Introduction to Spectral Theory of Linear Operators

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Subsection 1

Bounded Self-Adjoint Linear Operators

The Hilbert Adjoint Operator

- Let H be a complex Hilbert space.
- Let $T : H \to H$ be a bounded linear operator on H.
- The **Hilbert-adjoint operator** $T^* : H \to H$ is defined to be the operator satisfying

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \quad \text{for all } x, y \in H.$$

- From the general theory of Hilbert Spaces, we know the following facts:
  - T^* exists;
  - T^* is a bounded linear operator;
  - T^* is of norm $\|T^*\| = \|T\|$;
  - T^* is unique.

Self-Adjoint or Hermitian Operators

- Let H be a complex Hilbert space.
- Let $T: H \to H$ be a bounded linear operator on H.
- T is said to be **self-adjoint** or **Hermitian** if

$$T = T^*.$$

- Then $\langle Tx, y \rangle = \langle x, T^* y \rangle$ becomes

$$\langle Tx, y \rangle = \langle x, Ty \rangle.$$

- If T is self-adjoint, then $\langle Tx, x \rangle$ is real for all $x \in H$.
- Since H being complex, this condition implies self-adjointness.

Eigenvalues and Eigenvectors

Theorem (Eigenvalues and Eigenvectors)

Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Then:

(a) All the eigenvalues of T (if they exist) are real.

(b) Eigenvectors corresponding to different eigenvalues are orthogonal.

(a) Let λ be any eigenvalue of T and x a corresponding eigenvector.
Then $x \neq 0$ and $Tx = \lambda x$.
Using the self-adjointness of T, we get

$$\lambda\langle x,x\rangle = \langle \lambda x,x\rangle = \langle Tx,x\rangle = \langle x,Tx\rangle = \langle x,\lambda x\rangle = \overline{\lambda}\langle x,x\rangle.$$

Note that, since $x \neq 0$. $\langle x,x\rangle = \|x\|^2 \neq 0$.
So dividing by $\langle x,x\rangle$ gives $\lambda = \overline{\lambda}$.
We conclude that λ is real.

Eigenvalues and Eigenvectors (Cont'd)

(b) Let λ and μ be eigenvalues of T.

Let x and y be corresponding eigenvectors.

Then $Tx = \lambda x$ and $Ty = \mu y$.

Note that T is self-adjoint and μ is real.

So we get

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, Ty\rangle = \langle x, \mu y\rangle = \mu\langle x, y\rangle.$$

Since $\lambda \neq \mu$, $\langle x, y\rangle = 0$.

This shows that x and y are orthogonal.

Characterization of the Resolvent Set

Theorem (Resolvent Set)

Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Then a number λ belongs to the resolvent set $\rho(T)$ of T if and only if there exists a $c > 0$, such that for every $x \in H$,

$$\|T_\lambda x\| \geq c\|x\|, \quad \text{where } T_\lambda = T - \lambda I.$$

(a) If $\lambda \in \rho(T)$, then $R_\lambda = T_\lambda^{-1} : H \to H$ exists and is bounded.

Since $R_\lambda \neq 0$, $\|R_\lambda\| = k$, where $k > 0$.

Now $I = R_\lambda T_\lambda$. So, for every $x \in H$, we have

$$\|x\| = \|R_\lambda T_\lambda x\| \leq \|R_\lambda\| \|T_\lambda x\| = k\|T_\lambda x\|.$$

This gives $\|T_\lambda x\| \geq c\|x\|$, where $c = \frac{1}{k}$.

Characterization of the Resolvent Set (Converse (i))

(b) Suppose $\|T_\lambda x\| \geq c\|x\|$, $c > 0$, holds for all $x \in H$. We prove:
  (i) $T_\lambda : H \to T_\lambda(H)$ is bijective;
  (ii) $T_\lambda(H)$ is dense in H;
  (iii) $T_\lambda(H)$ is closed in H.

Then $T_\lambda(H) = H$ and $R_\lambda = T_\lambda^{-1}$ is bounded by the Bounded Inverse Theorem.

(i) We must show that $T_\lambda x_1 = T_\lambda x_2$ implies $x_1 = x_2$.

As T_λ is linear, if $T_\lambda x_1 = T_\lambda x_2$, then

$$0 = \|T_\lambda x_1 - T_\lambda x_2\| = \|T_\lambda(x_1 - x_2)\| \geq c\|x_1 - x_2\|.$$

Since $c > 0$, we get $\|x_1 - x_2\| = 0$.

So $x_1 = x_2$.

Since x_1, x_2 were arbitrary, $T_\lambda : H \to T_\lambda(H)$ is bijective.

Characterization of the Resolvent Set (Converse (ii))

(ii) We show $x_0 \perp \overline{T_\lambda(H)}$ implies $x_0 = 0$.
Then, by the Projection Theorem, $\overline{T_\lambda(H)} = H$.
Let $x_0 \perp \overline{T_\lambda(H)}$. Then $x_0 \perp T_\lambda(H)$.
Hence, for all $x \in H$, $0 = \langle T_\lambda x, x_0\rangle = \langle Tx, x_0\rangle - \lambda\langle x, x_0\rangle$.
Since T is self-adjoint,

$$\langle x, Tx_0\rangle = \langle Tx, x_0\rangle = \langle x, \overline{\lambda} x_0\rangle.$$

Hence, $Tx_0 = \overline{\lambda} x_0$.
A solution is $x_0 = 0$. Moreover, $x_0 \neq 0$ is impossible.
Indeed, that would mean that $\overline{\lambda}$ is an eigenvalue of T.
Then, $\lambda = \overline{\lambda}$ and $Tx_0 - \lambda x_0 = T_\lambda x_0 = 0$.
Since $c > 0$, by hypothesis, $0 = \|T_\lambda x_0\| \geq c\|x_0\| > 0$.
As x_0 was any vector orthogonal to $\overline{T_\lambda(H)}$, $\overline{T_\lambda(H)}^\perp = \{0\}$.
Hence $\overline{T_\lambda(H)} = H$. I.e., $T_\lambda(H)$ is dense in H.

Characterization of the Resolvent Set (Converse (iii))

(iii) We prove $y \in \overline{T_\lambda(H)}$ implies $y \in T_\lambda(H)$.

Then $T_\lambda(H)$ is closed and $T_\lambda(H) = H$ by Part (ii).

Let $y \in \overline{T_\lambda(H)}$.

Then, there is a sequence (y_n) in $T_\lambda(H)$, which converges to y.

Since $y_n \in T_\lambda(H)$, we have $y_n = T_\lambda x_n$, for some $x_n \in H$.

By the hypothesis,

$$\|x_n - x_m\| \leq \frac{1}{c}\|T_\lambda(x_n - x_m)\| = \frac{1}{c}\|y_n - y_m\|.$$

Since (y_n) converges, (x_n) is Cauchy.

Since H is complete, (x_n) converges, say, $x_n \to x$.

Characterization of the Resolvent Set ((iii) Cont'd)

- Since T is continuous, so is T_λ.

