Introduction to Spectral Theory of Linear Operators

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1. Spectral Theory of Linear Operators in Normed Spaces
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   - Further Properties of Resolvent and Spectrum
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Subsection 1

Spectral Theory in Finite Dimensional Normed Spaces

Linear Operators On Normed Spaces

- Let X be a finite dimensional normed space.
- Let $T : X \to X$ be a linear operator.
- We know that we can represent T by matrices (which depend on the choice of bases for X).
- Then the spectral theory of T is essentially matrix eigenvalue theory.
- For a given (real or complex) n-rowed square matrix $A = (\alpha_{jk})$, the concepts of *eigenvalues* and *eigenvectors* are defined in terms of the equation

$$Ax = \lambda x.$$

Eigenvalues, Eigenvectors, Eigenspaces and Spectrum

Definition (Eigenvalues, Eigenvectors, Eigenspaces, Spectrum, Resolvent Set of a Matrix)

An **eigenvalue** of a square matrix $A = (\alpha_{jk})$ is a number λ, such that

$$Ax = \lambda x$$

has a solution $x \neq 0$. This x is called an **eigenvector** of A corresponding to that eigenvalue λ.

- The eigenvectors corresponding to that eigenvalue λ and the zero vector form a vector subspace of X which is called the **eigenspace** of A corresponding to that eigenvalue λ.
- The set $\sigma(A)$ of all eigenvalues of A is called the **spectrum** of A.
- The complement $\rho(A) = \mathbb{C} - \sigma(A)$ of the spectrum of A in the complex plane is called the **resolvent set** of A.

Characteristic Equation, Determinant and Polynomial

- Let I be the $n \times n$ unit matrix.
- $Ax = \lambda x$ can be written $(A - \lambda I)x = 0$.
- This is a homogeneous system of n linear equations in n unknowns $\xi_1, \ldots, \xi_n$, the components of x.
- The determinant of the coefficients is $\det(A - \lambda I)$.
- This determinant must be zero in order to have a solution $x \neq 0$.
- This gives the **characteristic equation** of A:

$$\det(A - \lambda I) = \begin{vmatrix} \alpha_{11} - \lambda & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} - \lambda & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} - \lambda \end{vmatrix} = 0.$$

- $\det(A - \lambda I)$ is called the **characteristic determinant** of A.
- By developing it we obtain a polynomial in λ of degree n, the **characteristic polynomial** of A.

The Eigenvalue Theorem

Theorem (The Eigenvalue Theorem)

The eigenvalues of an $n \times n$ square matrix $A = (\alpha_{jk})$ are given by the solutions of the characteristic equation $\det(A - \lambda I) = 0$ of A. Hence A has at least one eigenvalue (and at most n numerically different eigenvalues).

- We have proven the first statement.

  Recall that, by the Fundamental Theorem of Algebra and the Factorization Theorem, a polynomial of degree $n > 0$, with coefficients in $\mathbb{C}$, has a root in $\mathbb{C}$ (and at most n numerically different roots).

  This yields the second statement.
- Note that roots may be complex even if A is real.

Example

- Consider the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

  We find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \quad \begin{aligned} &\Rightarrow (5-\lambda)(2-\lambda) - 4 = 0 \\ &\Rightarrow \lambda^2 - 7\lambda + 6 = 0 \\ &\Rightarrow (\lambda - 1)(\lambda - 6) = 0 \\ &\Rightarrow \lambda = 1 \text{ or } \lambda = 6. \end{aligned}$$

  Thus, the spectrum is $\{1, 6\}$.

Example (Cont'd)

- We found the eigenvalues of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

  The eigenvectors of A corresponding to 1 and 6 are obtained from

  $$\begin{cases} 4\xi_1 + 4\xi_2 = 0 \\ \xi_1 + \xi_2 = 0 \end{cases} \quad \text{and} \quad \begin{cases} -\xi_1 + 4\xi_2 = 0 \\ \xi_1 - 4\xi_2 = 0 \end{cases},$$

  respectively.

  Observe that in each case we need only one of the two equations.

  So $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ are eigenvectors of A corresponding to the eigenvalues 1 and 6, respectively.

Eigenvalues and Spectrum of an Operator

- Let X be a normed space of dimension n.
- Consider again a linear operator $T: X \to X$.
- Let $e = \{e_1, \ldots, e_n\}$ be any basis for X.
- Let $T_e = (\alpha_{jk})$ be the matrix representing T with respect to the basis e (whose elements are kept in the given order).
- The eigenvalues of the matrix T_e are called the **eigenvalues** *of the operator* T.
- The spectrum of the matrix T_e is called the **spectrum** of T.
- The resolvent set of T_e is called the **resolvent set** of T.

Eigenvalues of an Operator

Theorem (Eigenvalues of an Operator)

All matrices representing a given linear operator $T : X \to X$ on a finite dimensional normed space X relative to various bases for X have the same eigenvalues.

- We examine the effect of the transition from one basis for X to another.

  Let $e = (e_1, \ldots, e_n)$ and $\widetilde{e} = (\widetilde{e}_1, \ldots, \widetilde{e}_n)$ be any bases for X, written as row vectors.

  By the definition of a basis, each e_j is a linear combination of the $\widetilde{e}_k$'s and conversely.

  We can write this $\widetilde{e} = eC$ or $\widetilde{e}^\top = C^\top e^\top$, where C is a nonsingular $n \times n$ square matrix.

Eigenvalues of an Operator (Cont'd)

- Every $x \in X$ has a unique representation with respect to each of the two bases. Say,

$$x = \sum \xi_j e_j = e x_1 \quad \text{and} \quad x = \sum \tilde{\xi}_k \tilde{e}_k = \tilde{e} x_2,$$

where $x_1 = (\xi_j)$ and $x_2 = (\tilde{\xi}_k)$ are column vectors.

We get, $e x_1 = \tilde{e} x_2 = e C x_2$. Hence $x_1 = C x_2$.

Similarly, suppose $Tx = y = e y_1 = \tilde{e} y_2$. Then we have $y_1 = C y_2$.

Now, if T_1 and T_2 denote the matrices which represent T with respect to e and $\tilde{e}$, respectively, then $y_1 = T_1 x_1$ and $y_2 = T_2 x_2$.

Therefore, we obtain

$$C T_2 x_2 = C y_2 = y_1 = T_1 x_1 = T_1 C x_2.$$

Eigenvalues of an Operator (Conclusion)

- We obtained $CT_2x_2 = T_1Cx_2$.

  Premultiplying by C^{-1}, we obtain the transformation law

  $$T_2 = C^{-1}T_1C,$$

  with C determined by the bases and independent of T.

  Using $\det(C^{-1})\det(C) = 1$, we can now show that the characteristic determinants of T_2 and T_1 are equal.

  $$\begin{aligned} \det(T_2 - \lambda I) &= \det(C^{-1}T_1C - \lambda C^{-1}IC) \\ &= \det(C^{-1}(T_1 - \lambda I)C) \\ &= \det(C^{-1})\det(T_1 - \lambda I)\det C \\ &= \det(T_1 - \lambda I). \end{aligned}$$

  Equality of the eigenvalues of T_1 and T_2 now follows from the Eigenvalue Theorem.

