Introduction to Spectral Theory of Linear Operators

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Subsection 1

Compact Linear Operators on Normed Spaces

Compact Linear Operators

Definition (Compact Linear Operator)

Let X, Y be normed spaces. An operator $T : X \to Y$ is called a **compact linear operator** (or **completely continuous linear operator**) if:

- T is linear;
- For every bounded subset M of X, the image $T(M)$ is **relatively compact**, i.e., the closure $\overline{T(M)}$ is compact.

- The theory of compact linear operators emerged from the theory of **integral equations** of the form

$$(T - \lambda I)x(s) = y(s), \text{ where } Tx(s) = \int_a^b k(s,t)x(t)dt.$$

  In this equation:
  - $\lambda \in \mathbb{C}$ is a parameter;
  - y and the **kernel** k are given functions (subject to certain conditions);
  - x is the unknown function.

The Continuity Lemma

Lemma (Continuity)

Let X and Y be normed spaces. Then:

(a) Every compact linear operator $T: X \to Y$ is bounded, hence continuous.

(b) If $\dim X = \infty$, the identity operator $I: X \to X$ (which is continuous) is not compact.

(a) The unit sphere $U = \{x \in X : \|x\| = 1\}$ is bounded.
Since T is compact, $\overline{T(U)}$ is compact.
By the Compactness Lemma, $\overline{T(U)}$ is bounded.
So $\sup_{\|x\|=1} \|Tx\| < \infty$. Hence, T is bounded and, so, continuous.

(b) Of course, the closed unit ball $M = \{x \in X : \|x\| \le 1\}$ is bounded.
If $\dim X = \infty$, then M cannot be compact.
Thus, $I(M) = M = \overline{M}$ is not relatively compact.

Compactness Criterion

Theorem (Compactness Criterion)

Let X and Y be normed spaces and $T : X \to Y$ a linear operator. Then T is compact if and only if it maps every bounded sequence (x_n) in X onto a sequence (Tx_n) in Y which has a convergent subsequence.

- If T is compact and (x_n) is bounded.

  Then the closure of (Tx_n) in Y is compact.

  Thus, (Tx_n) contains a convergent subsequence.

  Conversely, assume that every bounded sequence (x_n) contains a subsequence (x_{n_k}), such that (Tx_{n_k}) converges in Y.

Compactness Criterion (Cont'd)

- Consider any bounded subset $B \subseteq X$.

  Let (y_n) be any sequence in $T(B)$.

  Then $y_n = Tx_n$, for some $x_n \in B$.

  Moreover, (x_n) is bounded since B is bounded.

  By assumption, (Tx_n) contains a convergent subsequence.

  Hence, $T(B)$ is compact because (y_n) in $T(B)$ was arbitrary.

  By definition, this shows that T is compact.
- By the Compactness Criterion, if $T_1, T_2 : X \to Y$ are two compact linear operators:
  - The sum $T_1 + T_2$ is compact;
  - The product αT_1 is compact, α any scalar.

  So the compact linear operators from X into Y form a vector space.

Finite Dimensionality of Domain or Range

Theorem (Finite Dimensionality of Domain or Range)

Let X and Y be normed spaces and $T : X \to Y$ a linear operator. Then:

(a) If T is bounded and $\dim T(X) < \infty$, the operator T is compact.

(b) If $\dim X < \infty$, the operator T is compact.

(a) Let (x_n) be any bounded sequence in X.
The inequality $\|Tx_n\| \le \|T\| \|x_n\|$ shows that (Tx_n) is bounded.
Since $\dim T(X) < \infty$, (Tx_n) is relatively compact.
It follows that (Tx_n) has a convergent subsequence.
By the Compactness Criterion, the operator T is compact.

(b) Follows from (a) by noting that $\dim X < \infty$ implies boundedness of T and $\dim T(X) \le \dim X$.

- An operator $T \in B(X, Y)$, with $\dim T(X) < \infty$, is often called an **operator of finite rank**.

Sequence of Compact Linear Operators

Theorem (Sequence of Compact Linear Operators)

Let (T_n) be a sequence of compact linear operators from a normed space X into a Banach space Y. If (T_n) is uniformly operator convergent, say, $\|T_n - T\| \to 0$, then the limit operator T is compact.

- Using a "diagonal method", we show that, for any bounded sequence (x_m) in X, the image (Tx_m) has a convergent subsequence.
  The conclusion then follows by the Compactness Criterion.
  - Since T_1 is compact, (x_m) has a subsequence $(x_{1,m})$, such that $(T_1 x_{1,m})$ is Cauchy;
  - Since T_2 is compact, $(x_{1,m})$ has a subsequence $(x_{2,m})$ such that $(T_2 x_{2,m})$ is Cauchy.
    ...

  The "diagonal sequence" $(y_m) = (x_{m,m})$ is a subsequence of (x_m), such that, for every fixed n, the sequence $(T_n y_m)_{m\in\mathbb{N}}$ is Cauchy.
  (x_m) is bounded, say, $\|x_m\| \le c$, for all m. Hence $\|y_m\| \le c$, for all m.

Sequence of Compact Linear Operators (Cont'd)

- Let $\varepsilon > 0$. Since $T_m \to T$, there is an $n = p$, such that

$$\|T - T_p\| < \frac{\varepsilon}{3c}.$$

Since $(T_p y_m)_{m\in\mathbb{N}}$ is Cauchy, there is an N, such that

$$\|T_p y_j - T_p y_k\| < \frac{\varepsilon}{3}, \text{ for all } j,k > N.$$

Hence, we obtain for $j,k > N$,

$$\begin{array}{rcl} \|Ty_j - Ty_k\| & \leq & \|Ty_j - T_p y_j\| + \|T_p y_j - T_p y_k\| + \|T_p y_k - Ty_k\| \\ & \leq & \|T - T_p\| \|y_j\| + \frac{\varepsilon}{3} + \|T_p - T\| \|y_k\| \\ & < & \frac{\varepsilon}{3c} c + \frac{\varepsilon}{3} + \frac{\varepsilon}{3c} c = \varepsilon. \end{array}$$

This shows that (Ty_m) is Cauchy. Since Y is complete, it converges. But (y_m) is a subsequence of the arbitrary bounded sequence (x_m). So, by the Compactness Criterion, T is compact.

Necessity of Uniform Operator Convergence

- The preceding theorem becomes false if we replace uniform operator convergence by strong operator convergence $\|T_n x - T_x\| \to 0$.

  Consider $T_n : \ell^2 \to \ell^2$ defined, for all $x = (\xi_j) \in \ell^2$, by

  $$T_n x = (\xi_1, \ldots, \xi_n, 0, 0, \ldots).$$

  Since T_n is linear and bounded, T_n is compact.

  Clearly, for all $x = (\xi_j) \in \ell^2$,

  $$T_n x \to x = Ix.$$

  However, I is not compact, since $\dim \ell^2 = \infty$.

Example

- Use the theorem to prove compactness of $T:\ell^2\to\ell^2$ defined by $y=(\eta_j)=Tx$, where $\eta_j=\frac{\xi_j}{j}$, for $j=1,2,\ldots$.

  T is linear. If $x=(\xi_j)\in\ell^2$, then $y=(\eta_j)\in\ell^2$. Let $T_n:\ell^2\to\ell^2$ be defined by

$$T_nx=\left(\xi_1,\frac{\xi_3}{2},\frac{\xi_3}{3},\ldots,\frac{\xi_n}{n},0,0,\ldots\right).$$

  T_n is linear and bounded, and is compact. Furthermore,

$$\begin{array}{rcl}\|(T-T_n)x\|^2 & = & \sum_{j=n+1}^{\infty}|\eta_j|^2=\sum_{j=n+1}^{\infty}\frac{1}{j^2}|\xi_j|^2\\ & \leq & \frac{1}{(n+1)^2}\sum_{j=n+1}^{\infty}|\xi_j|^2\leq\frac{\|x\|^2}{(n+1)^2}.\end{array}$$

  Taking the supremum over all x of norm 1, we get $\|T-T_n\|\leq\frac{1}{n+1}$. Hence, $T_n\to T$. So T is compact by the theorem.

The Weak Convergence Theorem

Theorem (Weak Convergence)

Let X and Y be normed spaces and $T : X \to Y$ a compact linear operator. Suppose that (x_n) in X is weakly convergent, say, $x_n \xrightarrow{w} x$. Then (Tx_n) is strongly convergent in Y and has the limit $y = Tx$.

- We write $y_n = Tx_n$ and $y = Tx$.

  Claim: $y_n \xrightarrow{w} y$.

  Let g be any bounded linear functional on Y. We define a functional f on X by setting $f(z) = g(Tz)$, for all $z \in X$. f is linear. f is bounded. Since T is compact, it is bounded. Moreover,

  $$|f(z)| = |g(Tz)| \leq \|g\| \|Tz\| \leq \|g\| \|T\| \|z\|.$$

  By definition, $x_n \xrightarrow{w} x$ implies $f(x_n) \to f(x)$.

  Hence by definition, $g(Tx_n) \to g(Tx)$. I.e., $g(y_n) \to g(y)$.

The Weak Convergence Theorem (Cont'd)

Claim: $y_n \to y$.

Assume this does not hold.

Then (y_n) has a subsequence (y_{n_k}), such that, for some $\eta > 0$,

$$\|y_{n_k} - y\| \geq \eta.$$

Since (x_n) is weakly convergent, (x_n) is bounded.

So (x_{n_k}) is also bounded.

Compactness of T implies that (Tx_{n_k}) has a convergent subsequence, say, $(\widetilde{y}_j)$. Let $\widetilde{y}_j \to \widetilde{y}$.

A fortiori, $\widetilde{y}_j \xrightarrow{w} \widetilde{y}$. Hence, $\widetilde{y} = y$.

Consequently, $\|\widetilde{y}_j - y\| \to 0$.

But $\|\widetilde{y}_j - y\| \geq \eta > 0$, a contradiction.

Subsection 2

Further Properties of Compact Linear Operators

Total Boundedness

Definition (ε-net, Total Boundedness)

Let B be a subset of a metric space X and let $\varepsilon > 0$ be given.

- A set $M_\varepsilon \subseteq X$ is called an **ε-net for B** if, for every point $z \in B$, there is a point of M_ε at a distance from z less than ε.
- The set B is said to be **totally bounded** if, for every $\varepsilon > 0$, there is a *finite* ε-net $M_\varepsilon \subseteq X$ for B, where "finite" means that M_ε is a finite set (that is, consists of finitely many points).

- Consequently, total boundedness of B means that:

  For every given $\varepsilon > 0$, the set B is contained in the union of finitely many open balls of radius ε.

The Total Boundedness Lemma

Lemma (Total Boundedness)

Let B be a subset of a metric space X. Then:

(a) If B is relatively compact, B is totally bounded.

(b) If B is totally bounded and X is complete, B is relatively compact.

(c) If B is totally bounded, for every $\varepsilon > 0$ it has a finite ε-net $M_\varepsilon \subseteq B$.

(d) If B is totally bounded, B is separable.

(a) Assume that B is relatively compact.
We show that, for any $\varepsilon_0 > 0$, there exists a finite ε_0-net for B.
If $B = \emptyset$, then $\emptyset$ is an ε_0-net for B.
Suppose $B \neq \emptyset$. Pick any $x_1 \in B$.
If $d(x_1, z) < \varepsilon_0$, for all $z \in B$, then $\{x_1\}$ is an ε_0-net for B.
Otherwise, let $x_2 \in B$ be such that $d(x_1, x_2) \geq \varepsilon_0$.
If, for all $z \in B$, $d(x_j, z) < \varepsilon_0$, $j = 1$ or 2, then $\{x_1, x_2\}$ is an ε_0-net for B.