  Hence, $y_n = T_\lambda x_n \to T_\lambda x$.

  By definition, $T_\lambda x \in T_\lambda(H)$.

  Since the limit is unique, $T_\lambda x = y$.

  Hence, $y \in T_\lambda(H)$.

  Since $y \in \overline{T_\lambda(H)}$ was arbitrary, $T_\lambda(H)$ is closed.

  We thus have $T_\lambda(H) = H$ by Part (ii).

  This means that $R_\lambda = T_\lambda^{-1}$ is defined on all of H.

  Moreover, by the Bounded Inverse Theorem, it is bounded.

  Hence, $\lambda \in \rho(T)$.

The Spectrum Theorem

Theorem (Spectrum)

The spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space H is real.

- Using the theorem, we show that a $\lambda = \alpha + i\beta$, α, β real, with $\beta \neq 0$ must belong to $\rho(T)$. It will follow that $\sigma(T) \subseteq \mathbb{R}$.

  For every $x \neq 0$ in H, we have $\langle T_\lambda x, x\rangle = \langle Tx, x\rangle - \lambda\langle x, x\rangle$.

  Since $\langle x, x\rangle$ and $\langle Tx, x\rangle$ are real,

  $$\overline{\langle T_\lambda x, x\rangle} = \langle Tx, x\rangle - \overline{\lambda}\langle x, x\rangle.$$

  By subtraction,

  $$\overline{\langle T_\lambda x, x\rangle} - \langle T_\lambda x, x\rangle = (\lambda - \overline{\lambda})\langle x, x\rangle = 2i\beta\|x\|^2.$$

The Spectrum Theorem (Cont'd)

- We found
$$\overline{\langle T_\lambda x, x\rangle} - \langle T_\lambda x, x\rangle = 2i\beta\|x\|^2.$$
The left side is $-2i\mathrm{Im}\langle T_\lambda x, x\rangle$, where Im is the imaginary part.
The latter cannot exceed the absolute value.
Dividing by 2, taking absolute values and applying the Schwarz inequality, we obtain
$$|\beta|\|x\|^2 = |\mathrm{Im}\langle T_\lambda x, x\rangle| \le |\langle T_\lambda x, x\rangle| \le \|T_\lambda x\|\|x\|.$$
Division by $\|x\| \neq 0$ gives $|\beta|\|x\| \le \|T_\lambda x\|$.
If $\beta \neq 0$, then, by a previous theorem, $\lambda \in \rho(T)$.
Hence, if $\lambda \in \sigma(T)$, $\beta = 0$. So λ is real.

Subsection 2

Further Properties of Bounded Self-Adjoint Operators

Spectrum

Theorem (Spectrum)

The spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space H lies in the closed interval $[m, M]$ on the real axis, where

$$m = \inf_{\|x\|=1} \langle Tx, x\rangle, \qquad M = \sup_{\|x\|=1} \langle Tx, x\rangle.$$

- By a previous result, $\sigma(T)$ lies on the real axis.

  We show that any real $\lambda = M + c$, with $c > 0$, belongs to the resolvent set $\rho(T)$.

  Suppose $x \neq 0$ and $v = \|x\|^{-1}x$.

  Then $x = \|x\| v$ and

$$\langle Tx, x\rangle = \|x\|^2 \langle Tv, v\rangle \leq \|x\|^2 \sup_{\|\widetilde{v}\|=1} \langle T\widetilde{v}, \widetilde{v}\rangle = \langle x, x\rangle M.$$

Spectrum (Cont'd)

- Hence, $-\langle Tx,x\rangle \geq -\langle x,x\rangle M$.

  By the Schwarz inequality, we obtain

$$\begin{array}{rcl} \|T_\lambda x\|\|x\| & \geq & -\langle T_\lambda x,x\rangle \\ & = & -\langle Tx,x\rangle + \lambda\langle x,x\rangle \\ & \geq & (-M+\lambda)\langle x,x\rangle \\ & = & c\|x\|^2, \end{array}$$

  where $c = \lambda - M > 0$ by assumption.

  Division by $\|x\|$ yields $\|T_\lambda x\| \geq c\|x\|$.

  Hence, by the Resolvent Set Theorem, $\lambda \in \rho(T)$.

  For a real $\lambda < m$ the idea of proof is the same.

Norm

Theorem (Norm)

For any bounded self-adjoint linear operator T on a complex Hilbert space H we have

$$\|T\| = \max(|m|,|M|) = \sup_{\|x\|=1} |\langle Tx,x\rangle|.$$

- Let $K = \sup_{\|x\|=1} |\langle Tx,x\rangle|$. By the Schwarz inequality,

$$K = \sup_{\|x\|=1} |\langle Tx,x\rangle| \le \sup_{\|x\|=1} \|Tx\| \|x\| = \|T\|.$$

We show, next, that $\|T\| \le K$.

Suppose, first, $Tz = 0$, for all z of norm 1. Then $T = 0$.

In this case, there is nothing to prove.

Norm (Cont'd)

- Consider, next, a z of norm 1, such that $Tz \neq 0$.

  Set $v = \|Tz\|^{1/2} z$ and $w = \|Tz\|^{-1/2} Tz$.

  Then $\|v\|^2 = \|w\|^2 = \|Tz\|$.

  We now set $y_1 = v + w$ and $y_2 = v - w$.

  Then, since T is self-adjoint,

$$\begin{aligned}
\langle Ty_1, y_1\rangle - \langle Ty_2, y_2\rangle &= \langle Tv + Tw, v + w\rangle - \langle Tv - Tw, v - w\rangle \\
&= \langle Tv, v\rangle + \langle Tv, w\rangle + \langle Tw, v\rangle + \langle Tw, w\rangle \\
&\quad - \langle Tv, v\rangle + \langle Tv, w\rangle + \langle Tw, v\rangle - \langle Tw, w\rangle \\
&= 2(\langle Tv, w\rangle + \langle Tw, v\rangle) \\
&= 2(\langle \|Tz\|^{1/2} Tz, \|Tz\|^{-1/2} Tz\rangle \\
&\qquad + \langle \|Tz\|^{-1/2} T^2 z, \|Tz\|^{1/2} z\rangle) \\
&= 2(\langle Tz, Tz\rangle + \langle T^2 z, z\rangle) \\
&= 4\|Tz\|^2.
\end{aligned}$$

Norm (Cont'd)

- Now for every $y \neq 0$ and $x = \|y\|^{-1}y$, we have $y = \|y\|x$. Moreover,

$$|\langle Ty, y\rangle| = \|y\|^2 |\langle Tx, x\rangle| \leq \|y\|^2 \sup_{\|\tilde{x}\|=1} |\langle T\tilde{x}, \tilde{x}\rangle| = K\|y\|^2.$$

So, by the triangle inequality and straightforward calculation,

$$\begin{aligned} |\langle Ty_1, y_1\rangle - \langle Ty_2, y_2\rangle| &\leq |\langle Ty_1, y_1\rangle| + |\langle Ty_2, y_2\rangle| \\ &\leq K(\|y_1\|^2 + \|y_2\|^2) \\ &= K(\|v+w\|^2 + \|v-w\|^2) \\ &= 2K(\|v\|^2 + \|w\|^2) \\ &= 4K\|Tz\|. \end{aligned}$$

Hence $4\|Tz\|^2 \leq 4K\|Tz\|$. So $\|Tz\| \leq K$.