Similar Matrices

- An $n \times n$ matrix T_2 is said to be **similar** to an $n \times n$ matrix T_1, if there exists a nonsingular matrix C, such that

$$T_2 = C^{-1} T_1 C.$$

- T_1 and T_2 are then called **similar matrices**.
- In terms of this concept, our proof shows that:
  - (i) Two matrices representing the same linear operator T on a finite dimensional normed space X relative to any two bases for X are similar.
  - (ii) Similar matrices have the same eigenvalues.

Existence of Eigenvalues and Determinant of an Operator

Existence Theorem (Eigenvalues)

A linear operator on a finite dimensional complex normed space $X \neq \{0\}$ has at least one eigenvalue.

- This follows from the Eigenvalue Theorem and the preceding theorem.
- Note that, with $\lambda = 0$, $\det(T_2 - \lambda I) = \det(T_1 - \lambda I)$ gives

$$\det T_2 = \det T_1.$$

Hence, the value of the determinant is an intrinsic property of T.

We call it the **determinant** of the operator T and denote it by $\det T$.

Subsection 2

Basic Concepts

The Operator T_λ Associated With An Operator T

- We now consider normed spaces of any dimension.
- Let $X \neq \{0\}$ be a complex normed space.
- Let $T : \mathscr{D}(T) \to X$ be a linear operator with domain $\mathscr{D}(T) \subseteq X$.
- With T we associate the operator

$$T_\lambda = T - \lambda I,$$

where:

  - λ is a complex number;
  - I is the identity operator on $\mathscr{D}(T)$.

The Resolvent of an Operator T

- If T_λ has an inverse, we denote it by $R_\lambda(T)$,

$$R_\lambda(T) = T_\lambda^{-1} = (T - \lambda I)^{-1}.$$

- We call $R_\lambda(T)$ the **resolvent operator** of T or, simply, the **resolvent** of T.
- Instead of $R_\lambda(T)$ we also write simply R_λ if the operator T is clear from context.
- The name "resolvent" is appropriate, since $R_\lambda(T)$ helps to *solve* the equation $T_\lambda x = y$.

  Indeed, suppose $R_\lambda(T)$ exists.

  Then

$$x = T_\lambda^{-1} y = R_\lambda(T) y.$$

Regular Value, Resolvent Set and Spectrum

Definition (Regular Value, Resolvent Set, Spectrum)

Let $X \neq \{0\}$ be a complex normed space and $T : \mathscr{D}(T) \to X$ a linear operator with domain $\mathscr{D}(T) \subseteq X$.

- A **regular value** λ of T is a complex number such that:
  - (R1) $R_\lambda(T)$ exists;
  - (R2) $R_\lambda(T)$ is bounded;
  - (R3) $R_\lambda(T)$ is defined on a set which is dense in X.
- The **resolvent set** $\rho(T)$ of T is the set of all regular values λ of T.
- Its complement $\sigma(T) = \mathbb{C} - \rho(T)$ in the complex plane $\mathbb{C}$ is called the **spectrum** of T.
- A $\lambda \in \sigma(T)$ is called a **spectral value** of T.

Partition of the Spectrum

Definition (Point, Continuous and Residual Spectrum)

Let $X \neq \{0\}$ be a complex normed space and $T : \mathscr{D}(T) \to X$ a linear operator with domain $\mathscr{D}(T) \subseteq X$.
The spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows:

- The **point spectrum** or **discrete spectrum** $\sigma_p(T)$ is the set such that $R_\lambda(T)$ does not exist.

  A $\lambda \in \sigma_p(T)$ is called an **eigenvalue** of T.
- The **continuous spectrum** $\sigma_c(T)$ is the set such that $R_\lambda(T)$ exists and satisfies (R3) but not (R2), that is, $R_\lambda(T)$ is unbounded.
- The **residual spectrum** $\sigma_r(T)$ is the set such that $R_\lambda(T)$ exists (bounded or not) but does not satisfy (R3), i.e., the domain of $R_\lambda(T)$ is not dense in X.

Summary of the Defining Conditions

- Some of the sets defined above may be empty.

  For instance, $\sigma_c(T) = \sigma_r(T) = \emptyset$ in the finite dimensional case.
- Recall the conditions
  - (R1) $R_\lambda(T)$ exists;
  - (R2) $R_\lambda(T)$ is bounded;
  - (R3) $R_\lambda(T)$ is defined on a set which is dense in X.
- The various cases can be summarized as follows:

Satisfied	Not Satisfied	λ Belongs to
$(R1)$ $(R2)$ $(R3)$		$\rho(T)$
	$(R1)$	$\sigma_p(T)$
$(R1)$ $(R3)$	$(R2)$	$\sigma_c(T)$
$(R1)$	$(R3)$	$\sigma_r(T)$

Eigenvalues, Eigenvectors and Eigenspaces

- The four sets in the table are disjoint and their union is the whole complex plane:

$$\mathbb{C} = \rho(T) \cup \sigma(T) = \rho(T) \cup \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

- If the resolvent $R_\lambda(T)$ exists, it is linear.
- $R_\lambda(T) : \mathscr{R}(T) \to \mathscr{D}(T)$ exists if and only if $T_\lambda x = 0$ implies $x = 0$.

  I.e., $R_\lambda(T)$ exists if and only if the null space of T_λ is $\{0\}$.
- Hence, if $T_\lambda x = (T - \lambda I)x = 0$, for some $x \neq 0$, then $\lambda \in \sigma_p(T)$, by definition. That is, λ is an eigenvalue of T.
- The vector x is then called an **eigenvector** of T (or **eigenfunction** of T if X is a function space) corresponding to the eigenvalue λ.
- The subspace of $\mathscr{D}(T)$ consisting of 0 and all eigenvectors of T corresponding to an eigenvalue λ of T is called the **eigenspace** of T corresponding to that eigenvalue λ.

Operator with a Spectral Value not an Eigenvalue

- If X is infinite dimensional, then T can have spectral values which are not eigenvalues.
- On the Hilbert sequence space $X = \ell^2$ we define a linear operator $T : \ell^2 \to \ell^2$ by

$$(\xi_1, \xi_2, \dots) \mapsto (0, \xi_1, \xi_2, \dots),$$

where $x = (\xi_j) \in \ell^2$. T is called the **right-shift operator**.
Note that T is bounded (with $\|T\| = 1$).

$$\|Tx\|^2 = \sum_{j=1}^{\infty} |\xi_j|^2 = \|x\|^2.$$

The operator $R_0(T) = T^{-1} : T(X) \to X$ exists.
It is the **left-shift operator**, given by

$$(\xi_1, \xi_2, \dots) \mapsto (\xi_2, \xi_3, \dots).$$

The Right-Shift Operator (Cont'd)

- To conclude, note that $R_0(T)$ does not satisfy (R3).

  Indeed, $T(X)$ is not dense in X.

  $T(X)$ is the subspace Y consisting of all $y = (\eta_j)$, with $\eta_1 = 0$.

  By definition, $\lambda = 0$ is a spectral value of T.

  However, $\lambda = 0$ is not an eigenvalue.

  $Tx = 0$ implies $x = 0$ and 0 is not an eigenvector.

Connection with Bounded Inverse Theorem

- Recall the

Open Mapping Theorem, Bounded Inverse Theorem

A bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping. Hence, if T is bijective, T^{-1} is continuous and thus bounded.