The Total Boundedness Lemma Part (a) (Cont'd)

- Otherwise, let $z = x_3 \in B$ be a point not satisfying the inequality.

  If, for all $z \in B$, $d(x_j, z) < \varepsilon_0$, $j = 1,2$ or 3, then $\{x_1, x_2, x_3\}$ is an ε_0-net for B. Otherwise we continue by selecting an $x_4 \in B$, etc.

  We assert the existence of a positive integer n, such that the set $\{x_1, \ldots, x_n\}$ obtained after n such steps is an ε_0-net for B.

  If there were no such n, our construction would yield a sequence (x_j) satisfying $d(x_j, x_k) \geq \varepsilon_0$, for $j \neq k$.

  Obviously, (x_j) could not have a subsequence which is Cauchy.

  Hence, (x_j) could not have a subsequence which converges in X.

  Since, by construction, (x_j) lies in B, this contradicts the relative compactness of B.

  Hence, there must be a finite ε_0-net for B.

  Since $\varepsilon_0 > 0$ was arbitrary, B is totally bounded.

The Total Boundedness Lemma Part (b)

(b) Let B be totally bounded and X complete.

Let (x_n) be an arbitrary sequence in B.

We show that (x_n) has a subsequence which converges in X.

By assumption, B has a finite ε-net for $\varepsilon = 1$.

Hence, B is contained in the union of finitely many open balls of radius 1.

From these balls we can pick a ball B_1 which contains infinitely many terms of (x_n) (counting repetitions).

Let $(x_{1,n})$ be the subsequence of (x_n) which lies in B_1.

Similarly, by assumption, B is also contained in the union of finitely many balls of radius $\varepsilon = \frac{1}{2}$.

From these balls, we can pick a ball B_2 which contains a subsequence $(x_{2,n})$ of the subsequence $(x_{1,n})$.

Inductively, choose $\varepsilon = \frac{1}{3}, \frac{1}{4}, \ldots$ and set $y_n = x_{n,n}$.

The Total Boundedness Lemma Part (b) (Cont'd)

- Now, for every given $\varepsilon > 0$, there is an N (depending on ε), such that all y_n with $n > N$ lie in a ball of radius ε.

  Hence (y_n) is Cauchy.

  Since X is complete, it converges in X, say, $y_n \to y \in X$.

  Also, $y_n \in B$ implies $y \in \overline{B}$.

  By the definition of the closure, for every sequence (z_n) in $\overline{B}$, there is a sequence (x_n) in B which satisfies $d(x_n, z_n) \le \frac{1}{n}$, for every n.

  Since (x_n) is in B, it has a subsequence which converges in $\overline{B}$, as we have just shown.

  Hence, since $d(x_n, z_n) \le \frac{1}{n}$, (z_n) also has a subsequence which converges in $\overline{B}$.

  So $\overline{B}$ is compact and B is relatively compact.

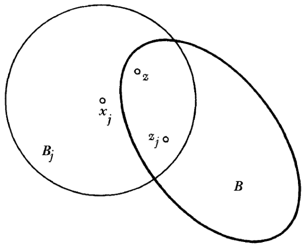
The Total Boundedness Lemma Part (c)

(c) If B is totally bounded, for every $\varepsilon > 0$, it has a finite ε-net $M_\varepsilon \subseteq B$.
The case $B = \emptyset$ is obvious.
Let $B \neq \emptyset$. By assumption, for given $\varepsilon > 0$, there is a finite ε_1-net $M_{\varepsilon_1} \subseteq X$ for B, where $\varepsilon_1 = \frac{\varepsilon}{2}$. Hence B is contained in the union of finitely many balls of radius ε_1 with the elements of M_{ε_1} as centers.

Let $B_1, \ldots, B_n$ be those balls which intersect B, and let $x_1, \ldots, x_n$ be their centers. We select a point $z_j \in B \cap B_j$.
We claim that

$$M_\varepsilon = \{z_1, \ldots, z_n\} \subseteq B$$

is an ε-net for B.



For every $z \in B$, there is a B_j containing z. Moreover,

$$d(z, z_j) \leq d(z, x_j) + d(x_j, z_j) < \varepsilon_1 + \varepsilon_1 = \varepsilon.$$

The Total Boundedness Lemma Part (d)

(d) If B is totally bounded, B is separable.

Suppose B is totally bounded.

Then, by Part (c), the set B contains a finite ε-net $M_{1/n}$ for itself, where $\varepsilon = \varepsilon_n = \frac{1}{n}$, $n = 1, 2, \ldots$.

The union M of all these nets is countable.

Moreover, M is dense in B.

In fact, for any given $\varepsilon > 0$, there is an n, such that $\frac{1}{n} < \varepsilon$.

Hence, for any $z \in B$, there is an $a \in M_{1/n} \subseteq M$, such that $d(z, a) < \varepsilon$.

This proves that B is separable.

Total Boundedness and Boundedness

- Total boundedness implies boundedness.
- The converse does not generally hold.

  Consider the metric space ℓ^2.

  Let U be the closed unit ball

  $$U = \{x : \|x\| \leq 1\} \subseteq \ell^2.$$

  - U is bounded.
  - U is not totally bounded.

    ℓ^2 is infinite dimensional and complete.

    So U is not compact.

    Hence, it is not totally bounded.

Separability of Range

Theorem (Separability of Range)

The range $\mathscr{R}(T)$ of a compact linear operator $T : X \to Y$ is separable, where X and Y are normed spaces.

- Consider the ball $B_n = B(0; n) \subseteq X$.

  Since T is compact, the image $C_n = T(B_n)$ is relatively compact.

  By Parts (a) and (d) of the Lemma, C_n is separable.

  The norm of any $x \in X$ is finite. So, for any x, there exists n sufficiently large, such that $\|x\| < n$. Hence, $x \in B_n$.

  Consequently, $X = \bigcup_{n=1}^{\infty} B_n$ and $T(X) = \bigcup_{n=1}^{\infty} T(B_n) = \bigcup_{n=1}^{\infty} C_n$.

  Since C_n is separable, it has a countable dense subset D_n.

  Moreover, the union $D = \bigcup_{n=1}^{\infty} D_n$ is countable.

  But $T(X) = \bigcup_{n=1}^{\infty} C_n$. So D is dense in the range $\mathscr{R}(T) = T(X)$.

Compact Extension

Theorem (Compact Extension)

A compact linear operator $T: X \to Y$ from a normed space X into a Banach space Y has a compact linear extension $\widetilde{T}: \widehat{X} \to Y$, where $\widehat{X}$ is the completion of X.

- We may regard X as a subspace of $\widehat{X}$.

  Since T is bounded, it has a bounded linear extension $\widetilde{T}: \widehat{X} \to Y$.

  We show that compactness of T implies $\widetilde{T}$ is also compact.

  Let $(\widehat{x}_n)$ be an arbitrary bounded sequence in $\widehat{X}$.

  We show that $(\widetilde{T}\widehat{x}_n)$ has a convergent subsequence.

  X is dense in $\widehat{X}$.

  So there is a sequence (x_n) in X, such that $\widehat{x}_n - x_n \to 0$.

  Clearly, (x_n) is bounded, too.

Compact Extension (Cont'd)

- Since T is compact, (Tx_n) has a convergent subsequence (Tx_{n_k}).

  Suppose $Tx_{n_k} \to y_0 \in Y$.

  Now $\widehat{x}_n - x_n \to 0$ implies $\widehat{x}_{n_k} - x_{n_k} \to 0$.

  Since $\widehat{T}$ is linear and bounded, it is continuous. Thus,

  $$\widetilde{T}\widehat{x}_{n_k} - Tx_{n_k} = \widetilde{T}(\widehat{x}_{n_k} - x_{n_k}) \to \widetilde{T}0 = 0.$$

  Since $Tx_{n_k} \to y_0 \in Y$, $\widetilde{T}\widehat{x}_{n_k} \to y_0$.

  We showed that the arbitrary bounded sequence $(\widehat{x}_n)$ has a subsequence $(\widehat{x}_{n_k})$, such that $(\widetilde{T}\widehat{x}_{n_k})$ converges. So $\widetilde{T}$ is compact.

The Adjoint Operator Theorem

- The adjoint operator of a compact linear operator is itself compact.

Theorem (Adjoint Operator)

Let $T : X \to Y$ be a linear operator. If T is compact, so is its adjoint operator $T^\times : Y' \to X'$, where X and Y are normed spaces and X' and Y' the dual spaces of X and Y.

- Let B be a subset of Y' which is bounded, say $\|g\| \leq c$, for all $g \in B$.

  We show that the image $T^\times(B) \subseteq X'$ is totally bounded.

  Since X' is complete, by Part (b) of the Total Boundedness Lemma, it will then follow that $T^\times(B)$ is relatively compact.

The Adjoint Operator Theorem (Cont'd)

- We must show, for any fixed $\varepsilon_0 > 0$, $T^\times(B)$ has a finite ε_0-net.
  Since T is compact, the image $T(U)$ of the unit ball $U = \{x \in X : \|x\| \leq 1\}$ is relatively compact.
  Hence $T(U)$ is totally bounded.
  Thus, there is a finite ε_1-net $M \subseteq T(U)$ for $T(U)$, where $\varepsilon_1 = \frac{\varepsilon_0}{4c}$.
  This means that U contains points $x_1, \ldots, x_n$, such that, for each $x \in U$, there exists some j, such that $\|Tx - Tx_j\| < \frac{\varepsilon_0}{4c}$.
  We define a linear operator $A : Y' \to \mathbb{R}^n$ by

$$Ag = (g(Tx_1), g(Tx_2), \ldots, g(Tx_n)).$$

  g is bounded by assumption.
  T is bounded by the Continuity Lemma.
  Hence, A is compact by the Finite Dimensionality Lemma.
  Since B is bounded, $A(B)$ is relatively compact.
  Hence, $A(B)$ is totally bounded.

The Adjoint Operator Theorem (Cont'd)

- Thus, $A(B)$ contains a finite ε_2-net $\{Ag_1,\ldots,Ag_m\}$ for itself, where $\varepsilon_2 = \frac{\varepsilon_0}{4}$. This means that, for each $g \in B$, there exists k, such that

$$\|Ag - Ag_k\|_0 < \frac{\varepsilon_0}{4},$$

where $\|\cdot\|_0$ is the norm on $\mathbb{R}^n$.
We show that $\{T^\times g_1,\ldots,T^\times g_m\}$ is the desired ε_0-net for $T^\times(B)$.
Since $\|Ag - Ag_k\|_0 < \frac{\varepsilon_0}{4}$, for all j and all $g \in B$, there is a k, such that

$$|g(Tx_j) - g_k(Tx_j)|^2 \le \sum_{j=1}^n |g(Tx_j) - g_k(Tx_j)|^2 = \|A(g - g_k)\|_0^2 < (\frac{\varepsilon_0}{4})^2.$$

Let $x \in U$ be arbitrary. Then, there is a j, for which $\|Tx - Tx_j\| < \frac{\varepsilon_0}{4c}$.
Let $g \in B$. Then, there is a k, such that

$$\|Ag - Ag_k\|_0 < \frac{\varepsilon_0}{4} \quad \text{and} \quad |g(Tx_j) - g_k(Tx_j)|^2 < (\frac{\varepsilon_0}{4})^2.$$

The Adjoint Operator Theorem (Conclusion)

- Thus,

$$\begin{array}{rcl} |g(Tx)-g_k(Tx)| & \leq & |g(Tx)-g(Tx_j)|+|g(Tx_j)-g_k(Tx_j)| \\ & & +|g_k(Tx_j)-g_k(Tx)| \\ & < & \|g\|\|Tx-Tx_j\|+\frac{\varepsilon_0}{4}+\|g_k\|\|Tx_j-Tx\| \\ & \leq & c\frac{\varepsilon_0}{4c}+\frac{\varepsilon_0}{4}+c\frac{\varepsilon_0}{4c}<\varepsilon_0. \end{array}$$

  Since this holds for every $x\in U$ and since by the definition of $T^\times$ we have $g(Tx)=(T^\times g)(x)$, etc., we finally obtain

$$\begin{array}{rcl} \|T^\times g-T^\times g_k\| & = & \sup_{\|x\|=1}|(T^\times(g-g_k))(x)| \\ & = & \sup_{\|x\|=1}|g(Tx)-g_k(Tx)|<\varepsilon_0. \end{array}$$

  This shows that $\{T^\times g_1,\ldots,T^\times g_m\}$ is an ε_0-net for $T^\times(B)$.
  Since $\varepsilon_0>0$ was arbitrary, $T^\times(B)$ is totally bounded.
  Hence, by the Total Boundedness Lemma, it is relatively compact.
  Since B was any bounded subset of Y', we get compactness of $T^\times$.