Taking the supremum over all z of norm 1, we obtain $\|T\| \leq K$.

m and M as Spectral Values

Theorem (m and M as Spectral Values)

Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space $H \neq \{0\}$. Let $m = \inf_{\|x\|=1} \langle Tx, x \rangle$, $M = \sup_{\|x\|=1} \langle Tx, x \rangle$. Then m and M are spectral values of T.

- We show that $M \in \sigma(T)$.

  By the spectral mapping theorem, the spectrum of $T + kI$, k a real constant, is obtained from that of T by a translation.

  Moreover, $M \in \sigma(T)$ iff $M + k \in \sigma(T + kI)$.

  Hence, we may assume $0 \leq m \leq M$, without loss of generality.

  By the previous theorem, we have $M = \sup_{\|x\|=1} \langle Tx, x \rangle = \|T\|$.

  By the definition of a supremum, there is a sequence (x_n), such that

  $$\|x_n\| = 1, \ \langle Tx_n, x_n \rangle = M - \delta_n, \ \delta_n \geq 0 \text{ and } \delta_n \to 0.$$

m and M as Spectral Values (Cont'd)

- Then $\|Tx_n\| \leq \|T\| \|x_n\| = \|T\| = M$.

  Since T is self-adjoint,

$$\begin{array}{rcl} \|Tx_n - Mx_n\|^2 & = & \langle Tx_n - Mx_n, Tx_n - Mx_n \rangle \\ & = & \|Tx_n\|^2 - 2M\langle Tx_n, x_n \rangle + M^2 \|x_n\|^2 \\ & \leq & M^2 - 2M(M - \delta_n) + M^2 \\ & = & 2M\delta_n \to 0. \end{array}$$

  Hence, there is no positive c, such that

$$\|T_M x_n\| = \|Tx_n - Mx_n\| \geq c = c\|x_n\|, \quad \|x_n\| = 1.$$

  By a preceding theorem, $\lambda = M$ is not in the resolvent set of T.

  Hence, $M \in \sigma(T)$.

  For $\lambda = m$, the proof is similar.

The Residual Spectrum

Theorem (Residual Spectrum)

The residual spectrum $\sigma_r(T)$ of a bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert space H is empty.

- We show that the assumption $\sigma_r(T) \neq \varnothing$ leads to a contradiction.
  Let $\lambda \in \sigma_r(T)$. By the definition of $\sigma_r(T)$, we have:
  - The inverse of T_λ exists;
  - Its domain $\mathscr{D}(T_\lambda^{-1})$ is not dense in H.

  By the projection theorem, some $y \neq 0$ in H is orthogonal to $\mathscr{D}(T_\lambda^{-1})$.
  But $\mathscr{D}(T_\lambda^{-1})$ is the range of T_λ. Hence, $\langle T_\lambda x, y\rangle = 0$, for all $x \in H$.
  Since λ is real and T is self-adjoint, we have $\langle x, T_\lambda y\rangle = 0$, for all x.
  Taking $x = T_\lambda y$, we get $\|T_\lambda y\|^2 = 0$. So $T_\lambda y = Ty - \lambda y = 0$.
  Since $y \neq 0$, this shows that λ is an eigenvalue of T.
  But this contradicts $\lambda \in \sigma_r(T)$. Hence, $\sigma_r(T) = \varnothing$.

Subsection 3

Positive Operators

Positive Operators on Hilbert Spaces

- We consider the set of all bounded self-adjoint linear operators on a complex Hilbert space H.
- If T is self-adjoint, $\langle Tx, x\rangle$ is real.
- So we may introduce on this set a partial ordering $\leq$ by defining

$$T_1 \leq T_2 \quad \text{if and only if} \quad \langle T_1 x, x\rangle \leq \langle T_2 x, x\rangle, \text{ for all } x \in H.$$

- A bounded self-adjoint linear operator $T : H \to H$ is said to be **positive**, written $T \geq 0$, if and only if $\langle Tx, x\rangle \geq 0$, for all $x \in H$.
- The operator is "nonnegative", but "positive" is the usual term.
- Note that $T_1 \leq T_2$ iff $0 \leq T_2 - T_1$.

Product of Positive Operators

- The sum of positive operators is positive.
- We know that a product (composite) of bounded self-adjoint linear operators is self-adjoint if and only if the operators commute.

Theorem (Product of Positive Operators)

If two bounded self-adjoint linear operators S and T on a Hilbert space H are positive and commute ($ST = TS$), then their product ST is positive.

- We must show that $\langle STx, x\rangle \geq 0$, for all $x \in H$.

  If $S = 0$, this holds.

  Let $S \neq 0$. We proceed in two steps:

  (a) We consider $S_1 = \frac{1}{\|S\|}S$, $S_{n+1} = S_n - S_n^2$, $n = 1, 2, \ldots$. We prove by induction that $0 \leq S_n \leq I$.

  (b) We prove that $\langle STx, x\rangle \geq 0$, for all $x \in H$.

Product of Positive Operators Part (a)

(a) First, we show that the inequality holds for $n = 1$.

The assumption $0 \leq S$ implies $0 \leq S_1$.

By an application of the Schwarz inequality and $\|Sx\| \leq \|S\|\|x\|$, we get

$$\begin{aligned} \langle S_1 x, x\rangle &= \tfrac{1}{\|S\|}\langle Sx, x\rangle \\ &\leq \tfrac{1}{\|S\|}\|Sx\|\|x\| \\ &\leq \|x\|^2 \\ &= \langle Ix, x\rangle. \end{aligned}$$

Product of Positive Operators Part (a) (Cont'd)

- Suppose the inequality holds for an $n = k$, i.e., $0 \leq S_k \leq I$.

  Thus, $0 \leq I - S_k \leq I$.

  Since S_k is self-adjoint, for every $x \in H$, $y = S_k x$,

$$\langle S_k^2(I - S_k)x, x\rangle = \langle (I - S_k)S_k x, S_k x\rangle = \langle (I - S_k)y, y\rangle \geq 0.$$

  By definition this proves $S_k^2(I - S_k) \geq 0$. Similarly, $S_k(I - S_k)^2 \geq 0$.

  By addition and simplification,

$$0 \leq S_k^2(I - S_k) + S_k(I - S_k)^2 = S_k - S_k^2 = S_{k+1}.$$

  Finally, note that $S_k^2 \geq 0$ and $I - S_k \geq 0$.

  Adding, we get $0 \leq I - S_k + S_k^2 = I - S_{k+1}$. Hence, $S_{k+1} \leq I$.