- From this we derive that if:
  - X is complete;
  - $T : X \to X$ is bounded and linear;
  - For some λ the resolvent $R_\lambda(T)$ exists and is defined on X;

  then for that λ the resolvent is bounded.

The Domain of R_λ

Lemma (Domain of R_λ)

Let X be a complex Banach space, $T : X \to X$ a linear operator, and $\lambda \in \rho(T)$. Assume that:

(a) T is closed or
(b) T is bounded.

Then $R_\lambda(T)$ is defined on the whole space X and is bounded.

(a) Since T is closed, so is $T_\lambda = T - \lambda I$. Hence $R_\lambda = T_\lambda^{-1}$ is closed.
R_λ is bounded by (R2). Hence its domain $\mathscr{D}(R_\lambda)$ is closed.
Now (R3) implies $\mathscr{D}(R_\lambda) = \overline{\mathscr{D}(R_\lambda)} = X$.

(b) Since $\mathscr{D}(T) = X$ is closed, T is closed.
So the statement follows from Part (a).

Subsection 3

Spectral Properties of Bounded Linear Operators

Invertibility of $I - T$

Theorem (Inverse)

Let $T \in B(X,X)$, where X is a Banach space. If $\|T\| < 1$, then $(I - T)^{-1}$ exists as a bounded linear operator on the whole space X and

$$(I - T)^{-1} = \sum_{j=0}^{\infty} T^j = I + T + T^2 + \cdots,$$

where the series on the right is convergent in the norm on $B(X,X)$.

- We have $\|T^j\| \leq \|T\|^j$.

  The geometric series $\sum \|T\|^j$ converges for $\|T\| < 1$.

  Hence the series $\sum_{j=0}^{\infty} T^j$ is absolutely convergent for $\|T\| < 1$.

  Since X is complete, so is $B(X,X)$.

  Absolute convergence, thus, implies convergence.

Invertibility of $I - T$ (Cont'd)

- We denote by S the sum of the series

$$\sum_{j=0}^{\infty} T^j = I + T + T^2 + \cdots .$$

It remains to show that $S = (I - T)^{-1}$.

We calculate

$$(I - T)(I + T + \cdots + T^n) = (I + T + \cdots + T^n)(I - T) = I - T^{n+1}.$$

We now let $n \to \infty$.

Then $T^{n+1} \to 0$, because $\| T \| < 1$.

We thus obtain $(I - T)S = S(I - T) = I$.

This shows that $S = (I - T)^{-1}$.

Closedness of the Spectrum

Theorem (The Spectrum is Closed)

The resolvent set $\rho(T)$ of a bounded linear operator T on a complex Banach space X is open. Hence, the spectrum $\sigma(T)$ is closed.

- If $\rho(T) = \varnothing$, it is open. Let $\rho(T) \neq \varnothing$.

  For a fixed $\lambda_0 \in \rho(T)$ and any $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} T - \lambda I &= T - \lambda_0 I - (\lambda - \lambda_0) I \\ &= (T - \lambda_0 I)[I - (\lambda - \lambda_0)(T - \lambda_0 I)^{-1}]. \end{aligned}$$

  Let V denote the operator in the brackets. Then

$$V = I - (\lambda - \lambda_0) R_{\lambda_0}.$$

  Moreover, we can write $T_\lambda = T_{\lambda_0} V$.

Closedness of the Spectrum (Cont'd)

- We obtained $T_\lambda = T_{\lambda_0} V$, where $V = I - (\lambda - \lambda_0) R_{\lambda_0}$.
  Now $\lambda_0 \in \rho(T)$ and T is bounded.
  By a previous lemma, $R_{\lambda_0} = T_{\lambda_0}^{-1} \in B(X, X)$.
  The theorem shows that V has an inverse in $B(X, X)$, for all λ, such that $\|(\lambda - \lambda_0) R_{\lambda_0}\| < 1$, i.e., $|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|}$, given by

$$V^{-1} = \sum_{j=0}^{\infty} [(\lambda - \lambda_0) R_{\lambda_0}]^j = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j R_{\lambda_0}^j.$$

  But $T_{\lambda_0}^{-1} = R_{\lambda_0} \in B(X, X)$. So, for $|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|}$, T_λ has an inverse

$$R_\lambda = T_\lambda^{-1} = (T_{\lambda_0} V)^{-1} = V^{-1} R_{\lambda_0}.$$

  Hence, $|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|}$ represents a neighborhood of λ_0 consisting of regular values λ of T. Since $\lambda_0 \in \rho(T)$ was arbitrary, $\rho(T)$ is open.
  So $\sigma(T) = \mathbb{C} - \rho(T)$ is closed.

Representation Theorem for the Resolvent

- In the preceding proof we have also obtained a basic representation of the resolvent by a power series in powers of λ.

Theorem (Representation for the Resolvent)

Let T be a bounded linear operator on a complex Banach space X. For every $\lambda_0 \in \rho(T)$, the resolvent $R_\lambda(T)$ has the representation

$$R_\lambda = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j R_{\lambda_0}^{j+1},$$

the series being absolutely convergent for every λ in the open disk given by $|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|}$ in the complex plane. This disk is a subset of $\rho(T)$.

The Spectrum Theorem

Theorem (Spectrum)

The spectrum $\sigma(T)$ of a bounded linear operator $T: X \to X$ on a complex Banach space X is compact and lies in the disk given by $\lambda \leq \|T\|$. Hence, the resolvent set $\rho(T)$ of T is not empty.

- Let $\lambda \neq 0$ and $\kappa = \frac{1}{\lambda}$. By the theorem, we obtain the representation

$$R_\lambda = (T - \lambda I)^{-1} = -\frac{1}{\lambda}(I - \kappa T)^{-1} = -\frac{1}{\lambda}\sum_{j=0}^{\infty}(\kappa T)^j = -\frac{1}{\lambda}\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}T\right)^j.$$

The series converges for λ such that $\|\frac{1}{\lambda}T\| = \frac{\|T\|}{\lambda} < 1$ i.e., $|\lambda| > \|T\|$.
The same theorem also shows that any such λ is in $\rho(T)$.
Hence the spectrum $\sigma(T) = \mathbb{C} - \rho(T)$ must lie in the disk $|\lambda| \leq \|T\|$.
So $\sigma(T)$ is bounded. But $\sigma(T)$ is closed. Hence $\sigma(T)$ is compact.

The Spectral Radius

- Since for a bounded linear operator T on a complex Banach space the spectrum is bounded, it seems natural to ask for the smallest disk about the origin which contains the whole spectrum.

Definition (Spectral Radius)

The spectral radius $r_\sigma(T)$ of an operator $T \in B(X,X)$ on a complex Banach space X is the radius

$$r_\sigma(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

of the smallest closed disk centered at the origin of the complex λ-plane and containing $\sigma(T)$.

- It is obvious that for the spectral radius of a bounded linear operator T on a complex Banach space we have $r_\sigma(T) \leq \|T\|$.
- Moreover, we will prove that $r_\sigma(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|}$.