Subsection 3

Spectral Properties of Compact Linear Operators

The Eigenvalues Theorem

Theorem (Eigenvalues)

The set of eigenvalues of a compact linear operator $T : X \to X$ on a normed space X is countable (perhaps finite or even empty), and the only possible point of accumulation is $\lambda = 0$.

- It suffices to show, for all real $k > 0$, the set of all $\lambda \in \sigma_p(T)$, such that $|\lambda| \geq k$ is finite. Suppose not for some $k_0 > 0$.

  Then there is a sequence (λ_n) of infinitely many distinct eigenvalues, such that $|\lambda_n| \geq k_0$. Also $Tx_n = \lambda_n x_n$, for some $x_n \neq 0$.

  The set of all the x_n's is linearly independent.

  Let $M_n = \text{span}\{x_1, \ldots, x_n\}$.

  Then, every $x \in M_n$ has a unique representation

$$x = \alpha_1 x_1 + \cdots + \alpha_n x_n.$$

The Eigenvalues Theorem (Cont'd)

- Apply $T - \lambda_n I$ to get

$$(T - \lambda_n I)x = \alpha_1 (T - \lambda_n I)x_1 + \cdots + \alpha_n (T - \lambda_n I)x_n.$$

Use $Tx_j = \lambda_j x_j$ to get

$$(T - \lambda_n I)x = \alpha_1 (\lambda_1 - \lambda_n)x_1 + \cdots + \alpha_{n-1}(\lambda_{n-1} - \lambda_n)x_{n-1}.$$

Note that x_n no longer occurs.

So $(T - \lambda_n I)x \in M_{n-1}$, for all $x \in M_n$.

The M_n's are closed.

By Riesz's Lemma, there exists a sequence (y_n), such that:

- $y_n \in M_n$;
- $\|y_n\| = 1$;
- $\|y_n - x\| \geq \frac{1}{2}$, for all $x \in M_{n-1}$.

The Eigenvalues Theorem (Cont'd)

- We show that

$$\|Ty_n - Ty_m\| \geq \frac{1}{2}k_0, \quad n > m.$$

So (Ty_n) has no convergent subsequence because $k_0 > 0$.
This contradicts the compactness of T since (y_n) is bounded.
By adding and subtracting a term we can write $Ty_n - Ty_m = \lambda_n y_n - \widetilde{x}$, where $\widetilde{x} = \lambda_n y_n - Ty_n + Ty_m$.
Let $m < n$. We show that $\widetilde{x} \in M_{n-1}$.
Since $m \leq n-1$, we have

$$y_m \in M_m \subseteq M_{n-1} = \operatorname{span}\{x_1, \ldots, x_{n-1}\}.$$

Since $Tx_j = \lambda_j x_j$, $Ty_m \in M_{n-1}$.
Since $(T - \lambda_n I)x \in M_{n-1}$,

$$\lambda_n y_n - Ty_n = -(T - \lambda_n I)y_n \in M_{n-1}.$$

The Eigenvalues Theorem (Conclusion)

- We have $Ty_m \in M_{n-1}$ and $\lambda_n y_n - Ty_n \in M_{n-1}$.
  Together, $\widetilde{x} = \lambda_n y_n - Ty_n + Ty_m \in M_{n-1}$.
  Thus, also $x = \lambda_n^{-1}\widetilde{x} \in M_{n-1}$.
  Hence, since $|\lambda_n| \geq k_0$,
  $$\|\lambda_n y_n - \widetilde{x}\| = |\lambda_n| \|y_n - x\| \geq \frac{1}{2}|\lambda_n| \geq \frac{1}{2}k_0.$$
  We conclude $\|Ty_n - Ty_m\| \geq \frac{1}{2}k_0$.
  Hence the assumption that there are infinitely many eigenvalues satisfying $\|\lambda_n\| \geq k_0$, for some $k_0 > 0$ must be false.
- It follows that, if a compact linear operator on a normed space has infinitely many eigenvalues, we can arrange these eigenvalues in a sequence converging to zero.

Compactness of Product

Lemma (Compactness of Product)

Let $T : X \to X$ be a compact linear operator and $S : X \to X$ a bounded linear operator on a normed space X. Then TS and ST are compact.

- Let $B \subseteq X$ be any bounded set.

  Since S is a bounded operator, $S(B)$ is a bounded set.

  Since T is compact, the set $TS(B) = T(S(B))$ is relatively compact.

  Hence TS is a compact linear operator.

  We prove that ST is also compact.

  Let (x_n) be any bounded sequence in X.

  By a previous result, (Tx_n) has a convergent subsequence (Tx_{n_k}).

  Thus, since S is bounded, (STx_n) converges.

  Hence, ST is compact.

Null Space Theorem

Theorem (Null Space)

Let $T: X \to X$ be a compact linear operator on a normed space X. Then, for every $\lambda \neq 0$, the null space $\mathcal{N}(T_\lambda)$ of $T_\lambda = T - \lambda I$ is finite dimensional.

- We know that, if the closed unit ball in a normed space X is compact, then the space is finite dimensional.
  So we show that the closed unit ball M in $\mathcal{N}(T_\lambda)$ is compact.
  Let (x_n) be in M. Then (x_n) is bounded ($\|x_n\| \leq 1$).
  By a previous result, (Tx_n) has a convergent subsequence (Tx_n).
  Now $x_n \in M \subseteq \mathcal{N}(T_\lambda)$ implies $T_\lambda x_n = Tx_n - \lambda x_n = 0$.
  So, since $\lambda \neq 0$, $x_n = \lambda^{-1} T x_n$.
  Consequently, $(x_{n_k}) = (\lambda^{-1} T x_{n_k})$ also converges.
  The limit is in M, since M is closed.
  Hence M is compact because (x_n) was arbitrary in M.
  This proves $\dim \mathcal{N}(T_\lambda) < \infty$.

Null Spaces Corollary

Corollary (Null Spaces)

Let $T: X \to X$ be a compact linear operator on a normed space X. Then, for every $\lambda \neq 0$, $\dim \mathcal{N}(T_\lambda^n) < \infty$, $n = 1, 2, \ldots$, and

$$\{0\} = \mathcal{N}(T_\lambda^0) \subseteq \mathcal{N}(T_\lambda) \subseteq \mathcal{N}(T_\lambda^2) \subseteq \cdots .$$

- Since T_λ is linear, it maps 0 onto 0.
  Hence, $T_\lambda^n x = 0$ implies $T_\lambda^{n+1} x = 0$.
  This yields the second conclusion.

Null Spaces Corollary (Cont'd)

- We prove, next, $\dim \mathscr{N}(T_\lambda^n) < \infty$.
  By the binomial theorem,

$$\begin{aligned} T_\lambda^n &= (T - \lambda I)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} T^k (-\lambda)^{n-k} \\ &= (-\lambda)^n I + T \sum_{k=1}^{n} \binom{n}{k} T^{k-1} (-\lambda)^{n-k}. \end{aligned}$$

  This can be written

$$T_\lambda^n = W - \mu I,$$

  with:
  - $\mu = -(-\lambda)^n$;
  - $W = TS = ST$, where S denotes the sum on the right.

  T is compact. Since T is bounded, S is bounded, by a previous result.
  Hence, W is compact by a previous lemma.
  Now we obtain the result by applying the preceding theorem.

The Range Theorem

- Recall that for a bounded linear operator, the null space is always closed but the range need not be closed.

Theorem (Range)

Let $T : X \to X$ be a compact linear operator on a normed space X. Then, for every $\lambda \neq 0$, the range of $T_\lambda = T - \lambda I$ is closed.

- We assume that the range $T_\lambda(X)$ is not closed. We derive a contradiction by proceeding as follows:
  - (a) We consider a y in the closure of $T_\lambda(X)$ but not in $T_\lambda(X)$.
    We let $(T_\lambda x_n)$ be a sequence converging to y.
    We show that $x_n \notin \mathscr{N}(T_\lambda)$ but $\mathscr{N}(T_\lambda)$ contains a sequence (z_n), such that $\|x_n - z_n\| < 2\delta_n$, where δ_n is the distance from x_n to $\mathscr{N}(T_\lambda)$.
  - (b) We show that $a_n \to \infty$, where $a_n = \|x_n - z_n\|$.
  - (c) We obtain the anticipated contradiction by considering the sequence (w_n), where $w_n = a_n^{-1}(x_n - z_n)$.

The Range Theorem Part (a)

(a) Suppose that $T_\lambda(X)$ is not closed. Then there is a $y \in \overline{T_\lambda(X)}$, $y \notin T_\lambda(X)$ and a sequence (x_n) in X, such that $y_n = T_\lambda x_n \to y$.

Since $T_\lambda(X)$ is a vector space, $0 \in T_\lambda(X)$.

Since $y \notin T_\lambda(X)$, $y \neq 0$.

This implies $y_n \neq 0$ and $x_n \notin \mathcal{N}(T_\lambda)$, for all sufficiently large n.

Without loss of generality we may assume that this holds for all n.

Since $\mathcal{N}(T_\lambda)$ is closed, the distance δ_n from x_n to $\mathcal{N}(T_\lambda)$ is positive,

$$\delta_n = \inf_{z \in \mathcal{N}(T_\lambda)} \|x_n - z\| > 0.$$

By the definition of an infimum, there is a sequence (z_n) in $\mathcal{N}(T_\lambda)$, such that

$$a_n = \|x_n - z_n\| < 2\delta_n.$$

The Range Theorem Part (b)

(b) We show that $a_n = \|x_n - z_n\| \overset{n\to\infty}{\longrightarrow} \infty$. Suppose this does not hold.
Then $(x_n - z_n)$ has a bounded subsequence.
Since T is compact, $(T(x_n - z_n))$ has a convergent subsequence.
From $T_\lambda = T - \lambda I$ and $\lambda \neq 0$, we have $I = \lambda^{-1}(T - T_\lambda)$.
Since $z_n \in \mathcal{N}(T_\lambda)$, we have $T_\lambda z_n = 0$.
So we get

$$x_n - z_n = \frac{1}{\lambda}(T - T_\lambda)(x_n - z_n) = \frac{1}{\lambda}[T(x_n - z_n) - T_\lambda x_n].$$

$(T(x_n - z_n))$ has a convergent subsequence and $(T_\lambda x_n)$ converges.
Hence, $(x_n - z_n)$ has a convergent subsequence, say, $x_{n_k} - z_{n_k} \to v$.
Since T is compact, T is continuous. Thus, so is T_λ.
Hence, by a preceding theorem, $T_\lambda(x_{n_k} - z_{n_k}) \to T_\lambda v$.
Since $z_n \in \mathcal{N}(T_\lambda)$, $T_\lambda z_{n_k} = 0$.
So, since $y_n = T_\lambda x_n \to y$, we have $T_\lambda(x_{n_k} - z_{n_k}) = T_\lambda x_{n_k} \to y$.
Hence, $T_\lambda v = y$. Thus $y \in T_\lambda(X)$. This contradicts $y \notin T_\lambda(X)$.