Product of Positive Operators Part (b)

(b) We now show that $\langle STx, x\rangle \geq 0$, for all $x \in H$.

We have

$$\begin{array}{rcl} S_1 & = & S_1^2 + S_2 \\ & = & S_1^2 + S_2^2 + S_3 \\ & = & \cdots \\ & = & S_1^2 + S_2^2 + \cdots + S_n^2 + S_{n+1}. \end{array}$$

Since $S_{n+1} \geq 0$, this implies

$$S_1^2 + \cdots + S_n^2 = S_1 - S_{n+1} \leq S_1.$$

By the self-adjointness of S_j and the definition of $\leq$, we get

$$\sum_{j=1}^{n} \|S_j x\|^2 = \sum_{j=1}^{n} \langle S_j x, S_j x\rangle = \sum_{j=1}^{n} \langle S_j^2 x, x\rangle \leq \langle S_1 x, x\rangle.$$

Since n is arbitrary, the infinite series $\|S_1 x\|^2 + \|S_2 x\|^2 + \cdots$ converges.

Hence $\|S_n x\| \to 0$. Therefore, $S_n x \to 0$.

Product of Positive Operators Part (b) (Cont'd)

- We obtained:
  - $S_1^2 + \cdots + S_n^2 = S_1 - S_{n+1}$;
  - $S_n x \to 0$.

  Hence,
  $$\left(\sum_{j=1}^{n} S_j^2\right) x = (S_1 - S_{n+1}) x \to S_1 x.$$
  All the S_j's commute with T, since they are sums and products of $S_1 = \frac{1}{\|S\|} S$ and S and T commute.
  Using $S = \|S\| S_1$, the preceding formula, $T \geq 0$ and the continuity of the inner product, we obtain, for every $x \in H$ and $y_j = S_j x$,
  $$\begin{array}{rcl} \langle STx, x\rangle & = & \|S\| \langle TS_1 x, x\rangle \\ & = & \|S\| \lim_{n\to\infty} \sum_{j=1}^{n} \langle TS_j^2 x, x\rangle \\ & = & \|S\| \lim_{n\to\infty} \sum_{j=1}^{n} \langle Ty_j, y_j\rangle \\ & \geq & 0. \end{array}$$

Monotone Sequences

Definition (Monotone Sequence)

A **monotone sequence** (T_n) of self-adjoint linear operators T_n on a Hilbert space H is a sequence (T_n) satisfying one of the following:

- It is **monotone increasing**, that is,

$$T_1 \leq T_2 \leq T_3 \leq \cdots;$$

- It is **monotone decreasing**, that is,

$$T_1 \geq T_2 \geq T_3 \geq \cdots.$$

The Monotone Sequence Theorem

Theorem (Monotone Sequence)

Let (T_n) be a sequence of bounded self-adjoint linear operators on a complex Hilbert space H, such that

$$T_1 \leq T_2 \leq \cdots \leq T_n \leq \cdots \leq K,$$

where K is a bounded self-adjoint linear operator on H.
Suppose that any T_j commutes with K and with every T_m.
Then (T_n) is strongly operator convergent ($T_n x \to Tx$, for all $x \in H$).
The limit operator T is linear, bounded, self-adjoint and satisfies $T \leq K$.

- We consider $S_n = K - T_n$ and prove:
  - (a) The sequence $(\langle S_n^2 x, x \rangle)$ converges, for every $x \in H$.
  - (b) $T_n x \to Tx$, where T is linear and self-adjoint, and is bounded by the Uniform Boundedness Theorem.

The Monotone Sequence Theorem Part (a)

(a) Clearly, $S_n = K - T_n$ is self-adjoint. We have

$$S_m^2 - S_n S_m = (S_m - S_n) S_m = (T_n - T_m)(K - T_m).$$

Let $m < n$. Then $T_n - T_m$ and $K - T_m$ are positive. Since these operators commute, by the theorem, their product is positive. Hence on the left, $S_m^2 - S_n S_m \geq 0$. I.e., $S_m^2 \geq S_n S_m$, for $m < n$. Similarly,

$$S_n S_m - S_n^2 = S_n(S_m - S_n) = (K - T_n)(T_n - T_m) \geq 0.$$

So $S_n S_m \geq S_n^2$. Taken together, $S_m^2 \geq S_n S_m \geq S_n^2$, $m < n$. By definition, using the self-adjointness of S_n, we have

$$\langle S_m^2 x, x \rangle \geq \langle S_n S_m x, x \rangle \geq \langle S_n^2 x, x \rangle = \langle S_n x, S_n x \rangle = \|S_n x\|^2 \geq 0.$$

This shows that $(\langle S_n^2 x, x \rangle)$, with fixed x, is a monotone decreasing sequence of nonnegative numbers. Hence, it converges.

The Monotone Sequence Theorem Part (b)

(b) We show that $(T_n x)$ converges.

By assumption, every T_n commutes with every T_m and with K.

Hence, the S_j's all commute.

These operators are self-adjoint.

For $m < n$, we have $-2\langle S_m S_n x, x\rangle \le -2\langle S_n^2 x, x\rangle$.

Thus, we obtain

$$\begin{array}{rcl} \|S_m x - S_n x\|^2 & = & \langle (S_m - S_n)x, (S_m - S_n)x\rangle \\ & = & \langle (S_m - S_n)^2 x, x\rangle \\ & = & \langle S_m^2 x, x\rangle - 2\langle S_m S_n x, x\rangle + \langle S_n^2 x, x\rangle \\ & \le & \langle S_m^2 x, x\rangle - \langle S_n^2 x, x\rangle. \end{array}$$

From this and Part (a), $(S_n x)$ is Cauchy.

It converges since H is complete.

The Monotone Sequence Theorem Part (b) (Cont'd)

- Now $T_n = K - S_n$.

  Since $(S_n x)$ converges, $(T_n x)$ also converges.

  Clearly, the limit depends on x.

  So we can write $T_n x \to Tx$, for every $x \in H$.

  Hence, this defines an operator $T : H \to H$, which is linear.

  T is self-adjoint because T_n is self-adjoint and the inner product is continuous.

  Since $(T_n x)$ converges, it is bounded for every $x \in H$.

  The Uniform Boundedness Theorem now implies that T is bounded.

  Finally, $T \leq K$ follows from $T_n \leq K$.

Subsection 4

Square Roots of a Positive Operator

Positive Square Root

- Let T be self-adjoint.
- Then T^2 is positive, since $\langle T^2x, x\rangle = \langle Tx, Tx\rangle \geq 0$.
- The converse problem consists of, given a positive operator T, finding a self-adjoint A such that $A^2 = T$.

Definition (Positive Square Root)

Let $T : H \to H$ be a positive bounded self-adjoint linear operator on a complex Hilbert space H. Then a bounded self-adjoint linear operator A is called a **square root** of T if

$$A^2 = T.$$

If, in addition, $A \geq 0$, then A is called a **positive square root** of T, denoted by $A = T^{1/2}$.