Subsection 4

Further Properties of Resolvent and Spectrum

Resolvent Equations

Theorem (Resolvent Equation, Commutativity)

Let X be a complex Banach space, $T \in B(X,X)$ and $\lambda, \mu \in \rho(T)$. Then:

(a) The resolvent R_λ of T satisfies the **Hilbert relation** or **resolvent equation**

$$R_\mu - R_\lambda = (\mu - \lambda) R_\mu R_\lambda, \quad \lambda, \mu \in \rho(T).$$

(b) R_λ commutes with any $S \in B(X,X)$ which commutes with T.

(c) We have $R_\lambda R_\mu = R_\mu R_\lambda$, $\lambda, \mu \in \rho(T)$.

(a) We showed the range of T is all of X.

Hence, $I = T_\lambda R_\lambda$, where I is the identity operator on X.

Also $I = R_\mu T_\mu$.

Resolvent Equations (Cont'd)

- Consequently,

$$\begin{aligned} R_\mu - R_\lambda &= R_\mu(T_\lambda R_\lambda) - (R_\mu T_\mu)R_\lambda \\ &= R_\mu(T_\lambda - T_\mu)R_\lambda \\ &= R_\mu[T - \lambda I - (T - \mu I)]R_\lambda \\ &= (\mu - \lambda)R_\mu R_\lambda. \end{aligned}$$

(b) By assumption, $ST = TS$. Hence, $ST_\lambda = T_\lambda S$.
Using $I = T_\lambda R_\lambda = R_\lambda T_\lambda$, we thus obtain

$$R_\lambda S = R_\lambda S T_\lambda R_\lambda = R_\lambda T_\lambda S R_\lambda = S R_\lambda.$$

(c) R_μ commutes with T by Part (b).
Hence, R_λ commutes with R_μ by Part (b).

Eigenvalues of Matrices formed by Polynomials

- If λ is an eigenvalue of a matrix A, then $Ax = \lambda x$ for some $x \neq 0$.
- Application of A gives

$$A^2 x = A\lambda x = \lambda A x = \lambda^2 x.$$

- Continuing we get, for every positive integer m, $A^m x = \lambda^m x$.
- I.e., if λ is an eigenvalue of A, then λ^m is an eigenvalue of A^m.
- More generally, if λ is an eigenvalue of A,

$$p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \cdots + \alpha_0$$

is an eigenvalue of the matrix

$$p(A) = \alpha_n A^n + \alpha_{n-1} A^{n-1} + \cdots + \alpha_0 I.$$

- We will show that this property extends to complex Banach spaces of any dimension, using the fact that a bounded linear operator has a nonempty spectrum (shown later by methods of complex analysis).

Notation

- Consider a polynomial

$$p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0.$$

- Define

$$p(\sigma(T)) = \{\mu \in \mathbb{C} : \mu = p(\lambda), \lambda \in \sigma(T)\}.$$

- Thus, $p(\sigma(T))$ is the set of all complex numbers μ, such that $\mu = p(\lambda)$, for some $\lambda \in \sigma(T)$.
- The set $p(\rho(T))$ is defined similarly

$$p(\rho(T)) = \{\mu \in \mathbb{C} : \mu = p(\lambda), \lambda \in \rho(T)\}.$$

Spectral Mapping Theorem for Polynomials

Spectral Mapping Theorem for Polynomials

Let X be a complex Banach space, $T \in B(X,X)$ and

$$p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \cdots + \alpha_0.$$

Then $\sigma(p(T)) = p(\sigma(T))$, i.e., the spectrum $\sigma(p(T))$ of the operator $p(T) = \alpha_n T^n + \alpha_{n-1} T^{n-1} + \cdots + \alpha_0 I$ consists precisely of all those values which the polynomial p assumes on the spectrum $\sigma(T)$ of T.

- We assume that $\sigma(T) \neq \emptyset$.
  - The case $n = 0$ is trivial. Then $p(\sigma(T)) = \{\alpha_0\} = \sigma(p(T))$.
  - Let $n > 0$.
    - In Part (a), we prove $\sigma(p(T)) \subseteq p(\sigma(T))$.
    - In Part (b), we prove $p(\sigma(T)) \subseteq \sigma(p(T))$.

Spectral Mapping Theorem for Polynomials Part (a)

(a) For simplicity we write $S = p(T)$ and $S_\mu = p(T) - \mu I$, $\mu \in \mathbb{C}$.

If S_μ^{-1} exists, the formula for S_μ shows that S_μ^{-1} is the resolvent operator of $p(T)$.

We keep μ fixed.

Since X is complex, the polynomial given by $s_\mu(\lambda) = p(\lambda) - \mu$ must factor completely into linear terms. Suppose

$$s_\mu(\lambda) = p(\lambda) - \mu = \alpha_n(\lambda - \gamma_1)(\lambda - \gamma_2)\cdots(\lambda - \gamma_n),$$

where $\gamma_1, \ldots, \gamma_n$ are the zeros of s_μ.

Corresponding to this, we have

$$S_\mu = p(T) - \mu I = \alpha_n(T - \gamma_1 I)(T - \gamma_2 I)\cdots(T - \gamma_n I).$$

Spectral Mapping Theorem for Polynomials Part (a Cont'd)

- Suppose each γ_j is in $\rho(T)$.

  Then each $T - \gamma_j I$ has a bounded inverse which, by previous results, is defined on all of X.

  The same holds for S_μ and

  $$S_\mu^{-1} = \frac{1}{\alpha_n}(T - \gamma_n I)^{-1} \cdots (T - \gamma_1 I)^{-1}.$$

  Hence in this case, $\mu \in \rho(p(T))$.

  From this we conclude that $\mu \in \sigma(p(T))$ implies $\gamma_j \in \sigma(T)$, for some j.

  Now we get $s_\mu(\gamma_j) = p(\gamma_j) - \mu = 0$.

  Thus, $\mu = p(\gamma_j) \in p(\sigma(T))$.

  Since $\mu \in \sigma(p(T))$ was arbitrary, $\sigma(p(T)) \subseteq p(\sigma(T))$.

Spectral Mapping Theorem for Polynomials Part (b)

(b) Let $\kappa \in p(\sigma(T))$.

By definition, this means that $\kappa = p(\beta)$, for some $\beta \in \sigma(T)$.

There are now two possibilities:

(A) $T - \beta I$ has no inverse;

(B) $T - \beta I$ has an inverse.

Spectral Mapping Theorem for Polynomials Part (b)(A)

(A) From $\kappa = p(\beta)$ we have $p(\beta) - \kappa = 0$.
Hence, β is a zero of the polynomial given by $s_\kappa(\lambda) = p(\lambda) - \kappa$.
So we can write

$$s_\kappa(\lambda) = p(\lambda) - \kappa = (\lambda - \beta) g(\lambda),$$

where $g(\lambda)$ is the product of the other $n-1$ linear factors and α_n.
Corresponding to this representation we have

$$S_\kappa = p(T) - \kappa I = (T - \beta I) g(T).$$

The factors of $g(T)$ all commute with $T - \beta I$.
So we also have $S_\kappa = g(T)(T - \beta I)$.
If S_κ had an inverse, we would now get

$$I = (T - \beta I) g(T) S_\kappa^{-1} = S_\kappa^{-1} g(T)(T - \beta I).$$

Then $T - \beta I$ would have an inverse, contradicting our assumption.
So $\kappa \in \sigma(p(T))$.

