty)$.
- $\mathscr{D}(D)$ contains the sequence (e_n) involving the Hermite polynomials.
- The sequence (e_n) is total (i.e., its span is dense) in $L^2(-\infty,+\infty)$.
- Hence, $\mathscr{D}(D)$ is dense in $L^2(-\infty,+\infty)$.

Unboundedness of the Differentiation Operator

Lemma (Differentiation Operator)

The differentiation operator D is unbounded.

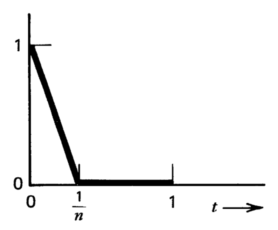
- D is an extension of $D_0 = D|_Y$, where $Y = \mathscr{D}(D) \cap L^2[0,1]$ and $L^2[0,1]$ is regarded as a subspace of $L^2(-\infty, +\infty)$.

  Hence, if D_0 is unbounded, so is D.

  We show that D_0 is unbounded.

  Let

$$x_n(t) = \begin{cases} 1 - nt, & \text{if } 0 \le t \le \frac{1}{n} \\ 0, & \text{if } \frac{1}{n} < t \le 1 \end{cases}.$$



Unboundedness of the Differentiation Operator (Cont'd)

- We defined
$$x_n(t) = \begin{cases} 1 - nt, & \text{if } 0 \le t \le \frac{1}{n} \\ 0, & \text{if } \frac{1}{n} < t \le 1 \end{cases}.$$
The derivative is
$$x_n'(t) = \begin{cases} -n, & \text{if } 0 < t < \frac{1}{n} \\ 0, & \text{if } \frac{1}{n} < t < 1 \end{cases}.$$
We calculate
$$\|x_n\|^2 = \int_0^1 |x_n(t)|^2 dt = \frac{1}{3n}.$$
Moreover,
$$\|D_0 x_n\|^2 = \int_0^1 |x_n'(t)|^2 dt = n.$$
The quotient $\frac{\|D_0 x_n\|}{\|x_n\|} = n\sqrt{3} > n$. So D_0 is unbounded.

Remarks on the Differentiation Operator

- The differentiation operator is unbounded, even if considered for $L^2[a,b]$, where $[a,b]$ is a compact interval.

Theorem (Self-Adjointness)

The differentiation operator D is self-adjoint.

- A proof of this theorem requires some tools from the theory of Lebesgue integration.
- We finally mention the following properties:
  - D does not have eigenvalues;
  - The spectrum $\sigma(D)$ is all of $\mathbb{R}$.