The Positive Square Root Theorem

Theorem (Positive Square Root)

Every positive bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert space H has a positive square root A, which is unique. This operator A commutes with every bounded linear operator on H which commutes with T.

- We proceed in three steps:
  (a) We show that if the theorem holds under the additional assumption $T \leq I$, it also holds without that assumption.
  (b) We obtain the existence of the operator $A = T^{1/2}$ from $A_n x \to Ax$, where $A_0 = 0$ and $A_{n+1} = A_n + \frac{1}{2}(T - A_n^2)$, $n = 0, 1, \ldots$. We also prove the commutativity stated in the theorem.
  (c) We prove uniqueness of the positive square root.

Positive Square Root Part (a)

(a) If $T = 0$, we can take $A = T^{1/2} = 0$.
Let $T \neq 0$. By the Schwarz inequality,

$$\langle Tx, x\rangle \leq \|Tx\| \|x\| \leq \|T\| \|x\|^2.$$

Dividing by $\|T\| \neq 0$ and setting $Q = \frac{1}{\|T\|} T$, we obtain

$$\langle Qx, x\rangle \leq \|x\|^2 = \langle Ix, x\rangle.$$

I.e., $Q \leq I$.
Suppose Q has a unique positive square root $B = Q^{1/2}$. Then $B^2 = Q$.
Moreover, we have

$$(\|T\|^{1/2} B)^2 = \|T\| B^2 = \|T\| Q = T.$$

So a square root of $T = \|T\| Q$ is $\|T\|^{1/2} B$. Also, uniqueness of $Q^{1/2}$ implies uniqueness of the positive square root of T.
Hence, it suffices to prove the theorem under the additional assumption $T \leq I$.

Positive Square Root Part (b)

(b) (Existence) Consider

$$\begin{aligned} A_0 &= 0; \\ A_{n+1} &= A_n + \tfrac{1}{2}(T - A_n^2), \quad n = 0, 1, \ldots. \end{aligned}$$

Since $A_0 = 0$, we have

$$A_1 = \frac{1}{2}T, \quad A_2 = T - \frac{1}{8}T^2, \quad \text{etc..}$$

Each A_n is a polynomial in T.

Hence, the A_n's are self-adjoint and all commute.

They also commute with every operator that T commutes with.

We now prove:

(i) $A_n \le I$, $n = 0, 1, \ldots$;

(ii) $A_n \le A_{n+1}$, $n = 0, 1, \ldots$;

(iii) $A_n x \to Ax$, $A = T^{1/2}$;

(iv) $ST = TS$ implies $AS = SA$, where S is a bounded linear operator on H.

Positive Square Root Part (b) (i)

(i) We have $A_0 \leq I$.

Let $n > 0$.

Since $I - A_{n-1}$ is self-adjoint,

$$(I - A_{n-1})^2 \geq 0.$$

Also, $T \leq I$ implies $I - T \geq 0$.

From this, we obtain

$$\begin{array}{rcl} 0 & \leq & \frac{1}{2}(I - A_{n-1})^2 + \frac{1}{2}(I - T) \\ & = & I - A_{n-1} - \frac{1}{2}(T - A_{n-1}^2) \\ & = & I - A_n. \end{array}$$

Positive Square Root Part (b) (ii)

(ii) We use induction.

We have

$$0 = A_0 \leq A_1 = \frac{1}{2}T.$$

We show that $A_{n-1} \leq A_n$, for any fixed n, implies $A_n \leq A_{n+1}$.

We calculate directly

$$\begin{aligned} A_{n+1} - A_n &= A_n + \tfrac{1}{2}(T - A_n^2) - A_{n-1} - \tfrac{1}{2}(T - A_{n-1}^2) \\ &= (A_n - A_{n-1})[I - \tfrac{1}{2}(A_n + A_{n-1})]. \end{aligned}$$

Here $A_n - A_{n-1} \geq 0$, by hypothesis, and the bracket is ≥ 0 by (i).

Hence, $A_{n+1} - A_n \geq 0$.

Positive Square Root Part (b) (iii) and (iv)

(iii) (A_n) is monotone by (ii) and $A_n \leq I$ by (i).

Hence, a previous theorem implies the existence of a bounded self-adjoint linear operator A, such that $A_n x \to Ax$, for all $x \in H$.

Since $(A_n x)$ converges,

$$\frac{1}{2}(Tx - A_n^2 x) = A_{n+1}x - A_n x \to 0.$$

Hence, $Tx - A^2 x = 0$, for all x. I.e., $T = A^2$.

Also $A \geq 0$, because $0 = A_0 \leq A_n$ by (ii).

I.e., $\langle A_n x, x \rangle \geq 0$, for every $x \in H$.

By the continuity of the inner product, $\langle Ax, x \rangle \geq 0$, for every $x \in H$.

(iv) We know that $ST = TS$ implies $A_n S = S A_n$.

I.e., $A_n S x = S A_n x$, for all $x \in H$.

Letting $n \to \infty$, we obtain (iv).

Positive Square Root Part (c)

(c) (Uniqueness) Let both A and B be positive square roots of T.
Then $A^2 = B^2 = T$. Also

$$BT = BB^2 = B^2B = TB.$$

So, by (iv), $AB = BA$.
Let $x \in H$ be arbitrary and $y = (A - B)x$.
Then $\langle Ay, y\rangle \geq 0$ and $\langle By, y\rangle \geq 0$ because $A \geq 0$ and $B \geq 0$.
Using $AB = BA$ and $A^2 = B^2$, we obtain

$$\langle Ay, y\rangle + \langle By, y\rangle = \langle (A+B)y, y\rangle = \langle (A^2 - B^2)x, y\rangle = 0.$$

Hence $\langle Ay, y\rangle = \langle By, y\rangle = 0$.

Positive Square Root Part (c) (Cont'd)

- Since $A \geq 0$ and A is self-adjoint, it has itself a positive square root C, that is, $C^2 = A$ and C is self-adjoint.

  We thus obtain

  $$0 = \langle Ay, y \rangle = \langle C^2 y, y \rangle = \langle Cy, Cy \rangle = \|Cy\|^2.$$

  So $Cy = 0$. Moreover,

  $$Ay = C^2 y = C(Cy) = 0.$$

  Similarly, $By = 0$. Hence, $(A - B)y = 0$.

  Using $y = (A - B)x$, we thus have, for all $x \in H$,

  $$\|Ax - Bx\|^2 = \langle (A - B)^2 x, x \rangle = \langle (A - B)y, x \rangle = 0.$$

  This shows that $Ax - Bx = 0$, for all $x \in H$. So $A = B$.

Subsection 5

Projection Operators

Orthogonal Projections

- A Hilbert space H can be represented as the direct sum of a closed subspace Y and its orthogonal complement $Y^{\perp}$:

$$\begin{array}{rcl} H & = & Y \oplus Y^{\perp}; \\ x & = & y + z, \qquad y \in Y, z \in Y^{\perp}. \end{array}$$

- Since the sum is direct, y is unique, for any given $x \in H$.
- Hence this representation defines a linear operator

$$\begin{array}{rrcl} P: & H & \to & H \\ & x & \mapsto & y = Px. \end{array}$$

- P is called an **orthogonal projection** or **projection** on H.
- More specifically, P is called the **projection of H onto Y**.