Spectral Mapping Theorem for Polynomials Part (b)(B)

(B) Suppose that $\kappa = p(\beta)$, for some $\beta \in \sigma(T)$, but $(T - \beta I)^{-1}$ exists.

Suppose that the range of $T - \beta I$ was X.

Then, $(T - \beta I)^{-1}$ would be bounded by the Bounded Inverse Theorem.

Thus, $\beta \in \rho(T)$, which would contradict $\beta \in \sigma(T)$.

It follows that for the range of $T - \beta I$, we must have

$$\mathscr{R}(T - \beta I) \neq X.$$

Since $S_\kappa = (T - \beta I)g(T)$, we now get $\mathscr{R}(S_\kappa) \neq X$.

This shows that $\kappa \in \sigma(p(T))$, since $\kappa \in \rho(p(T))$ would imply that $\mathscr{R}(S_\kappa) = X$ by a preceding lemma.

Linear Independence of Eigenvectors

Theorem (Linear Independence)

Eigenvectors $x_1,\ldots,x_n$ corresponding to different eigenvalues $\lambda_1,\ldots,\lambda_n$ of a linear operator T on a vector space X constitute a linearly independent set.

- Towards a contradiction, assume that $\{x_1,\ldots,x_n\}$ is linearly dependent. Let x_m be the first of the vectors which is a linear combination of its predecessors, say, $x_m = \alpha_1 x_1 + \cdots + \alpha_{m-1} x_{m-1}$.
  Then $\{x_1,\ldots,x_{m-1}\}$ is linearly independent.
  Apply $T - \lambda_m I$ on both sides:
  $$(T-\lambda_m I)x_m = \sum_{j=1}^{m-1} \alpha_j (T-\lambda_m I)x_j = \sum_{j=1}^{m-1} \alpha_j (\lambda_j - \lambda_m) x_j.$$
  Since x_m is an eigenvector corresponding to λ_m, the left side is zero.
  By the linear independence of $\{x_1,\ldots,x_{m-1}\}$, $\alpha_j(\lambda_j - \lambda_m) = 0$.
  Hence, $\alpha_j = 0$, $j = 1,\ldots,m-1$. But then $x_m = 0$, contradicting $x_m \neq 0$, x_m being an eigenvector.

Subsection 5

Use of Complex Analysis in Spectral Theory

Domains in the Complex Plane

- A metric space is said to be **connected** if it is not the union of two disjoint nonempty open subsets.
- A subset of a metric space is said to be **connected** if it is connected regarded as a subspace.
- By a **domain** G in the complex plane $\mathbb{C}$ we mean an open connected subset G of $\mathbb{C}$.
- It can be shown that an open subset G of $\mathbb{C}$ is connected if and only if every pair of points of G can be joined by a broken line consisting of finitely many straight line segments all points of which belong to G.

Holomorphic or Analytic Functions

- A complex valued function h of a complex variable λ is said to be **holomorphic** (or **analytic**) on a domain G of the complex λ-plane if h is defined and differentiable on G, that is, the derivative h' of h, defined by
$$h'(\lambda) = \lim_{\Delta\lambda \to 0} \frac{h(\lambda + \Delta\lambda) - h(\lambda)}{\Delta\lambda}$$
exists for every $\lambda \in G$.
- The function h is said to be **holomorphic at a point** $\lambda_0 \in \mathbb{C}$ if h is holomorphic on some ε-neighborhood of λ_0.
- The function h is holomorphic on G if and only if, at every $\lambda_0 \in G$, it has a power series representation
$$h(\lambda) = \sum_{j=0}^{\infty} c_j (\lambda - \lambda_0)^j,$$
with a nonzero radius of convergence.

Operator Functions

- By a **vector valued function** or **operator function** we mean a mapping
$$\begin{array}{rccc} S: & \Lambda & \to & B(X,X) \\ & \lambda & \mapsto & S_\lambda \end{array}$$
where Λ is any subset of the complex λ-plane.
- We write S_λ instead of $S(\lambda)$, to have a notation similar to R_λ.
- S being given, we may choose any $x \in X$, so that we get a mapping $\Lambda \to X$; $\lambda \mapsto S_\lambda x$.
- We may also choose $x \in X$ and any $f \in X'$ to get a mapping of Λ into the complex plane, namely,
$$\begin{array}{ccc} \Lambda & \to & \mathbb{C} \\ \lambda & \mapsto & f(S_\lambda x). \end{array}$$

Local Holomorphy and Holomorphy

Definition (Local Holomorphy, Holomorphy)

Let Λ be an open subset of $\mathbb{C}$ and X a complex Banach space. Then the operator function $S : \Lambda \to B(X,X)$ is said to be:

- **locally holomorphic** on Λ if, for every $x \in X$ and $f \in X'$, the function h, defined by

$$h(\lambda) = f(S_\lambda x)$$

  is holomorphic at every $\lambda_0 \in \Lambda$ in the usual sense;
- **holomorphic** on Λ if S is locally holomorphic on Λ and Λ is a domain;
- **holomorphic at a point** $\lambda_0 \in \mathbb{C}$ if S is holomorphic on some ε-neighborhood of λ_0.

Holomorphy and the Resolvent

- The resolvent set $\rho(T)$ of a bounded linear operator T is open but may not always be a domain.
- Thus, in general, it is the union of disjoint domains (disjoint connected open sets).
- We will see that the resolvent is holomorphic at every point of $\rho(T)$.
  - Hence in any case it is locally holomorphic on $\rho(T)$;
  - It is holomorphic on $\rho(T)$ if and only if $\rho(T)$ is connected, so that $\rho(T)$ is a single domain.

Remarks on the Definition

- Recall that we defined three kinds of convergence in connection with bounded linear operators.
- Accordingly, we can define three corresponding kinds of derivative S'_λ of S_λ with respect to λ by the formulas:

$$\begin{array}{rcl} \left\| \frac{1}{\Delta\lambda}[S_{\lambda+\Delta\lambda} - S_\lambda] - S'_\lambda \right\| & \to & 0 \\ \left\| \frac{1}{\Delta\lambda}[S_{\lambda+\Delta\lambda}x - S_\lambda x] - S'_\lambda x \right\| & \to & 0, \quad x \in X \\ \left| \frac{1}{\Delta\lambda}[f(S_{\lambda+\Delta\lambda}x) - f(S_\lambda x)] - f(S'_\lambda x) \right| & \to & 0, \quad x \in X, f \in X'. \end{array}$$

- The existence of the derivative in the sense of the last formula for all λ in a domain Λ means that h defined by $h(\lambda) = f(S_\lambda x)$ is a holomorphic function on Λ in the usual sense, i.e., our definition of the derivative.
- It can be shown that the existence of this derivative (for every $x \in X$ and every $f \in X'$) implies the existence of the other two kinds of derivative.

Holomorphy of R_λ

Theorem (Holomorphy of R_λ)

The resolvent $R_\lambda(T)$ of a bounded linear operator $T : X \to X$ on a complex Banach space X is holomorphic at every point λ_0 of the resolvent set $\rho(T)$ of T. Hence, it is locally holomorphic on $\rho(T)$.