The Range Theorem Part (c)

(c) In Part (b) it was shown that $a_n = \|x_n - z_n\|$ is divergent.

Set $w_n = \frac{1}{a_n}(x_n - z_n)$. Then $\|w_n\| = 1$.

Since $a_n \to \infty$, whereas $T_\lambda z_n = 0$ and $(T_\lambda x_n)$ converges, we get

$$T_\lambda w_n = \frac{1}{a_n} T_\lambda x_n \to 0.$$

Using $I = \lambda^{-1}(T - T_\lambda)$, we obtain $w_n = \frac{1}{\lambda}(Tw_n - T_\lambda w_n)$.

Now T is compact and (w_n) is bounded.

So (Tw_n) has a convergent subsequence.

Furthermore, $(T_\lambda w_n)$ converges.

So (w_n) has a convergent subsequence, say $w_{n_j} \to w$.

A comparison with $T_\lambda w_n \to 0$ implies that $T_\lambda w = 0$.

Hence, $w \in \mathcal{N}(T_\lambda)$.

Since $z_n \in \mathcal{N}(T_\lambda)$, also $u_n = z_n + a_n w \in \mathcal{N}(T_\lambda)$.

The Range Theorem Part (c) (Cont'd)

- We showed that $u_n \in \mathcal{N}(T_\lambda)$.

  Hence, for the distance from x_n to u_n, we must have $\|x_n - u_n\| \geq \delta_n$.

  Now recall that:

  - $a_n < 2\delta_n$;
  - $w_n = \frac{1}{a_n}(x_n - z_n)$;
  - $u_n = z_n + a_n w$.

  So we get

$$\begin{aligned} \delta_n &\leq \|x_n - z_n - a_n w\| = \|a_n w_n - a_n w\| \\ &= a_n \|w_n - w\| < 2\delta_n \|w_n - w\|. \end{aligned}$$

  Dividing by $2\delta_n > 0$, we have $\frac{1}{2} < \|w_n - w\|$.

  This contradicts $w_{n_j} \to w$.

The Ranges Corollary

Coronary (Ranges)

Under the assumptions in the theorem, the range of T_λ^n is closed for every $n = 0, 1, 2, \ldots$. Furthermore,

$$X = T_\lambda^0(X) \supseteq T_\lambda(X) \supseteq T_\lambda^2(X) \supseteq \cdots.$$

- Note that W in the proof of the Null Space Theorem is compact.

  So the first statement follows from the Range Theorem.

  The second statement follows by induction.
  - We have
    $$T_\lambda^0(X) = I(X) = X \supseteq T_\lambda(X).$$
  - Assume $T_\lambda^{n-1}(X) \supseteq T_\lambda^n(X)$.

    Applying T_λ, we get $T_\lambda^n(X) \supseteq T_\lambda^{n+1}(X)$.

Subsection 4

Further Spectral Properties of Compact Linear Operators

Compact Linear Operators: Null Spaces and Ranges

- For now, concerning a compact linear operator T on a normed space X and $\lambda \neq 0$, we know the following facts:
  - The null spaces $\mathscr{N}(T_\lambda^n)$, $n = 1, 2, \ldots$, are finite dimensional and satisfy

$$\mathscr{N}(T_\lambda^n) \subseteq \mathscr{N}(T_\lambda^{n+1});$$

  - The ranges $T_\lambda^n(X)$ are closed and satisfy

$$T_\lambda^n(X) \supseteq T_\lambda^{n+1}(X).$$

Null Spaces Lemma

Lemma (Null Spaces)

Let $T : X \to X$ be a compact linear operator on a normed space X, and let $\lambda \neq 0$. Then there exists a smallest integer r (depending on λ) such that from $n = r$ on, the null spaces $\mathscr{N}(T_\lambda^n)$ are all equal, and if $r > 0$, the inclusions $\mathscr{N}(T_\lambda^0) \subseteq \mathscr{N}(T_\lambda) \subseteq \cdots \subseteq \mathscr{N}(T_\lambda^r)$ are all proper.

- Let us write $\mathscr{N}_n = \mathscr{N}(T_\lambda^n)$, for simplicity.

  The idea of the proof is as follows.

  (a) We assume that $\mathscr{N}_m = \mathscr{N}_{m+1}$, for no m and derive a contradiction, using Riesz's Lemma.

  (b) We show that $\mathscr{N}_m = \mathscr{N}_{m+1}$ implies $\mathscr{N}_n = \mathscr{N}_{n+1}$, for all $n > m$.

Null Spaces Lemma Part (a)

(a) We know that $\mathscr{N}_m \subseteq \mathscr{N}_{m+1}$. Suppose that $\mathscr{N}_m = \mathscr{N}_{m+1}$, for no m.

Then $\mathscr{N}_n$ is a proper subspace of $\mathscr{N}_{n+1}$, for every n.

Since these null spaces are closed, Riesz's Lemma implies the existence of a sequence (y_n), such that:

- $y_n \in \mathscr{N}_n$;
- $\|y_n\| = 1$;
- $\|y_n - x\| \geq \frac{1}{2}$, for all $x \in \mathscr{N}_{n-1}$.

We show that

$$\|Ty_n - Ty_m\| \geq \frac{1}{2}|\lambda|, \quad m < n.$$

Then (Ty_n) has no convergent subsequence because $|\lambda| > 0$.

This contradicts the compactness of T since (y_n) is bounded.

Null Spaces Lemma Part (a) (Cont'd)

- From $T_\lambda = T - \lambda I$, we have:
  - $T = T_\lambda + \lambda I$;
  - $Ty_n - Ty_m = \lambda y_n - \widetilde{x}$, where $\widetilde{x} = T_\lambda y_m + \lambda y_m - T_\lambda y_n$.

  Let $m < n$. We show that $\widetilde{x} \in \mathscr{N}_{n-1}$.

  Since $m \le n-1$, we clearly have $\lambda y_m \in \mathscr{N}_m \subseteq \mathscr{N}_{n-1}$.

  Also $y_m \in \mathscr{N}_m$ implies $0 = T_\lambda^m y_m = T_\lambda^{m-1}(T_\lambda y_m)$.

  That is, $T_\lambda y_m \in \mathscr{N}_{m-1} \subseteq \mathscr{N}_{n-1}$.

  Similarly, $y_n \in \mathscr{N}_n$ implies $T_\lambda y_n \in \mathscr{N}_{n-1}$.

  Together, $\widetilde{x} \in \mathscr{N}_{n-1}$. Also $x = \lambda^{-1}\widetilde{x} \in \mathscr{N}_{n-1}$.

  Hence
  $$\|Ty_n - Ty_m\| = \|\lambda y_n - \widetilde{x}\| = |\lambda| \|y_n - x\| \ge \frac{1}{2}|\lambda|.$$

  Our assumption that $\mathscr{N}_m = \mathscr{N}_{m+1}$, for no m is false.

  We must have $\mathscr{N}_m = \mathscr{N}_{m+1}$, for some m.

Null Spaces Lemma Part (b)

(b) We prove that $\mathcal{N}_m = \mathcal{N}_{m+1}$ implies $\mathcal{N}_n = \mathcal{N}_{n+1}$, for all $n > m$.

Suppose this does not hold.

Then $\mathcal{N}_n$ is a proper subspace of $\mathcal{N}_{n+1}$, for some $n > m$.

Consider an $x \in \mathcal{N}_{n+1} - \mathcal{N}_n$.

By definition, $T_\lambda^{n+1} x = 0$, but $T_\lambda^n x \neq 0$.

Since $n > m$, we have $n - m > 0$.

Set $z = T_\lambda^{n-m} x$. Then:

- $T_\lambda^{m+1} z = T_\lambda^{n+1} x = 0$;
- $T_\lambda^m z = T_\lambda^n x \neq 0$.

Hence, $z \in \mathcal{N}_{m+1}$, but $z \notin \mathcal{N}_m$.

So $\mathcal{N}_m$ is a proper subspace of $\mathcal{N}_{m+1}$.

This contradicts $\mathcal{N}_m = \mathcal{N}_{m+1}$.

The first statement is proved, where r is the smallest n, such that $\mathcal{N}_n = \mathcal{N}_{n+1}$. So, if $r > 0$, the inclusions in the lemma are proper.

The Ranges Lemma

Lemma (Ranges)

Let $T: X \to X$ be a compact linear operator on a normed space X, and let $\lambda \neq 0$. Then, there exists a smallest integer q (depending on λ) such that from $n = q$ on, the ranges $T_\lambda^n(X)$ are all equal and, if $q > 0$, the inclusions $T_\lambda^0(X) \supsetneq T_\lambda(X) \supsetneq \cdots \supsetneq T_\lambda^q(X)$ are all proper.

- We write $\mathscr{R}_n = T_\lambda^n(X)$. Suppose that $\mathscr{R}_s = \mathscr{R}_{s+1}$ for no s.
  Then $\mathscr{R}_{n+1}$ is a proper subspace of $\mathscr{R}_n$, for every n.
  Since these ranges are closed, by Riesz's Lemma, there exists a sequence (x_n), such that:
  - $x_n \in \mathscr{R}_n$;
  - $\|x_n\| = 1$;
  - $\|x_n - x\| \geq \frac{1}{2}$, for all $x \in \mathscr{R}_{n+1}$.

  Let $m < n$. Since $T = T_\lambda + \lambda I$, we can write

$$Tx_m - Tx_n = \lambda x_m - (-T_\lambda x_m + T_\lambda x_n + \lambda x_n).$$

The Ranges Lemma (Cont'd)

- We obtained $Tx_m - Tx_n = \lambda x_m - (-T_\lambda x_m + T_\lambda x_n + \lambda x_n)$.
  On the right side:
  - $\lambda x_m \in \mathscr{R}_m$;
  - $T_\lambda x_m \in \mathscr{R}_{m+1}$, since $x_m \in \mathscr{R}_m$;
  - $T_\lambda x_n + \lambda x_n \in \mathscr{R}_n \subseteq \mathscr{R}_{m+1}$, since $n > m$.

  Hence $Tx_m - Tx_n = \lambda(x_m - x)$, for all $x \in \mathscr{R}_{m+1}$.

  Consequently, $\|Tx_m - Tx_n\| = |\lambda| \|x_m - x\| \geq \frac{1}{2}|\lambda| > 0$.

  Since (x_n) is bounded and T is compact, (Tx_n) has a convergent subsequence. This contradicts the preceding inequality.

  So we have $\mathscr{R}_s = \mathscr{R}_{s+1}$, for some s.

  Let q be the smallest s such that $\mathscr{R}_s = \mathscr{R}_{s+1}$.

  Then, if $q > 0$, the inclusions stated in the lemma are proper.

  Furthermore, $\mathscr{R}_{q+1} = \mathscr{R}_q$ means that T_λ maps $\mathscr{R}_q$ onto itself.

  Hence, repeated application of T_λ gives $\mathscr{R}_{n+1} = \mathscr{R}_n$, for every $n > q$.