Orthogonal Projections (Cont'd)

- A linear operator $P : H \to H$ is a projection on H if there is a closed subspace Y of H, such that:
  - Y is the range of P;
  - $Y^{\perp}$ is the null space of P;
  - $P|_Y$ is the identity operator on Y.
- Note that, with this notation, we can now write

$$x = y + z = Px + (I - P)x.$$

- So the projection of H onto $Y^{\perp}$ is $I - P$.

The Projection Theorem

Theorem (Projection)

A bounded linear operator $P : H \to H$ on a Hilbert space H is a projection if and only if P is self-adjoint and idempotent (that is, $P^2 = P$).

(a) Suppose that P is a projection on H and denote $P(H)$ by Y.

For every $x \in H$ and $Px = y \in Y$, we have

$$P^2 x = Py = y = Px.$$

Hence, $P^2 = P$.

Let $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$, where $y_1, y_2 \in Y$ and $z_1, z_2 \in Y^\perp$.

Then, since $Y \perp Y^\perp$, $\langle y_1, z_2 \rangle = \langle y_2, z_1 \rangle = 0$. So we have

$$\langle Px_1, x_2 \rangle = \langle y_1, y_2 + z_2 \rangle = \langle y_1, y_2 \rangle = \langle y_1 + z_1, y_2 \rangle = \langle x_1, Px_2 \rangle.$$

Hence, P is self-adjoint.

The Projection Theorem (Converse)

(b) Conversely, suppose that $P^2 = P = P^*$ and denote $P(H)$ by Y.
Then, for every $x \in H$,

$$x = Px + (I - P)x.$$

The orthogonality $Y = P(H) \perp (I - P)(H)$ follows from

$$\langle Px, (I-P)v\rangle = \langle x, P(I-P)v\rangle = \langle x, Pv - P^2 v\rangle = \langle x, 0\rangle = 0.$$

We show Y is the null space $\mathcal{N}(I - P)$ of $I - P$.

- $Y \subseteq \mathcal{N}(I-P)$: $(I-P)Px = Px - P^2 x = 0$;
- $Y \supseteq \mathcal{N}(I-P)$: $(I-P)x = 0$ implies $x = Px$.

Hence, Y is closed.

Finally, writing $y = Px$, we have

$$Py = P^2 x = Px = y.$$

Therefore, $P|_Y$ is the identity operator on Y.

Spectral Representations

- We attempt to represent complicated linear operators on Hilbert spaces in terms of simple operators, such as projections.
- The resulting representation is called a **spectral representation** of the operator because the projections employed for that purpose are related to the spectrum of the operator.
- For a spectral representation of bounded self-adjoint linear operators:
  - The first step is a thorough investigation of general properties of projections.
  - The second step is the definition of projections suitable for that purpose.
    These are one-parameter families of projections, called **spectral families**.
  - The third step associates with a given bounded self-adjoint linear operator T a spectral family in a unique way.
    This is called the **spectral family associated with** T.

Positivity and Norm of Projections

Theorem (Positivity, Norm)

For any projection P on a Hilbert space H:

(a) $\langle Px, x\rangle = \|Px\|^2$;

(b) $P \geq 0$;

(c) $\|P\| \leq 1$; $\|P\| = 1$ if $P(H) \neq \{0\}$.

- (a) and (b) follow from

$$\langle Px, x\rangle = \langle P^2x, x\rangle = \langle Px, Px\rangle = \|Px\|^2 \geq 0.$$

By the Schwarz inequality,

$$\|Px\|^2 = \langle Px, x\rangle \leq \|Px\|\|x\|.$$

So $\frac{\|Px\|}{\|x\|} \leq 1$, for every $x \neq 0$. Hence, $\|P\| \leq 1$.

If $x \in P(H)$ and $x \neq 0$, $\frac{\|Px\|}{\|x\|} = 1$. This proves (c).

Product of Projections

Theorem (Product of Projections)

In connection with products (composites) of projections on a Hilbert space H, the following two statements hold:

(a) $P = P_1 P_2$ is a projection on H if and only if the projections P_1 and P_2 commute, that is, $P_1 P_2 = P_2 P_1$. Then P projects H onto $Y = Y_1 \cap Y_2$, where $Y_j = P_j(H)$.

(b) Two closed subspaces Y and V of H are orthogonal if and only if the corresponding projections satisfy $P_Y P_V = 0$.

(a) Suppose that $P_1 P_2 = P_2 P_1$.
Then P is self-adjoint, by a previous theorem.
Moreover, P is idempotent, since

$$P^2 = (P_1 P_2)(P_1 P_2) = P_1^2 P_2^2 = P_1 P_2 = P.$$

Hence P is a projection.

Product of Projections (Cont'd)

- For every $x \in H$, we have $Px = P_1(P_2x) = P_2(P_1x)$.

  Since P_1 projects H onto Y_1, we must have $P_1(P_2x) \in Y_1$. Similarly, $P_2(P_1x) \in Y_2$. Together, $Px \in Y_1 \cap Y_2$. Since $x \in H$ was arbitrary, this shows that P projects H into $Y = Y_1 \cap Y_2$.

  P projects H onto Y: Suppose $y \in Y$. Then $y \in Y_1$ and $y \in Y_2$. Thus, $Py = P_1P_2y = P_1y = y$.

  Conversely, suppose $P = P_1P_2$ is a projection defined on H.

  Then P is self-adjoint. By a previous theorem, $P_1P_2 = P_2P_1$.

(b) Suppose $Y \perp V$. Then $Y \cap V = \{0\}$. Hence, $P_YP_Vx = 0$, for all $x \in H$, by part (a). So $P_YP_V = 0$.

Conversely, suppose $P_YP_V = 0$. Then, for every $y \in Y$ and $v \in V$,

$$\langle y, v\rangle = \langle P_Y y, P_V v\rangle = \langle y, P_Y P_V v\rangle = \langle y, 0\rangle = 0.$$

Hence, $Y \perp V$.

Sum of Projections

Theorem (Sum of Projections)

Let P_1 and P_2 be projections on a Hilbert space H. Then:

(a) The sum $P = P_1 + P_2$ is a projection on H if and only if $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ are orthogonal.

(b) If $P = P_1 + P_2$ is a projection, P projects H onto $Y = Y_1 \oplus Y_2$.

(a) If $P = P_1 + P_2$ is a projection, $P = P^2$. Expanding, we get

$$\begin{aligned} P_1 + P_2 &= (P_1 + P_2)^2 \\ &= P_1^2 + P_1 P_2 + P_2 P_1 + P_2^2 \\ &= P_1 + P_1 P_2 + P_2 P_1 + P_2. \end{aligned}$$

Hence, $P_1 P_2 + P_2 P_1 = 0$.