- We proved that for every value $\lambda_0 \in \rho(T)$ the resolvent $R_\lambda(T)$ of an operator $T \in B(X, X)$ on a complex Banach space X has a power series representation

$$R_\lambda(T) = \sum_{j=0}^{\infty} R_{\lambda_0}(T)^{j+1}(\lambda - \lambda_0)^j,$$

which converges absolutely for each λ in the disk $|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|}$.

Holomorphy of R_λ (Cont'd)

- We have

$$R_\lambda(T) = \sum_{j=0}^{\infty} R_{\lambda_0}(T)^{j+1}(\lambda - \lambda_0)^j,$$

converging absolutely for each λ in the disk $|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|}$.

Take any $x \in X$ and $f \in X'$ and define h by

$$h(\lambda) = f(R_\lambda(T)x).$$

We obtain the power series representation

$$h(\lambda) = \sum_{j=0}^{\infty} c_j(\lambda - \lambda_0)^j, \quad c_j = f(R_{\lambda_0}(T)^{j+1}x).$$

This is absolutely convergent on the disk $|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|}$.

The Resolvent Theorem

- $\rho(T)$ is the largest set on which the resolvent of T is locally holomorphic.

Theorem (Resolvent)

If $T \in B(X,X)$, where X is a complex Banach space, and $\lambda \in \rho(T)$, then $\|R_\lambda(T)\| \geq \frac{1}{\delta(\lambda)}$, where $\delta(\lambda) = \inf_{s \in \sigma(T)} |\lambda - s|$ is the distance from λ to the spectrum $\sigma(T)$. Hence $\|R_\lambda(T)\| \to \infty$ as $\delta(\lambda) \to 0$.

- For every $\lambda_0 \in \rho(T)$, the disk $|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|}$ is a subset of $\rho(T)$.

  Hence, assuming $\sigma(T) \neq \emptyset$ (proof below), we see that the distance from λ_0 to the spectrum must at least equal the radius of the disk.

  That is, $\delta(\lambda_0) \geq \frac{1}{\|R_{\lambda_0}\|}$. This implies the conclusion.

Nonemptiness of the Spectrum

Theorem (Spectrum)

If $X \neq \{0\}$ is a complex Banach space and $T \in B(X,X)$, then $\sigma(T) \neq \emptyset$.

- By assumption, $X \neq \{0\}$.
  If $T = 0$, then $\sigma(T) = \{0\} \neq \emptyset$.
  Let $T \neq 0$. Then $\|T\| \neq 0$. We obtain the series
  $$R_\lambda = -\frac{1}{\lambda}\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}T\right)^j, \quad |\lambda| > \|T\|.$$
  This series converges for $\frac{1}{|\lambda|} < \frac{1}{\|T\|}$.
  So it converges absolutely for $\frac{1}{|\lambda|} < \frac{1}{2\|T\|}$, i.e., for $|\lambda| > 2\|T\|$.
  For these λ, by the formula for the sum of a geometric series,
  $$\|R_\lambda\| \leq \frac{1}{|\lambda|}\sum_{j=0}^{\infty}\left\|\frac{1}{\lambda}T\right\|^j = \frac{1}{|\lambda| - \|T\|} \leq \frac{1}{\|T\|}.$$

Nonemptiness of the Spectrum (Cont'd)

- We show that the assumption $\sigma(T) = \emptyset$ leads to a contradiction. $\sigma(T) = \emptyset$ implies $\rho(T) = \mathbb{C}$. Hence, R_λ is holomorphic for all λ. Consequently, for a fixed $x \in X$ and a fixed $f \in X'$, the function h defined by $h(\lambda) = f(R_\lambda x)$ is holomorphic on $\mathbb{C}$, i.e., h is an entire function. Since holomorphy implies continuity, h is continuous.

  Thus, h is bounded on the compact disk $|\lambda| \leq 2\|T\|$. But h is also bounded for $|\lambda| \geq 2\|T\|$, since $\|R_\lambda\| < \frac{1}{\|T\|}$, by the preceding inequality.

$$|h(\lambda)| = |f(R_\lambda x)| \leq \|f\|\|R_\lambda x\| \leq \|f\|\|R_\lambda\|\|x\| \leq \frac{\|f\|\|x\|}{\|T\|}.$$

  Hence h is bounded on $\mathbb{C}$. By Liouville's Theorem, which states that an entire function which is bounded on the whole complex plane is a constant, h is constant. Since $x \in X$ and $f \in X'$ in h were arbitrary, $h =$ const implies that R_λ is independent of λ. The same holds for $R_\lambda^{-1} = T - \lambda I$. But this is impossible.

The Spectral Radius Theorem

Theorem (Spectral Radius)

If T is a bounded linear operator on a complex Banach space, then for the spectral radius $r_\sigma(T)$ of T we have $r_\sigma(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|}$.

- We have $\sigma(T^n) = [\sigma(T)]^n$ by the Spectral Mapping Theorem. Thus, $r_\sigma(T^n) = [r_\sigma(T)]^n$. By the Spectrum Theorem, $r_\sigma(T^n) \leq \|T^n\|$. Therefore, for every n,

$$r_\sigma(T) = \sqrt[n]{r_\sigma(T^n)} \leq \sqrt[n]{\|T^n\|}.$$

Hence,

$$r_\sigma(T) \leq \underline{\lim}_{n\to\infty} \sqrt[n]{\|T^n\|} \leq \overline{\lim}_{n\to\infty} \sqrt[n]{\|T^n\|}.$$

The Spectral Radius Theorem (Cont'd)

- Claim: $\overline{\lim}_{n\to\infty}\sqrt[n]{\|T^n\|}\le r_\sigma(T)$.

  A power series $\sum c_n\kappa^n$ converges absolutely for $|\kappa|<r$ with radius of convergence r given by the well-known **Hadamard formula**

$$\frac{1}{r}=\overline{\lim}_{n\to\infty}\sqrt[n]{|c_n|}.$$

  Setting $\kappa=\frac{1}{\lambda}$, we get

$$R_\lambda=-\kappa\sum_{n=0}^{\infty}T^n\kappa^n.$$

  Then, writing $|c_n|=\|T^n\|$, we obtain

$$\left\|\sum_{n=0}^{\infty}T^n\kappa^n\right\|\le\sum_{n=0}^{\infty}\|T^n\||\kappa|^n=\sum_{n=0}^{\infty}|c_n||\kappa|^n.$$

The Spectral Radius Theorem (Cont'd)

- The Hadamard formula shows that we have absolute convergence for $|\kappa| < r$, hence for $|\lambda| = \frac{1}{|\kappa|} > \frac{1}{r} = \overline{\lim}_{n\to\infty} \sqrt[n]{\|T^n\|}$.

  We know that R_λ is locally holomorphic precisely on the resolvent set $\rho(T)$ in the complex λ-plane.

  To $\rho(T)$ there corresponds a set in the complex κ-plane, call it M.

  Then it is known from complex analysis that the radius of convergence r is the radius of the largest open circular disk about $\kappa = 0$ which lies entirely in M.

  Hence, $\frac{1}{r}$ is the radius of the smallest circle about $\lambda = 0$ in the λ-plane whose exterior lies entirely in $\rho(T)$.

  By definition, this means that $\frac{1}{r}$ is the spectral radius of T.

  Hence, $r_\sigma(T) = \frac{1}{r} = \overline{\lim}_{n\to\infty} \sqrt[n]{\|T^n\|}$.