Null Spaces and Ranges Theorem

Theorem (Null Spaces and Ranges)

Let $T : X \to X$ be a compact linear operator on a normed space X, and let $\lambda \neq 0$. Then there exists a smallest integer $n = r$ (depending on λ), such that

$$\mathcal{N}(T_\lambda^r) = \mathcal{N}(T_\lambda^{r+1}) = \mathcal{N}(T_\lambda^{r+2}) = \cdots$$
$$T_\lambda^r(X) = T_\lambda^{r+1}(X) = T_\lambda^{r+2}(X) = \cdots.$$

If $r > 0$, the following inclusions are proper:

$$\mathcal{N}(T_\lambda^0) \subseteq \mathcal{N}(T_\lambda) \subseteq \cdots \subseteq \mathcal{N}(T_\lambda^r) \text{ and } T_\lambda^0(X) \supseteq T_\lambda(X) \supseteq \cdots \supseteq T_\lambda^r(X).$$

- A previous lemma gives the conclusions for the kernels.
  The preceding lemma gives those for ranges with q instead of r.
  All we have to show is that $q = r$.
  Denoting, as before $\mathcal{N}_n = \mathcal{N}(T_\lambda^n)$ and $\mathcal{R}_n = T_\lambda^n(X)$, we show:
  (a) $q \geq r$;
  (b) $r \leq q$.

Null Spaces and Ranges Theorem Part (a)

(a) We have $\mathscr{R}_{q+1} = \mathscr{R}_q$. This means that $T_\lambda(\mathscr{R}_q) = \mathscr{R}_q$.

Hence, if $y \in \mathscr{R}_q$, $y = T_\lambda x$, for some $x \in \mathscr{R}_q$.

Claim: $T_\lambda x = 0$, $x \in \mathscr{R}_q$ implies $x = 0$.

Suppose not. Then $T_\lambda x_1 = 0$, for some nonzero $x_1 \in \mathscr{R}_q$.

By hypothesis, $x_1 = T_\lambda x_2$, for some $x_2 \in \mathscr{R}_q$.

Similarly, $x_2 = T_\lambda x_3$, for some $x_3 \in \mathscr{R}_q$, etc.

For every n, we thus obtain by substitution:

- $0 \neq x_1 = T_\lambda x_2 = \cdots = T_\lambda^{n-1} x_n$;
- $0 = T_\lambda x_1 = T_\lambda^n x_n$.

Hence, $x_n \notin \mathscr{N}_{n-1}$, but $x_n \in \mathscr{N}_n$.

We have $\mathscr{N}_{n-1} \subseteq \mathscr{N}_n$.

Our result shows that this inclusion is proper, for every n.

This is a contradiction.

Null Spaces and Ranges Theorem Part (a) (Cont'd)

- Recall that $\mathscr{R}_{q+1} = \mathscr{R}_q$.

  We prove that $\mathscr{N}_{q+1} = \mathscr{N}_q$.

  Then $q \geq r$, since r is the smallest integer for which we have equality.

  We have $\mathscr{N}_{q+1} \supseteq \mathscr{N}_q$.

  We prove that $\mathscr{N}_{q+1} \subseteq \mathscr{N}_q$. Equivalently,

  $$T_\lambda^{q+1} x = 0 \quad \text{implies} \quad T_\lambda^q x = 0.$$

  Suppose not. Then, for some x_0,

  $$y = T_\lambda^q x_0 \neq 0 \quad \text{but} \quad T_\lambda y = T_\lambda^{q+1} x_0 = 0.$$

  Hence $y \in \mathscr{R}_q$, $y \neq 0$, $T_\lambda y = 0$.

  This contradicts the Claim above.

Null Spaces and Ranges Theorem Part (b)

(b) We prove that $q \leq r$. If $q = 0$, this holds. Let $q \geq 1$.

We prove $q \leq r$ by showing that $\mathscr{N}_{q-1}$ is a proper subspace of $\mathscr{N}_q$.

Then $q \leq r$, since r is the smallest integer n, such that $\mathscr{N}_n = \mathscr{N}_{n+1}$.

By the definition of q, the inclusion $\mathscr{R}_q \subseteq \mathscr{R}_{q-1}$ is proper.

Let $y \in \mathscr{R}_{q-1} - \mathscr{R}_q$. Then $y \in \mathscr{R}_{q-1}$. So $y = T_\lambda^{q-1} x$, for some x.

Also $T_\lambda y \in \mathscr{R}_q = \mathscr{R}_{q+1}$ implies that $T_\lambda y = T_\lambda^{q+1} z$, for some z.

But $T_\lambda^q z \in \mathscr{R}_q$, whereas $y \notin \mathscr{R}_q$.

So $T_\lambda^{q-1}(x - T_\lambda z) = y - T_\lambda^q z \neq 0$.

Hence. $x - T_\lambda z \notin \mathscr{N}_{q-1}$.

But $x - T_\lambda z \in \mathscr{N}_q$ because $T_\lambda^q (x - T_\lambda z) = T_\lambda y - T_\lambda y = 0$.

This proves that $\mathscr{N}_{q-1} \neq \mathscr{N}_q$.

Hence, $\mathscr{N}_{q-1}$ is a proper subspace of $\mathscr{N}_q$. So $q \leq r$.

Spectrum of a Compact Operator on a Banach Space

Theorem (Eigenvalues)

Let $T: X \to X$ be a compact linear operator on a Banach space X. Then every spectral value $\lambda \neq 0$ of T (if it exists) is an eigenvalue of T.

- If $\mathcal{N}(T_\lambda) \neq \{0\}$, then λ is an eigenvalue of T.

  Suppose that $\mathcal{N}(T_\lambda) = \{0\}$, where $\lambda \neq 0$.

  Then $T_\lambda x = 0$ implies that $x = 0$ and $T_\lambda^{-1}: T_\lambda(X) \to X$ exists.

  Since $\{0\} = \mathcal{N}(I) = \mathcal{N}(T_\lambda^0) = \mathcal{N}(T_\lambda)$, we have $r = 0$.

  Hence, $X = T_\lambda^0(X) = T_\lambda(X)$.

  It follows that T_λ is bijective.

  Hence, since X is complete, by the Bounded Inverse Theorem, T_λ^{-1} is bounded.

  Therefore, by definition, $\lambda \in \rho(T)$.

The Value $\lambda = 0$

- Suppose $T : X \to X$ is a compact operator on a complex normed space X.
- If X is finite dimensional, then T has representations by matrices.

  It is clear that 0 may or may not belong to $\sigma(T) = \sigma_p(T)$.

  I.e., if $\dim X < \infty$, we may have $0 \notin \sigma(T)$. Then $0 \in \rho(T)$.
- However, if $\dim X = \infty$, then we must have $0 \in \sigma(T)$.

  In addition, all three cases

  $$0 \in \sigma_p(T), \qquad 0 \in \sigma_c(T), \qquad 0 \in \sigma_r(T)$$

  are possible.

Direct Sum Representation (Existence)

Theorem (Direct Sum)

Let $T: X \to X$ be a compact linear operator on a normed space X, and let $\lambda \neq 0$. Let r be the smallest integer (depending on λ), such that

$$\mathcal{N}(T_\lambda^r) = \mathcal{N}(T_\lambda^{r+1}) \quad \text{and} \quad T_\lambda^r(X) = T_\lambda^{r+1}(X).$$

Then X can be represented in the form

$$X = \mathcal{N}(T_\lambda^r) \oplus T_\lambda^r(X).$$

- Consider any $x \in X$. We must show that x has a unique representation of the form

$$x = y + z, \quad y \in \mathcal{N}_r, \; z \in \mathcal{R}_r,$$

where $\mathcal{N}_n = \mathcal{N}(T_\lambda^n)$ and $\mathcal{R}_n = T_\lambda^n(X)$.

Direct Sum Representation (Existence Cont'd)

- Let $z = T_\lambda^r x$. Then $z \in \mathscr{R}_r$.
  Now $\mathscr{R}_r = \mathscr{R}_{2r}$ by the previous theorem. Hence $z \in \mathscr{R}_{2r}$.
  So $z = T_\lambda^{2r} x_1$, for some $x_1 \in X$.
  Let $x_0 = T_\lambda^r x_1$. Then $x_0 \in \mathscr{R}_r$.
  Moreover,
  $$T_\lambda^r x_0 = T_\lambda^{2r} x_1 = z = T_\lambda^r x.$$
  This shows that $T_\lambda^r (x - x_0) = 0$. Hence, $x - x_0 \in \mathscr{N}_r$.
  So we get
  $$x = (x - x_0) + x_0,$$
  with $x - x_0 \in \mathscr{N}_r$ and $x_0 \in \mathscr{R}_r$.

Direct Sum Representation (Uniqueness)

- We show uniqueness.

  Assume, in addition to $x = (x - x_0) + x_0$,

  there exists $\widetilde{x}_0 \in \mathscr{R}_r$, with $x - \widetilde{x}_0 \in \mathscr{N}_r$.

  Let $v_0 = x_0 - \widetilde{x}_0$.

  Then $v_0 \in \mathscr{R}_r$, since $\mathscr{R}_r$ is a vector space.

  Hence $v_0 = T_\lambda^r v$, for some $v \in X$.

  Also

  $$v_0 = x_0 - \widetilde{x}_0 = (x - \widetilde{x}_0) - (x - x_0).$$

  Hence, $v_0 \in \mathscr{N}_r$ and $T_\lambda^r v_0 = 0$.

  Together, $T_\lambda^{2r} v = T_\lambda^r v_0 = 0$. Thus, $v \in \mathscr{N}_{2r} = \mathscr{N}_r$.

  This implies that $v_0 = T_\lambda^r v = 0$. That is, $v_0 = x_0 - \widetilde{x}_0 = 0$, or $x_0 = \widetilde{x}_0$.

  Therefore, the representation is unique, and the sum $\mathscr{N}_r + \mathscr{R}_r$ is indeed direct.

Subsection 5

Operator Equations Involving Compact Linear Operators

Fredholm Equations

- Let X be a normed space.
- Let $T : X \to X$ be a compact linear operator on X.
- Let $T^\times : X' \to X'$ be the adjoint operator of T.
- We will be dealing with the equations:
  (1) $Tx - \lambda x = y$, with $y \in X$ given and $\lambda \neq 0$;
  (2) The corresponding homogeneous equation $Tx - \lambda x = 0$, $\lambda \neq 0$;
  (3) Equations similar to (1) involving the adjoint operator $T^\times f - \lambda f = g$, where $g \in X'$ is given and $\lambda \neq 0$;
  (4) The corresponding homogeneous equation $T^\times f - \lambda f = 0$, $\lambda \neq 0$.
- $\lambda \in \mathbb{C}$ is arbitrary and fixed, not zero, and we shall study the existence of solutions x and f, respectively.

On the Solvability of (1)

Theorem (Solutions of (1))

Let $T : X \to X$ be a compact linear operator on a normed space X and let $\lambda \neq 0$. Then $Tx - \lambda x = y$ has a solution x if and only if y is such that $f(y) = 0$, for all $f \in X'$ satisfying $T^{\times} f - \lambda f = 0$. Hence, if the latter has only the trivial solution $f = 0$, then the former is solvable for any given $y \in X$.

(a) Suppose $Tx - \lambda x = y$ has a solution $x = x_0$, i.e., $y = Tx_0 - \lambda x_0 = T_{\lambda} x_0$. Let f be any solution of $T^{\times} f - \lambda f = 0$. Then we have

$$f(y) = f(Tx_0 - \lambda x_0) = f(Tx_0) - \lambda f(x_0).$$

Now, by the definition of the adjoint, $f(Tx_0) = (T^{\times} f)(x_0)$.

Hence, by the adjoint equation, $f(y) = (T^{\times} f)(x_0) - \lambda f(x_0) = 0$.

On the Solvability of (1) Converse

(b) Conversely, assume that y in $Tx - \lambda x = y$ satisfies $f(y) = 0$, for all $f \in X'$, such that $T^{\times} f - \lambda f = 0$.

Suppose $Tx - \lambda x = y$ has no solution.