Sum of Projections Part (a) (Cont'd)

- We obtained $P_1P_2 + P_2P_1 = 0$.
  Multiplying by P_2 on the left, we obtain $P_2P_1P_2 + P_2P_1 = 0$.
  Multiplying this by P_2 on the right, we have $2P_2P_1P_2 = 0$.
  So $P_2P_1 = 0$. Hence, $Y_1 \perp Y_2$.
  Conversely, suppose $Y_1 \perp Y_2$.
  Then $P_1P_2 = P_2P_1 = 0$.
  This yields $P_1P_2 + P_2P_1 = 0$.
  So we get $P^2 = P$.
  Since P_1 and P_2 are self-adjoint, so is $P = P_1 + P_2$.
  Hence, P is a projection.

Sum of Projections Part (b)

(b) We determine the closed subspace $Y \subseteq H$ onto which P projects. Since $P = P_1 + P_2$, we have, for every $x \in H$,

$$y = Px = P_1 x + P_2 x.$$

Here, $P_1 x \in Y_1$ and $P_2 x \in Y_2$.

Hence $y \in Y_1 \oplus Y_2$. So $Y \subseteq Y_1 \oplus Y_2$.

We show that $Y \supseteq Y_1 \oplus Y_2$.

Let $v \in Y_1 \oplus Y_2$ be arbitrary.

Then $v = y_1 + y_2$, with $y_1 \in Y_1$ and $y_2 \in Y_2$.

Applying P and using $Y_1 \perp Y_2$, we obtain

$$Pv = P_1(y_1 + y_2) + P_2(y_1 + y_2) = P_1 y_1 + P_2 y_2 = y_1 + y_2 = v.$$

Hence, $v \in Y$. So $Y \supseteq Y_1 \oplus Y_2$.

Subsection 6

Further Properties of Projections

Partial Order on the Set of all Projections

Theorem (Partial Order)

Let P_1 and P_2 be projections defined on a Hilbert space H. Denote by $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ the subspaces onto which H is projected by P_1 and P_2. Let $\mathscr{N}(P_1)$ and $\mathscr{N}(P_2)$ be the null spaces of these projections. Then the following conditions are equivalent:

(1) $P_2 P_1 = P_1 P_2 = P_1$;

(2) $Y_1 \subseteq Y_2$;

(3) $\mathscr{N}(P_1) \supseteq \mathscr{N}(P_2)$;

(4) $\|P_1 x\| \leq \|P_2 x\|$, for all $x \in H$;

(5) $P_1 \leq P_2$.

(1)⇒(4): We have $\|P_1\| \leq 1$. Hence (1) yields, for all $x \in H$,

$$\|P_1 x\| = \|P_1 P_2 x\| \leq \|P_1\| \|P_2 x\| \leq \|P_2 x\|.$$

Partial Order on the Set of all Projections (Cont'd)

(4)⇒(5): We have, for all $x \in H$,

$$\langle P_1 x, x \rangle = \| P_1 x \|^2 \leq \| P_2 x \|^2 = \langle P_2 x, x \rangle.$$

This shows that $P_1 \leq P_2$, by definition.

(5)⇒(3): Let $x \in \mathcal{N}(P_2)$. Then $P_2 x = 0$. By hypothesis,

$$\| P_1 x \|^2 = \langle P_1 x, x \rangle \leq \langle P_2 x, x \rangle = 0.$$

Hence, $P_1 x = 0$. So $x \in \mathcal{N}(P_1)$. This shows that $\mathcal{N}(P_1) \supseteq \mathcal{N}(P_2)$.

(3)⇒(2): Note that $\mathcal{N}(P_j)$ is the orthogonal complement of Y_j in H.

(2)⇒(1): For every $x \in H$, we have $P_1 x \in Y_1$.

Hence, by hypothesis, $P_1 x \in Y_2$. So $P_2(P_1 x) = P_1 x$. I.e., $P_2 P_1 = P_1$.

Since P_1 is self-adjoint, by a preceding result, $P_1 = P_2 P_1 = P_1 P_2$.

Difference of Projections

Theorem (Difference of Projections)

Let P_1 and P_2 be projections on a Hilbert space H. Then:

(a) The difference $P = P_2 - P_1$ is a projection on H if and only if $Y_1 \subseteq Y_2$, where $Y_j = P_j(H)$.

(b) If $P = P_2 - P_1$ is a projection, P projects H onto Y, where Y is the orthogonal complement of Y_1 in Y_2.

(a) If $P = P_2 - P_1$ is a projection, $P = P^2$. Expanding

$$\begin{aligned} P_2 - P_1 &= (P_2 - P_1)^2 \\ &= P_2^2 - P_2 P_1 - P_1 P_2 + P_1^2 \\ &= P_2 - P_2 P_1 - P_1 P_2 + P_1. \end{aligned}$$

Hence $P_1 P_2 + P_2 P_1 = 2P_1$.

Difference of Projections Part (a) (Cont'd)

- We got $P_1P_2 + P_2P_1 = 2P_1$.

  Multiplication by P_2 from left and right gives

  $$P_2P_1P_2 + P_2P_1 = 2P_2P_1 \quad \text{and} \quad P_1P_2 + P_2P_1P_2 = 2P_1P_2.$$

  Hence, we get

  $$P_2P_1P_2 = P_2P_1 \quad \text{and} \quad P_2P_1P_2 = P_1P_2.$$

  So $P_2P_1 = P_1P_2 = P_1$. Thus, $Y_1 \subseteq Y_2$.

  Conversely, suppose $Y_1 \subseteq Y_2$.

  Then $P_2P_1 = P_1P_2 = P_1$. This implies $P_1P_2 + P_2P_1 = 2P_1$.

  Thus, P is idempotent.

  Since P_1 and P_2 are self-adjoint, $P = P_2 - P_1$ is self-adjoint.

  So P is a projection.

Difference of Projections Part (b)

(b) $Y = P(H)$ consists of all vectors of the form

$$y = Px = P_2x - P_1x, \quad x \in H.$$

Since $Y_1 \subseteq Y_2$, by Part (a), we have $P_2P_1 = P_1$. Thus,

$$P_2y = P_2^2x - P_2P_1x = P_2x - P_1x = y.$$

This shows that $y \in Y_2$. Moreover,

$$P_1y = P_1P_2x - P_1^2x = P_1x - P_1x = 0.$$

This shows that $y \in \mathcal{N}(P_1) = Y_1^\perp$. So $Y \subseteq Y_2 \cap Y_1^\perp$.

Difference of Projections Part (b) (Cont'd)

- We show, next, that $Y \supseteq Y_2 \cap Y_1^\perp$.

  The projection of H onto $Y_1^\perp$ is $I - P_1$.

  So every $v \in Y_2 \cap Y_1^\perp$ is of the form $v = (I - P_1)y_2$, $y_2 \in Y_2$.