Subsection 6

Banach Algebras

Algebras

- An **algebra** A over a field K is a vector space A over K, such that for each ordered pair of elements $x,y \in A$, a unique product

$$xy \in A$$

is defined, satisfying, for all $x,y,z \in A$ and all scalars α:

(1) $(xy)z = x(yz)$;
(2a) $x(y+z) = xy + xz$;
(2b) $(x+y)z = xz + yz$;
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$.

- If $K = \mathbb{R}$ or $\mathbb{C}$, then A is said to be **real** or **complex**, respectively.

Algebras With Additional Properties

- A is said to be **commutative** (or **abelian**) if the multiplication is commutative, that is, if for all $x, y \in A$,

  (4) $xy = yx$.
- A is called an **algebra with identity** if A contains an element e, such that for all $x \in A$,

  (5) $ex = xe = x$.
- The element e is called an **identity** of A.
- If A has an identity, the identity is unique.

Normed Algebra, Banach Algebra

Definition (Normed Algebra, Banach Algebra)

A **normed algebra** A is a normed space which is an algebra, such that for all $x, y \in A$,

(6) $\|xy\| \leq \|x\| \|y\|$;

and if A has an identity e,

(7) $\|e\| = 1$.

A **Banach algebra** is a normed algebra which is complete, considered as a normed space.

- Property (6) relates multiplication and norm.
- We have

$$\begin{aligned} \|xy - x_0 y_0\| &= \|x(y - y_0) + (x - x_0) y_0\| \\ &\leq \|x\| \|y - y_0\| + \|x - x_0\| \|y_0\|. \end{aligned}$$

- So the product is a jointly continuous function of its factors.

Examples

- **Spaces $\mathbb{R}$ and $\mathbb{C}$**: The real line $\mathbb{R}$ and the complex plane $\mathbb{C}$ are commutative Banach algebras with identity $e=1$.
- **Space $C[a,b]$**: The space $C[a,b]$ is a commutative Banach algebra with identity ($e=1$), the product xy being defined as usual:

  $$(xy)(t)=x(t)y(t), \quad \text{for all } t\in[a,b].$$

  The subspace of $C[a,b]$ consisting of all polynomials is a commutative normed algebra with identity ($e=1$).
- **Matrices**: The vector space X of all complex $n\times n$ matrices ($n>1$, fixed) is a non-commutative algebra with identity I (the $n\times n$ unit matrix). By defining a norm on X, we obtain a Banach algebra.

Bounded Linear Operators

- **Space** $B(X,X)$: The Banach space $B(X,X)$ of all bounded linear operators on a complex Banach space $X \neq \{0\}$ is a Banach algebra.
  - The identity is I (the identity operator on X);
  - The multiplication is composition of operators, by definition.
- Relation (6) is
$$\|T_1 T_2\| \leq \|T_1\| \|T_2\|,$$
verified previously.
- $B(X,X)$ is not commutative, unless $\dim X = 1$.

Invertibility

- Let A be an algebra with identity.
- An $x \in A$ is said to be **invertible** if it has an inverse in A, i.e., if A contains an element, written x^{-1}, such that

$$x^{-1}x = xx^{-1} = e.$$

- If x is invertible, the inverse is unique.

  Suppose y and z are both inverses of x.

  Then, by definition, $yx = e = xz$.

  So we get

$$y = ye = y(xz) = (yx)z = ez = z.$$

Resolvent Set and Spectrum

Definition (Resolvent Set, Spectrum)

Let A be a complex Banach algebra with identity.

- The **resolvent set** $\rho(x)$ of an $x \in A$ is the set of all λ in the complex plane such that $x - \lambda e$ is invertible.
- The **spectrum** $\sigma(x)$ of x is the complement of $\rho(x)$ in the complex plane. Thus, $\sigma(x) = \mathbb{C} - \rho(x)$.
- Any $\lambda \in \sigma(x)$ is called a **spectral value** of x.

- Hence, the spectral values of $x \in A$ are those λ for which $x - \lambda e$ is not invertible.

Resolvent Set and Spectrum

Proposition

If X is a complex Banach space, then $B(X,X)$ is a Banach algebra. Then, the resolvent set of the operator $T \in B(X,X)$ agrees with its resolvent set as an element of the Banach algebra.

- Let $T \in B(X,X)$ and λ in the resolvent set $\rho(T)$. Then, by the present definition, $R_\lambda(T) = (T - \lambda I)^{-1}$ exists and is an element of $B(X,X)$. I.e., $R_\lambda(T)$ is a bounded linear operator defined on X. Hence, $\lambda \in \rho(T)$, with $\rho(T)$ as defined previously.

  Conversely, suppose that $\lambda \in \rho(T)$, with $\rho(T)$ defined as before. Then $R_\lambda(T)$ exists and is linear, bounded and defined on a dense subset of X. But, since T is bounded, we get that $R_\lambda(T)$ is defined on all of X. Hence $\lambda \in \rho(T)$, with $\rho(T)$ as defined presently.

Subsection 7

Further Properties of Banach Algebras

The Inverse Theorem

Theorem (Inverse)

Let A be a complex Banach algebra with identity e. If $x \in A$ satisfies $\|x\| < 1$, then $e - x$ is invertible, and

$$(e-x)^{-1} = e + \sum_{j=1}^{\infty} x^j.$$

- We have $\|x^j\| \leq \|x\|^j$. So $\sum \|x^j\|$ converges, since $\|x\| < 1$.
  Hence, the series in the formula converges absolutely.
  Since A is complete, the series converges.
  Let s denote its sum. We show that $s = (e-x)^{-1}$.
  $$(e-x)(e+x+\cdots+x^n) = (e+x+\cdots+x^n)(e-x) = e - x^{n+1}.$$
  We now let $n \to \infty$. Since $\|x\| < 1$, $x^{n+1} \to 0$.
  By continuity of multiplication, $(e-x)s = s(e-x) = e$.
  Hence, $s = (e-x)^{-1}$.

The Group of Invertible Elements

- Let A be a complex Banach algebra A with identity e
- Consider the subset G of all invertible elements of A.

  Claim: G is a group.

  $e \in G$.

  Suppose $x \in G$. Then x^{-1} exists and has an inverse $(x^{-1})^{-1} = x$. So x^{-1} is in G.

  Finally, suppose $x, y \in G$. Then $y^{-1}x^{-1}$ is the inverse of xy.

  $$(xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = xex^{-1} = e.$$

  Similarly, $(y^{-1}x^{-1})(xy) = e$.

  So $xy \in G$.

The Invertible Elements Theorem

Theorem (Invertible Elements)

Let A be a complex Banach algebra with identity. Then the set G of all invertible elements of A is an open subset of A. Hence, the subset $M = A - G$ of all non-invertible elements of A is closed.

- Let $x_0 \in G$. We have to show that every $x \in A$ sufficiently close to x_0, say, $\|x - x_0\| < \frac{1}{\|x_0^{-1}\|}$, belongs to G. Let $y = x_0^{-1}x$ and $z = e - y$. Then, we obtain

$$\begin{array}{rcl} \|z\| & = & \|-z\| = \|y - e\| = \|x_0^{-1}x - x_0^{-1}x_0\| \\ & = & \|x_0^{-1}(x - x_0)\| \leq \|x_0^{-1}\| \|x - x_0\| < 1. \end{array}$$

  Thus $\|z\| < 1$. So $e - z$ is invertible by the Inverse Theorem. Hence $e - z = y \in G$. But G is a group. So $x = x_0 x_0^{-1} x = x_0 y \in G$.