Then $y = T_{\lambda} x$, for no x. Hence $y \notin T_{\lambda}(X)$. We know $T_{\lambda}(X)$ is closed.

So the distance δ from y to $T_{\lambda}(X)$ is positive.

By a previous lemma, there exists an $\tilde{f} \in X'$, such that:

- $\tilde{f}(y) = \delta$;
- $\tilde{f}(z) = 0$, for every $z \in T_{\lambda}(X)$.

Since $z \in T_{\lambda}(X)$, we have $z = T_{\lambda} x$, for some $x \in X$. So we get

$$0 = \tilde{f}(z) = \tilde{f}(T_{\lambda} x) = \tilde{f}(Tx) - \lambda \tilde{f}(x) = (T^{\times} \tilde{f})(x) - \lambda \tilde{f}(x).$$

This holds for every $x \in X$, since $z \in T_{\lambda}(X)$ was arbitrary.

Hence, $\tilde{f}$ is a solution of $T^{\times} f - \lambda f = 0$.

By assumption, it satisfies $\tilde{f}(y) = 0$. This contradicts $\tilde{f}(y) = \delta > 0$.

Consequently, $Tx - \lambda x = y$ must have a solution.

The second statement of the theorem follows from the first.

Normal Solvability

- Let $A : X \to X$ be a bounded linear operator on a normed space X.
- Let $A^{\times}$ be the adjoint operator of A.
- Consider the equation

$$Ax = y, \quad y \text{ given.}$$

- Suppose that it has a solution $x \in X$ if and only if y satisfies $f(y) = 0$, for every solution $f \in X'$ of the equation

$$A^{\times} f = 0.$$

- Then $Ax = y$ is said to be **normally solvable**.
- The preceding theorem shows that $Tx - \lambda x = y$, with a compact linear operator T and $\lambda \neq 0$, is normally solvable.

Bound for Certain Solutions of (1)

Lemma (Bound for Certain Solutions of (1))

Let $T: X \to X$ be a compact linear operator on a normed space and let $\lambda \neq 0$ be given. Then there exists a real number $c > 0$, which is independent of y in $Tx - \lambda x = y$, and such that, for every y for which the equation has a solution, at least one of these solutions, call it $x = \widetilde{x}$, satisfies

$$\|\widetilde{x}\| \leq c\|y\|, \quad \text{where } y = T_\lambda \widetilde{x}.$$

- We subdivide the proof into two steps:
  (a) We show that if the equation with a given y has solutions at all, the set of these solutions contains a solution of minimum norm, call it $\widetilde{x}$.
  (b) We show that there is a $c > 0$, such that the norm inequality holds for a solution $\widetilde{x}$ of minimum norm corresponding to any $y = T_\lambda \widetilde{x}$, for which the equation has solutions.

Bound for Certain Solutions of (1) Part (a)

(a) Let x_0 be a solution of $Tx - \lambda x = y$.
If x is any other solution, then $z = x - x_0$ satisfies $Tx - \lambda x = 0$.
Hence, every solution can be written $x = x_0 + z$, where $z \in \mathcal{N}(T_\lambda)$.
Conversely, for every $z \in \mathcal{N}(T_\lambda)$, the sum $x_0 + z$ is a solution.
For a fixed x_0, the norm of x depends on z, $p(z) = \|x_0 + z\|$. Let

$$k = \inf_{z \in \mathcal{N}(T_\lambda)} p(z).$$

By the definition of an infimum, $\mathcal{N}(T_\lambda)$ contains a sequence (z_n), such that

$$p(z_n) = \|x_0 + z_n\| \stackrel{n\to\infty}{\longrightarrow} k.$$

Since $(p(z_n))$ converges, it is bounded. Moreover,

$$\|z_n\| = \|(x_0 + z_n) - x_0\| \le \|x_0 + z_n\| + \|x_0\| = p(z_n) + \|x_0\|.$$

So (z_n) is bounded.

Bound for Certain Solutions of (1) Part (a) (Cont'd)

- Since T is compact, (Tz_n) has a convergent subsequence.

  But $z_n \in \mathcal{N}(T_\lambda)$ means that $T_\lambda z_n = 0$.

  I.e., $Tz_n = \lambda z_n$, where $\lambda \neq 0$.

  Hence, (z_n) has a convergent subsequence, say, $z_{n_j} \to z_0$.

  Since $\mathcal{N}(T_\lambda)$ is closed, $z_0 \in \mathcal{N}(T_\lambda)$.

  Since p is continuous, $p(z_{n_j}) \to p(z_0)$.

  We thus obtain

  $$p(z_0) = \|x_0 + z_0\| = k.$$

  Thus, if $Tx - \lambda x = y$, with a given y, has solutions, the set of these solutions contains a solution $\tilde{x} = x_0 + z_0$ of minimum norm.

Bound for Certain Solutions of (1) Part (b)

(b) We show there is a $c > 0$ (independent of y) such that $\|\widetilde{x}\| \le c\|y\|$ holds for a solution $\widetilde{x}$ of minimum norm corresponding to any $y = T_\lambda \widetilde{x}$ for which $Tx - \lambda x = y$ is solvable.

Suppose not. Then there is a sequence (y_n), such that

$$\frac{\|\widetilde{x}_n\|}{\|y_n\|} \xrightarrow{n\to\infty} \infty,$$

where $\widetilde{x}_n$ is of minimum norm and satisfies $T_\lambda \widetilde{x}_n = y_n$.

Multiplication by an α shows that to αy_n, there corresponds $\alpha \widetilde{x}_n$ as a solution of minimum norm.

Thus, without loss of generality, we assume $\|\widetilde{x}_n\| = 1$.

Then $\|y_n\| \to 0$.

Now T is compact and $(\widetilde{x}_n)$ is bounded.

So $(T\widetilde{x}_n)$ has a convergent subsequence, say, $T\widetilde{x}_{n_j} \to v_0$.

If, for convenience, we write $v_0 = \lambda \widetilde{x}_0$, then $T\widetilde{x}_{n_j} \to \lambda \widetilde{x}_0$.

Bound for Certain Solutions of (1) Part (b) (Cont'd)

- Since $y_n = T_\lambda \widetilde{x}_n = T\widetilde{x}_n - \lambda \widetilde{x}_n$, we have $\lambda \widetilde{x}_n = T\widetilde{x}_n - y_n$.
  Using this and $\|y_n\| \to 0$, and noting $\lambda \neq 0$,

  $$\widetilde{x}_{n_j} = \frac{1}{\lambda}(T\widetilde{x}_{n_j} - y_{n_j}) \to \frac{1}{\lambda}(\lambda \widetilde{x}_0 - 0) = \widetilde{x}_0.$$

  Since T is continuous, $T\widetilde{x}_{n_j} \to T\widetilde{x}_0$.
  Hence $T\widetilde{x}_0 = \lambda \widetilde{x}_0$.
  Since $T_\lambda \widetilde{x}_n = y_n$, we see that $x = \widetilde{x}_n - \widetilde{x}_0$ satisfies $T_\lambda x = y_n$.
  Since $\widetilde{x}_n$ is of minimum norm,

  $$\|x\| = \|\widetilde{x}_n - \widetilde{x}_0\| \geq \|\widetilde{x}_n\| = 1.$$

  This contradicts $\widetilde{x}_{n_j} \to \widetilde{x}_0$.
  Hence, $c = \sup_{y \in T_\lambda(X)} \frac{\|\widetilde{x}\|}{\|y\|} < \infty$, where $y = T_\lambda \widetilde{x}$.

Solutions of (3)

Theorem (Solutions of (3))

Let $T : X \to X$ be a compact linear operator on a normed space X and let $\lambda \neq 0$. Then $T^\times f - \lambda f = g$ has a solution f if and only if g is such that $g(x) = 0$, for all $x \in X$, which satisfy $Tx - \lambda x = 0$. Hence, if the latter has only the trivial solution $x = 0$, then the former is solvable, for any $g \in X'$.

(a) Suppose $T^\times f - \lambda f = g$ has a solution f.

Let x be such that $Tx - \lambda x = 0$.

Then we have

$$g(x) = (T^\times f)(x) - \lambda f(x) = f(Tx - \lambda x) = f(0) = 0.$$

(b) Conversely, suppose g satisfies $g(x) = 0$, for all x, with $Tx - \lambda x = 0$.

We show that $T^\times f - \lambda f = g$ has a solution f.

Solutions of (3) (Cont'd)

- Consider any $x \in X$ and set $y = T_\lambda x$. Then $y \in T_\lambda(X)$.
  We may define a functional f_0 on $T_\lambda(X)$ by
  $$f_0(y) = f_0(T_\lambda x) = g(x).$$
  This definition is unambiguous.
  If $T_\lambda x_1 = T_\lambda x_2$, then $T_\lambda(x_1 - x_2) = 0$.
  So $x_1 - x_2$ is a solution of $Tx - \lambda x = 0$.
  Thus, $g(x_1 - x_2) = 0$ by assumption.
  f_0 is linear since T_λ and g are linear.
  We show that f_0 is bounded.
  By the preceding lemma, for every $y \in T_\lambda(X)$, at least one of the corresponding x's satisfies $\|x\| \le c\|y\|$, where c does not depend on y.
  Boundedness of f_0 can now be seen from
  $$|f_0(y)| = |g(x)| \le \|g\| \|x\| \le c\|g\| \|y\| = \widetilde{c}\|y\|,$$
  where $\widetilde{c} = c\|g\|$.

Solutions of (3) (Conclusion)

- By the Hahn-Banach Theorem, the functional f_0 has an extension f on X, which is a bounded linear functional defined on all of X.

  By the definition of f_0,

  $$f(Tx - \lambda x) = f(T_\lambda x) = f_0(T_\lambda x) = g(x).$$

  On the left, by the definition of adjoint, we have for all $x \in X$,

  $$f(Tx - \lambda x) = f(Tx) - \lambda f(x) = (T^\times f)(x) - \lambda f(x).$$

  Together with the preceding formula this shows that f is a solution of

  $$T^\times f - \lambda f = g.$$

  The second statement follows readily from the first one.

Subsection 6

Further Theorems of Fredholm Type

Review of Assumptions

- Let X be a normed space.
- We revisit compact linear operators $T : X \to X$ on X.
- Let $T^{\times}$ be the adjoint operator of T and $\lambda \neq 0$ be fixed.
- We present further results about the solvability of the following operator equations:
  (1) $Tx - \lambda x = y$, y given;
  (2) $Tx - \lambda x = 0$;
  (3) $T^{\times} f - \lambda f = g$, g given;
  (4) $T^{\times} f - \lambda f = 0$.

Solutions of $Tx - \lambda x = y$

Theorem (Solutions of (1))

Let $T : X \to X$ be a compact linear operator on a normed space X and let $\lambda \neq 0$. Then:

(a) $Tx - \lambda x = y$ has a solution x, for every $y \in X$, if and only if the homogeneous equation $Tx - \lambda x = 0$ has only the trivial solution $x = 0$. In this case the solution is unique, and T_λ has a bounded inverse.

(b) $T^\times f - \lambda f = g$ has a solution f, for every $g \in X'$, if and only if $T^\times f - \lambda f = 0$ has only the trivial solution $f = 0$. In this case the solution is unique.

(a) Suppose that for every $y \in X$, $Tx - \lambda x = y$ is solvable.
Assume that $x = 0$ is not the only solution of $Tx - \lambda x = 0$.
Then $Tx - \lambda x = 0$ has a solution $x_1 \neq 0$.
For any y, $Tx - \lambda x = y$ is solvable. So $T_\lambda x = x_1$ has a solution $x = x_2$.
For the same reason, there is an x_3, such that $T_\lambda x_3 = x_2$, etc.

Solutions of $Tx - \lambda x = y$ (Cont'd)

- By substitution, we thus have, for every $k = 2, 3, \ldots$,

$$0 \neq x_1 = T_\lambda x_2 = T_\lambda^2 x_3 = \cdots = T_\lambda^{k-1} x_k.$$

  Moreover, $0 = T_\lambda x_1 = T_\lambda^k x_k$.

  Hence, $x_k \in \mathcal{N}(T_\lambda^k)$ but $x_k \notin \mathcal{N}(T_\lambda^{k-1})$.

  This means that the null space $\mathcal{N}(T_\lambda^{k-1})$ is a proper subspace of $\mathcal{N}(T_\lambda^k)$, for all k.

  But this contradicts a previous theorem.

Solutions of $Tx - \lambda x = y$ (Converse)

- Conversely, suppose that $x = 0$ is the only solution of $Tx - \lambda x = 0$.

  Then, by a preceding result, $T^\times f - \lambda f = g$, with any g, is solvable.

  We know that $T^\times$ is compact.

  So we can apply the first part of the proof to $T^\times$ and conclude that $f = 0$ must be the only solution of $T^\times f - \lambda f = 0$.

  Solvability of $Tx - \lambda x = y$ follows by a previous theorem.

  Now note that the difference of two solutions of $Tx - \lambda x = y$ is a solution of $Tx - \lambda x = 0$. Clearly, such a unique solution $x = T_\lambda^{-1} y$ is the solution of minimum norm. Thus, the solution is unique.

  By a previous lemma, boundedness of T_λ^{-1} follows:

$$\|x\| = \|T_\lambda^{-1} y\| \le c\|y\|.$$

(b) This is a consequence of (a) and the fact that $T^\times$ is compact.

The Biorthogonal System Lemma

Lemma (Biorthogonal System)

Given a linearly independent set $\{f_1,\ldots,f_m\}$ in the dual space X' of a normed space X, there are elements $z_1,\ldots,z_m$ in X, such that

$$f_j(z_k)=\delta_{jk}=\begin{cases} 0, & \text{if } j\neq k \\ 1, & \text{if } j=k \end{cases}, \quad j,k=1,\ldots,m.$$

- The order being immaterial, it suffices to prove that there exists a z_m, such that $f_m(z_m)=1$, $f_j(z_m)=0$, $j=1,\ldots,m-1$.
  If $m=1$, by the linear independence, $f_1\neq 0$.
  So, $f_1(x_0)\neq 0$, for some x_0. Set $z_1=\alpha x_0$, $\alpha=\frac{1}{f_1(x_0)}$. Then $f_1(z_1)=1$.
  Let $m>1$ and assume the lemma holds for $m-1$.
  So X contains elements $z_1,\ldots,z_{m-1}$, such that

$$f_k(z_k)=1, \quad f_n(z_k)=0, \quad n\neq k, \quad k,n=1,\ldots,m-1.$$

The Biorthogonal System Lemma (Cont'd)

- Consider the set $M = \{x \in X : f_1(x) = 0, \ldots, f_{m-1}(x) = 0\}$.

  We show that M contains a $\widetilde{z}_m$, such that $f_m(\widetilde{z}_m) = \beta \neq 0$.

  This clearly yields the result, where $z_m = \beta^{-1}\widetilde{z}_m$.

  Suppose, to the contrary, that $f_m(x) = 0$, for all $x \in M$.

  We take any $x \in X$ and set

$$\widetilde{x} = x - \sum_{j=1}^{m-1} f_j(x) z_j.$$

  Then, for $k \leq m-1$,

$$f_k(\widetilde{x}) = f_k(x) - \sum_{j=1}^{m-1} f_j(x) f_k(z_j) = f_k(x) - f_k(x) = 0.$$

  This shows that $\widetilde{x} \in M$.

The Biorthogonal System Lemma (Conclusion)

- So, by our assumption, $f_m(\widetilde{x}) = 0$.

  By definition, we get

$$f_m(x) = f_m\left(\widetilde{x} + \sum_{j=1}^{m-1} f_j(x) z_j\right) = f_m(\widetilde{x}) + \sum_{j=1}^{m-1} f_j(x) f_m(z_j) = \sum_{j=1}^{m-1} \alpha_j f_j(x),$$

  where $\alpha_j = f_m(z_j)$. But $x \in X$ was arbitrary. So this is a representation of f_m as a linear combination of $f_1, \ldots, f_{m-1}$. This contradicts the linear independence of $\{f_1, \ldots, f_m\}$.

  Hence $f_m(x) = 0$, for all $x \in M$ is impossible.

  So M must contain a z_m such that

$$f_m(z_m) = 1, \quad f_j(z_m) = 0, \quad j = 1, \ldots, m-1.$$

Null Spaces of T_λ and $T_\lambda^\times$

Theorem (Null Spaces of T_λ and $T_\lambda^\times$)

Let $T: X \to X$ be a compact linear operator on a normed space X, and let $\lambda \neq 0$. Then, the equations $Tx - \lambda x = 0$ and $T^\times f - \lambda f = 0$ have the same number of linearly independent solutions.

- T and $T^\times$ are compact.

  So $\mathscr{N}(T_\lambda)$ and $\mathscr{N}(T_\lambda^\times)$ are finite dimensional, say

  $$\dim \mathscr{N}(T_\lambda) = n \quad \text{and} \quad \dim \mathscr{N}(T_\lambda^\times) = m.$$

  We subdivide the proof into three parts:

  (a) The case $m = n = 0$ and a preparation for $m > 0, n > 0$;
  (b) The proof that $n < m$ is impossible;
  (c) The proof that $n > m$ is impossible.

Null Spaces of T_λ and $T_\lambda^\times$ Part (a)

(a) If $n = 0$, the only solution of $Tx - \lambda x = 0$ is $x = 0$.

Then $T^\times f - \lambda f = g$ with any given g is solvable.

By a preceding result, this implies that $f = 0$ is the only solution of $T^\times f - \lambda f = 0$. Hence $m = 0$.

Similarly, from $m = 0$ it follows that $n = 0$.

Suppose $m > 0$ and $n > 0$.

Let $\{x_1, \ldots, x_n\}$ be a basis for $\mathcal{N}(T_\lambda)$.

Clearly, $x_1 \notin Y_1 = \mathrm{span}\{x_2, \ldots, x_n\}$.

By a previous lemma, there is a $\widetilde{g}_1 \in X'$, which is:

- Zero everywhere on Y_1;
- $\widetilde{g}_1(x_1) = \delta$, where $\delta > 0$ is the distance from x_1 to Y_1.

Hence $g_1 = \delta^{-1} \widetilde{g}_1$ satisfies

$$g_1(x_1) = 1 \quad \text{and} \quad g_1(x_2) = 0, \ldots, g_1(x_n) = 0.$$

Null Spaces of T_λ and $T_\lambda^\times$ Part (a) (Cont'd)

- Similarly, there is a g_2, such that

$$g_2(x_2) = 1 \quad \text{and} \quad g_2(x_j) = 0, \quad \text{for } j \neq 2, \text{ etc..}$$

Hence X' contains $g_1, \ldots, g_n$, such that

$$g_k(x_j) = \delta_{jk} = \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases}, \quad j, k = 1, \ldots, n.$$

Similarly, suppose $\{f_1, \ldots, f_m\}$ is a basis for $\mathscr{N}(T_\lambda^\times)$.

Then by the lemma, there are elements $z_1, \ldots, z_m$ of X, such that

$$f_j(z_k) = \delta_{jk}, \quad j, k = 1, \ldots, m.$$

Null Spaces of T_λ and $T_\lambda^\times$ Part (b) (Claim)

(b) We show that $n < m$ is impossible. Let $n < m$.
Define $S : X \to X$ by

$$Sx = Tx + \sum_{j=1}^{n} g_j(x) z_j.$$

S is compact since, by a previous result, $g_j(x)z_j$ represents a compact linear operator, and a sum of compact operators is compact.

Claim: $S_\lambda x_0 = Sx_0 - \lambda x_0 = 0$ implies $x_0 = 0$.

By the hypothesis, we have $f_k(S_\lambda x_0) = f_k(0) = 0$, for $k = 1, , \ldots, m$.

Hence, by the definition of S and of f_j, we obtain

$$\begin{aligned} 0 &= f_k(S_\lambda x_0) \\ &= f_k(T_\lambda x_0 + \textstyle\sum_{j=1}^{n} g_j(x_0) z_j) \\ &= f_k(T_\lambda x_0) + \textstyle\sum_{j=1}^{n} g_j(x_0) f_k(z_j) \\ &= (T_\lambda^\times f_k)(x_0) + g_k(x_0). \end{aligned}$$

Null Spaces of T_λ and $T_\lambda^\times$ Part (b) (Claim Cont'd)

- Since $f_k \in \mathcal{N}(T_\lambda^\times)$, we have $T_\lambda^\times f_k = 0$.

  Hence, by the preceding equation, $g_k(x_0) = 0$, $k = 1, \ldots, m$.

  This implies $Sx_0 = Tx_0$, by the definition of S.

  So $T_\lambda x_0 = S_\lambda x_0 = 0$, by the hypothesis.

  Hence $x_0 \in \mathcal{N}(T_\lambda)$.

  Since $\{x_1, \ldots, x_n\}$ is a basis for $\mathcal{N}(T_\lambda)$, $x_0 = \sum_{j=1}^n \alpha_j x_j$, where the α_j's are suitable scalars.

  Applying g_k, we have, for all $k = 1, \ldots, n$,

$$0 = g_k(x_0) = \sum_{j=1}^n \alpha_j g_k(x_j) = \alpha_k.$$

  Hence $x_0 = 0$.

Null Spaces of T_λ and $T_\lambda^\times$ Part (b) (Cont'd)

- A preceding theorem now implies that $S_\lambda x = y$, with any y, is solvable.
  We choose $y = z_{n+1}$.
  Let $x = v$ be a corresponding solution, i.e., $S_\lambda v = z_{n+1}$.
  We calculate

$$\begin{aligned} 1 &= f_{n+1}(z_{n+1}) \\ &= f_{n+1}(S_\lambda v) \\ &= f_{n+1}(T_\lambda v + \textstyle\sum_{j=1}^n g_j(v) z_j) \\ &= (T_\lambda^\times f_{n+1})(v) + \textstyle\sum_{j=1}^n g_j(v) f_{n+1}(z_j) \\ &= (T_\lambda^\times f_{n+1})(v). \end{aligned}$$

  Since we assumed $n < m$, we have $n+1 \leq m$ and $f_{n+1} \in \mathcal{N}(T_\lambda^\times)$.
  Hence $T_\lambda^\times f_{n+1} = 0$. This contradicts the preceding equation.
  Therefore, $n < m$ is impossible.

Null Spaces of T_λ and $T_\lambda^\times$ Part (c) (Claim)

(c) We show $n > m$ is also impossible. Let $n > m$.
Define $\widetilde{S} : X' \to X'$ by

$$\widetilde{S}f = T^\times f + \sum_{j=1}^{m} f(z_j)g_j.$$

By a previous theorem, $T^\times$ is compact.

Moreover, $\widetilde{S}$ is compact since $f(z_j)g_j$ represents a compact linear operator by a previous theorem.

Claim: $\widetilde{S}_\lambda f_0 = \widetilde{S}f_0 - \lambda f_0 = 0$ implies $f_0 = 0$.