  Since $x_0 \in G$ was arbitrary, this proves that G is open.

The Spectral Radius

- Define the **spectral radius** $r_\sigma(x)$ of an $x \in A$ by

$$r_\sigma(x) = \sup_{\lambda \in \sigma(x)} |\lambda|.$$

Theorem (Spectrum)

Let A be a complex Banach algebra with identity e. Then for any $x \in A$, the spectrum $\sigma(x)$ is compact, and the spectral radius satisfies

$$r_\sigma(x) \leq \|x\|.$$

- Suppose $|\lambda| > \|x\|$. Then $\|\lambda^{-1}x\| < 1$.
  So $e - \lambda^{-1}x$ is invertible.
  Hence, $-\lambda(e - \lambda^{-1}x) = x - \lambda e$ is invertible also.
  So we have $\lambda \in \rho(x)$. Hence $\sigma(x)$ is bounded.

The Spectral Radius (Cont'd)

- Claim: $\sigma(x)$ is closed, since $\rho(x) = \mathbb{C} - \sigma(x)$ is open.

  If $\lambda_0 \in \rho(x)$, then $x - \lambda_0 e$ is invertible. Thus, there is a neighborhood $N \subseteq A$ of $x - \lambda_0 e$ consisting wholly of invertible elements.

  Now for a fixed x, the mapping $\lambda \mapsto x - \lambda e$ is continuous.

  Hence, all $x - \lambda e$, with λ close to λ_0, say,

  $$|\lambda - \lambda_0| < \delta, \text{ with } \delta > 0,$$

  lie in N. So these $x - \lambda e$ are invertible.

  Thus, the corresponding λ belong to $\rho(x)$.

  But $\lambda_0 \in \rho(x)$ was arbitrary.

  So $\rho(x)$ is open. Hence, $\sigma(x) = \mathbb{C} - \rho(x)$ is closed.

- The theorem shows that $\rho(x) \neq \emptyset$.

Nonemptiness of the Spectrum

Theorem (Spectrum)

Let A be a complex Banach algebra with identity e. Then $\sigma(x) \neq \emptyset$.

- Let $\lambda, \mu \in \rho(x)$. We write

$$\begin{aligned} v(\lambda) &= (x-\lambda e)^{-1}; \\ w &= (\mu-\lambda)v(\lambda). \end{aligned}$$

Then

$$\begin{aligned} x-\mu e &= x-\lambda e-(\mu-\lambda)e \\ &= (x-\lambda e)e-(\mu-\lambda)(x-\lambda e)(x-\lambda e)^{-1} \\ &= (x-\lambda e)(e-w). \end{aligned}$$

Taking inverses, we have $v(\mu)=(e-w)^{-1}v(\lambda)$.
Suppose μ is so close to λ that $\|w\|<\frac{1}{2}$. Then

$$\|(e-w)^{-1}-e-w\| = \left\|\sum_{j=2}^{\infty} w^j\right\| \leq \sum_{j=2}^{\infty}\|w\|^j = \frac{\|w\|^2}{1-\|w\|} \leq 2\|w\|^2.$$

Nonemptiness of the Spectrum (Cont'd)

- We showed $v(\mu) = (e - w)^{-1} v(\lambda)$ and $\|(e - w)^{-1} - e - w\| \leq 2\|w\|^2$. From this, we get

$$\begin{array}{rcl} \|v(\mu) - v(\lambda) - (\mu - \lambda) v(\lambda)^2\| & = & \|(e - w)^{-1} v(\lambda) - (e + w) v(\lambda)\| \\ & \leq & \|v(\lambda)\| \|(e - w)^{-1} - (e + w)\| \\ & \leq & 2\|w\|^2 \|v(\lambda)\|. \end{array}$$

  $\|w\|^2$ contains a factor $|\mu - \lambda|^2$. Therefore,

$$\frac{\|w\|^2}{|\mu - \lambda|} \xrightarrow{\mu \to \lambda} 0.$$

  Hence, dividing the inequality by $|\mu - \lambda|$ and letting $\mu \to \lambda$,

$$\frac{1}{\mu - \lambda} [v(\mu) - v(\lambda)] \to v(\lambda)^2.$$

Nonemptiness of the Spectrum (Cont'd)

- Let $f \in A'$, where A' is the dual of A, considered as a Banach space.

  We define $h : \rho(x) \to \mathbb{C}$ by
  $$h(\lambda) = f(v(\lambda)).$$
  Since f is continuous, so is h.

  Applying f to the previous limit, we obtain
  $$\lim_{\mu \to \lambda} \frac{h(\mu) - h(\lambda)}{\mu - \lambda} = f(v(\lambda)^2).$$
  This shows that h is holomorphic at every point of $\rho(x)$.

  If $\sigma(x)$ were empty, then $\rho(x) = \mathbb{C}$.

  So h would be an entire function.

Nonemptiness of the Spectrum (Cont'd)

- Now we have
  - $v(\lambda) = -\lambda^{-1}(e - \lambda^{-1}x)^{-1}$;
  - $(e - \lambda^{-1}x)^{-1} \xrightarrow{|\lambda| \to \infty} e^{-1} = e$.

  So we obtain

$$|h(\lambda)| = |f(v(\lambda))| \leq \|f\| \|v(\lambda)\| = \|f\| \frac{1}{|\lambda|} \left\| (e - \frac{1}{\lambda}x)^{-1} \right\| \xrightarrow{|\lambda| \to \infty} 0.$$

  This shows that h would be bounded on $\mathbb{C}$.

  Hence, by Liouville's Theorem, it is a constant.

  So it is zero by the preceding relation.

  Since $f \in A'$ was arbitrary, $h(\lambda) = f(v(\lambda)) = 0$ implies $v(\lambda) = 0$.

  This is impossible since it gives

$$\|e\| = \|(x - \lambda e)v(\lambda)\| = \|0\| = 0.$$

  Hence, $\sigma(x) = \emptyset$ cannot hold.

Supplying an Algebra with an Identity

- The existence of an identity e is necessary.
- If A has no identity, we can supply A with an identity.

  Let $\widetilde{A}$ be the set of all ordered pairs (x,α), where $x \in A$ and α is a scalar. Define

$$\begin{aligned} (x,\alpha)+(y,\beta) &= (x+y,\alpha+\beta) \\ \beta(x,\alpha) &= (\beta x,\beta\alpha) \\ (x,\alpha)(y,\beta) &= (xy+\alpha y+\beta x,\alpha\beta) \\ \|(x,\alpha)\| &= \|x\|+|\alpha| \\ \widetilde{e} &= (0,1). \end{aligned}$$

  Then $\widetilde{A}$ is a Banach algebra with identity $\widetilde{e}$.
- The mapping $x \mapsto (x,0)$ is an isomorphism of A onto a subspace of $\widetilde{A}$, both regarded as normed spaces.
- This subspace has codimension 1. Identifying x with $(x,0)$, then $\widetilde{A}$ is A plus the one-dimensional space generated by $\widetilde{e}$.