Using the hypothesis, the definition of $\widetilde{S}$, the definition of adjoint operator and that of the g_k's we obtain for each $k = 1, \ldots, m$,

$$0 = (\widetilde{S}_\lambda f_0)(x_k) = (T_\lambda^\times f_0)(x_k) + \sum_{j=1}^{m} f_0(z_j)g_j(x_k) = f_0(T_\lambda x_k) + f_0(z_k).$$

Null Spaces of T_λ and $T_\lambda^\times$ Part (c) (Claim Cont'd)

- Recall that $\{x_1,\ldots,x_n\}$ is a basis for $\mathscr{N}(T_\lambda)$.
  Now $m<n$ implies that $x_k \in \mathscr{N}(T_\lambda)$, for $k=1,\ldots,m$.
  Hence, $f_0(T_\lambda x_k) = f_0(0)$.
  So $f_0(z_k)=0$, $k=1,\ldots,m$.
  Consequently, $\widetilde{S} f_0 = T^\times f_0$, by the definition of $\widetilde{S}$.
  By hypothesis, $T_\lambda^\times f_0 = \widetilde{S}_\lambda f_0 = 0$.
  Hence, $f_0 \in \mathscr{N}(T_\lambda^\times)$.
  But $\{f_1,\ldots,f_m\}$ is a basis for $\mathscr{N}(T_\lambda^\times)$.
  So $f_0 = \sum_{j=1}^m \beta_j f_j$, where the β_j's are suitable scalars.
  Thus, for each $k=1,\ldots,m$,
  $$0 = f_0(z_k) = \sum_{j=1}^m \beta_j f_j(z_k) = \beta_k.$$
  Hence $f_0 = 0$.

Null Spaces of T_λ and $T_\lambda^\times$ Part (c) (Cont'd)

- A preceding theorem now implies that $\widetilde{S}_\lambda f = g$, for any g, is solvable.

  We choose $g = g_{m+1}$.

  Let $f = h$ be a corresponding solution, i.e., $\widetilde{S}_\lambda h = g_{m+1}$.

  Using the definition of the g_k's and that of $\widetilde{S}$, we obtain

$$\begin{aligned} 1 &= g_{m+1}(x_{m+1}) \\ &= (\widetilde{S}_\lambda h)(x_{m+1}) \\ &= (T_\lambda^\times h)(x_{m+1}) + \textstyle\sum_{j=1}^m h(z_j) g_j(x_{m+1}) \\ &= (T_\lambda^\times h)(x_{m+1}) \\ &= h(T_\lambda(x_{m+1})). \end{aligned}$$

  The assumption $m < n$ implies $m + 1 \leq n$.

  So $x_{m+1} \in \mathcal{N}(T_\lambda)$. Hence, $h(T_\lambda x_{m+1}) = h(0) = 0$.

  This contradicts the previous equation and shows that $m < n$ is impossible.

The Eigenvalue Theorem

Theorem (Eigenvalues)

Let $T : X \to X$ be a compact linear operator on a normed space X. Then, if T has nonzero spectral values, every one of them must be an eigenvalue of T.

- If the resolvent $R_\lambda = T_\lambda^{-1}$ does not exist, $\lambda \in \sigma_p(T)$ by definition.

  Let $\lambda \neq 0$ and assume that $R_\lambda = T_\lambda^{-1}$ exists.

  Then $T_\lambda x = 0$ implies $x = 0$.

  This means that $Tx - \lambda x = 0$ has only the trivial solution.

  By a preceding theorem, $Tx - \lambda x = y$, with any y, is solvable.

  That is, R_λ is defined on all of X and is bounded.

  Hence, $\lambda \in \rho(T)$.

Subsection 7

Fredholm Alternative

Fredholm Alternative

Definition (Fredholm Alternative)

A bounded linear operator $A : X \to X$ on a normed space X is said to satisfy the **Fredholm alternative** if A is such that either (I) or (II) holds:

(I) The nonhomogeneous equations $Ax = y$, $A^{\times} f = g$ ($A^{\times} : X' \to X'$ the adjoint operator of A) have solutions x and f, respectively, for every given $y \in X$ and $g \in X'$, the solutions being unique.
The corresponding homogeneous equations $Ax = 0$, $A^{\times} f = 0$ have only the trivial solutions $x = 0$ and $f = 0$, respectively.

(II) The homogeneous equations $Ax = 0$, $A^{\times} f = 0$ have the same number of linearly independent solutions $x_1, \ldots, x_n$ and $f_1, \ldots, f_n$, $n \geq 1$, respectively.
The nonhomogeneous equations $Ax = y$, $A^{\times} f = g$ are not solvable for all y and g, respectively; they have a solution if and only if y and g are such that $f_k(y) = 0$, $g(x_k) = 0$, $k = 1, \ldots, n$, respectively.

The Fredholm Alternative Theorem

- Summarizing the results of the preceding two sections:

Theorem (Fredholm Alternative)

Let $T: X \to X$ be a compact linear operator on a normed space X, and let $\lambda \neq 0$. Then $T_\lambda = T - \lambda I$ satisfies the Fredholm alternative.

- In applications, instead of showing the existence of a solution directly, it is often simpler to prove that the homogeneous equation has only the trivial solution.
- Riesz's theory of compact linear operators was suggested by Fredholm's theory of integral equations of the second kind

$$x(s) - \mu \int_a^b k(s,t)x(t)dt = \tilde{y}(s)$$

- In fact Riesz's theory generalizes Fredholm's results, which predate the development of the theory of Hilbert and Banach spaces.

Fredholm Alternative for Integral Equations

- Consider again the integral equation

$$x(s) - \mu \int_a^b k(s,t)x(t)dt = \widetilde{y}(s).$$

- Set $\mu = \frac{1}{\lambda}$ and $\widetilde{y}(s) = -\frac{y(s)}{\lambda}$, where $\lambda \neq 0$.
- Then

$$x(s) - \frac{1}{\lambda}\int_a^b k(s,t)x(t)dt = -\frac{1}{\lambda}y(s).$$

- This gives

$$\int_a^b k(s,t)x(t)dt - \lambda x(s) = y(s).$$

- So we get

$$Tx - \lambda x = y, \quad \lambda \neq 0,$$

with T defined by $(Tx)(s) = \int_a^b k(s,t)x(t)dt$.

Fredholm Alternative for Integral Equations (Cont'd)

- We obtained

$$Tx - \lambda x = y, \quad \lambda \neq 0,$$

with T defined by

$$(Tx)(s) = \int_a^b k(s,t)x(t)dt.$$

- Now, the general theory applied to this T gives

Theorem (Fredholm Alternative for Integral Equations)

If k in is such that $T : X \to X$ is a compact linear operator on a normed space X, then the Fredholm alternative holds for T_λ. Thus, one of the two alternatives hold:

- The integral equation has a unique solution for all $y \in X$;
- The homogeneous equation corresponding to the integral equation has finitely many linearly independent nontrivial solutions x (i.e., $x \neq 0$).

Alternative (I): Neumann Series

- Suppose that T in $Tx - \lambda x = y$ is compact.
- Suppose λ is in the resolvent set $\rho(T)$ of T.
- Then the resolvent
$$R_\lambda(T) = (T - \lambda I)^{-1}$$
exists, is defined on all of X and is bounded.
- So, for every $y \in X$, we get the unique solution of $Tx - \lambda x = y$
$$x = R_\lambda(T)y.$$
- Since $R_\lambda(T)$ is linear, we get $R_\lambda(T)0 = 0$.
- This implies that the homogeneous equation $Tx - \lambda x = 0$ has only the trivial solution $x = 0$.
- Hence, $\lambda \in \rho(T)$ yields Case (I) of the Fredholm alternative.

Alternative (I): Neumann Series (Cont'd)

- Let $|\lambda| > \|T\|$.
- Assume X is a complex Banach space.
- Then we have $\lambda \in \rho(T)$.
- Furthermore,

$$R_\lambda(T) = -\frac{1}{\lambda}\left(I + \frac{1}{\lambda}T + \frac{1}{\lambda^2}T^2 + \cdots\right).$$

- Consequently, for the solution $x = R_\lambda(T)y$, we have the representation

$$x = -\frac{1}{\lambda}\left(y + \frac{1}{\lambda}Ty + \frac{1}{\lambda^2}T^2y + \cdots\right).$$

- This series is called a **Neumann series**.

Alternative (II)

- Case (II) of the Fredholm alternative is obtained if we take a nonzero $\lambda \in \sigma(T)$ (if such a λ exists), where $\sigma(T)$ is the spectrum of T.
- A previous theorem implies that λ is an eigenvalue.
- The dimension of the corresponding eigenspace is finite.
- It is equal to the dimension of the corresponding eigenspace of $T_\lambda^\times$.

Special Cases

- Two spaces of particular interest are $X = L^2[a,b]$ and $X = C[a,b]$.
- To apply the theorem, one needs conditions for the kernel k which are sufficient for T to be compact.
  - If $X = L^2[a,b]$, such a condition is that k be in $L^2(J \times J)$, where $J = [a,b]$. (This is a measure theoretic result.)
  - In the case $X = C[a,b]$, where $[a,b]$ is compact, continuity of k will imply compactness of T.

    We will obtain this result by applying Ascoli's Theorem.

Equicontinuous Sequences and Ascoli's Theorem

- A sequence (x_n) in $C[a,b]$ is said to be **equicontinuous** if, for every $\varepsilon > 0$, there is a $\delta > 0$, depending only on ε, such that, for all x_n and all $s_1, s_2 \in [a,b]$, satisfying $|s_1 - s_2| < \delta$, we have

$$|x_n(s_1) - x_n(s_2)| < \varepsilon.$$

- Note that in equicontinuity:
  - δ does not depend on n;
  - Each x_n is uniformly continuous on $[a,b]$.

Ascoli's Theorem (Equicontinuous Sequence)

A bounded equicontinuous sequence (x_n) in $C[a,b]$ has a subsequence which converges (in the norm on $C[a,b]$).

Compact Integral Operators

Theorem (Compact Integral Operator)

Let $J=[a,b]$ be any compact interval and suppose that k is continuous on $J\times J$. Then the operator $T:X\to X$ defined by $(Tx)(s)=\int_a^b k(s,t)x(t)dt$, where $X=C[a,b]$, is a compact linear operator.

- T is linear.

  Boundedness of T follows from

$$\|Tx\|=\max_{s\in J}\left|\int_a^b k(s,t)x(t)dt\right|\leq\|x\|\max_{s\in J}\int_a^b|k(s,t)|dt.$$

  This is of the form $\|Tx\|\leq\widetilde{c}\|x\|$.

  Let (x_n) be any bounded sequence in X, say, $\|x_n\|\leq c$, for all n.

  Let $y_n=Tx_n$. Then $\|y_n\|\leq\|T\|\|x_n\|$. Hence, (y_n) is also bounded.

Compact Integral Operators (Cont'd)

Claim: (y_n) is equicontinuous.

By hypothesis, the kernel k is continuous on $J \times J$.

Moreover, $J \times J$ is compact. Thus, k is uniformly continuous on $J \times J$.

Hence, given $\varepsilon > 0$, there is a $\delta > 0$, such that, for all $t \in J$ and all $s_1, s_2 \in J$, satisfying $|s_1 - s_2| < \delta$, we have $|k(s_1, t) - k(s_2, t)| < \frac{\varepsilon}{(b-a)c}$.

Consequently, for s_1, s_2 as before and every n,

$$\begin{aligned} |y_n(s_1) - y_n(s_2)| &= \left| \int_a^b [k(s_1, t) - k(s_2, t)] x_n(t) dt \right| \\ &< (b-a) \frac{\varepsilon}{(b-a)c} c = \varepsilon. \end{aligned}$$

This proves equicontinuity of (y_n).

Ascoli's Theorem implies that (y_n) has a convergent subsequence.

Since (x_n) was an arbitrary bounded sequence and $y_n = Tx_n$, compactness of T follows from a previous theorem.