  Using again $P_2 P_1 = P_1$, we obtain, since $P_2 y_2 = y_2$,

$$\begin{array}{rcl} Pv & = & (P_2 - P_1)(I - P_1)y_2 \\ & = & (P_2 - P_2 P_1 - P_1 + P_1^2)y_2 \\ & = & y_2 - P_1 y_2 \\ & = & Y_2 \cap Y_1^\perp. \end{array}$$

  This shows that $v \in Y$. Hence, $Y \supseteq Y_2 \cap Y_1^\perp$.

  We conclude that $Y = P(H) = Y_2 \cap Y_1^\perp$.

Monotone Increasing Sequence

Theorem (Monotone Increasing Sequence)

Let (P_n) be a monotone increasing sequence of projections P_n defined on a Hilbert space H. Then:

(a) (P_n) is strongly operator convergent, say, $P_n x \to Px$, for every $x \in H$, and the limit operator P is a projection defined on H.

(b) P projects H onto $P(H) = \overline{\bigcup_{n=1}^{\infty} P_n(H)}$.

(c) P has the null space $\mathcal{N}(P) = \bigcap_{n=1}^{\infty} \mathcal{N}(P_n)$.

(a) Let $m < n$. By assumption, $P_m \le P_n$. So $P_m(H) \subseteq P_n(H)$.
By the previous theorem, $P_n - P_m$ is a projection.

- Hence, for every fixed $x \in H$, we obtain

$$\begin{aligned} \|P_n x - P_m x\|^2 &= \|(P_n - P_m)x\|^2 = \langle (P_n - P_m)x, x\rangle \\ &= \langle P_n x, x\rangle - \langle P_m x, x\rangle = \|P_n x\|^2 - \|P_m x\|^2. \end{aligned}$$

Monotone Increasing Sequence Part (a) (Cont'd)

- Now $\|P_n\| \leq 1$. So $\|P_n x\| \leq \|x\|$, for every n.
  Hence $(\|P_n x\|)$ is a bounded sequence of numbers.
  $(\|P_n\|)$ is also monotone since (P_n) is monotone.
  Hence $(\|P_n x\|)$ converges.
  From this and the preceding equality, $(P_n x)$ is Cauchy.
  Since H is complete, $(P_n x)$ converges.
  The limit depends on x, say, $P_n x \to Px$.
  This defines an operator P on H.
  Linearity of P is obvious.
  Since $P_n x \to Px$ and the P_n's are bounded, self-adjoint and idempotent, P has the same properties.
  Hence, by the Projection Theorem, P is a projection.

Monotone Increasing Sequence Part (b)

(b) We determine $P(H)$. Let $m < n$. Then $P_m \leq P_n$.

This gives $P_n - P_m \geq 0$. So $\langle (P_n - P_m)x, x \rangle \geq 0$, by definition.

As $n \to \infty$, by continuity of the inner product, $\langle (P - P_m)x, x \rangle \geq 0$.

So $P_m \leq P$. Hence, $P_m(H) \subseteq P(H)$, for all m. So $\bigcup P_m(H) \subseteq P(H)$.

Now, for all m and all $x \in H$, $P_m x \in P_m(H) \subseteq \bigcup P_m(H)$.

Since $P_m x \to Px$, we see that $Px \in \overline{\bigcup P_m(H)}$.

Hence, $P(H) \subseteq \overline{\bigcup P_m(H)}$.

Taken together,

$$\bigcup P_m(H) \subseteq P(H) \subseteq \overline{\bigcup P_m(H)}.$$

Therefore, we have $P(H) = \mathcal{N}(I - P)$. So $P(H)$ is closed.

This proves (b).

Monotone Increasing Sequence Part (c)

(c) We determine $\mathcal{N}(P)$.

By Part (b) of the proof, for all n, $P(H) \supseteq P_n(H)$.

Using a preceding lemma, $\mathcal{N}(P) = P(H)^{\perp} \subseteq P_n(H)^{\perp}$.

Hence, $\mathcal{N}(P) \subseteq \bigcap P_n(H)^{\perp} = \bigcap \mathcal{N}(P_n)$.

On the other hand, suppose $x \in \bigcap \mathcal{N}(P_n)$.

Then $x \in \mathcal{N}(P_n)$, for every n. So $P_n x = 0$.

Moreover, $P_n x \to Px$ implies $Px = 0$.

I.e., $x \in \mathcal{N}(P)$.

Since $x \in \bigcap \mathcal{N}(P_n)$ was arbitrary, $\bigcap \mathcal{N}(P_n) \subseteq \mathcal{N}(P)$.

We, thus, obtain $\mathcal{N}(P) = \bigcap \mathcal{N}(P_n)$.

Subsection 7

Spectral Family

Self-Adjoint Operators on a Unitary Space

- Consider the unitary space (inner product space over $\mathbb{C}$) $H = \mathbb{C}^n$.
- Let $T : H \to H$ be a self-adjoint linear operator on H.
- Then T is bounded.
- Moreover, we may choose a basis for H and represent T by a Hermitian matrix which we denote simply by T.
- The spectrum of the operator consists of the eigenvalues of that matrix which are real.

Spectrum of Self-Adjoint Operators on a Unitary Space

- For simplicity, we assume that the matrix T has n different eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$.
- Then a previous theorem implies that T has an orthonormal set of n eigenvectors $x_1, x_2, \ldots, x_n$, where x_j corresponds to λ_j.
- We write these vectors as column vectors, for convenience.
- This is a basis for H.
- So every $x \in H$ has a unique representation

$$x = \sum_{j=1}^{n} \gamma_j x_j, \quad \gamma_j = \langle x, x_j \rangle = x^\top \overline{x}_j.$$

Spectral Representation of Self-Adjoint Operators

- We obtained the representation

$$x = \sum_{j=1}^{n} \gamma_j x_j, \quad \gamma_j = \langle x, x_j \rangle = x^\top \overline{x}_j.$$

- Since x_j is an eigenvector of T, $Tx_j = \lambda_j x_j$.
- Consequently, we obtain

$$Tx = \sum_{j=1}^{n} \lambda_j \gamma_j x_j.$$

- Thus, whereas T may act on x in a complicated way, it acts on each term of the sum in a very simple fashion.

Spectral Representation of Self-Adjoint Operators (Cont'd)

- We may define an operator

$$\begin{array}{rccc} P_j: & H & \to & H; \\ & x & \mapsto & \gamma_j x_j. \end{array}$$

- Obviously, P_j is the projection (orthogonal projection) of H onto the eigenspace of T corresponding to λ_j.
- We obtain

$$x = \sum_{j=1}^{n} P_j x.$$

- Hence, $I = \sum_{j=1}^{n} P_j$, with I the identity on H.
- We also have

$$Tx = \sum_{j=1}^{n} \lambda_j P_j x.$$

- Hence, $T = \sum_{j=1}^{n} \lambda_j P_j$.

The One-Parameter Family of Projections E_λ

- For any real λ, we define

$$E_\lambda = \sum_{\lambda_j \le \lambda} P_j, \quad \lambda \in \mathbb{R}.$$

- For any λ, the operator E_λ is the projection of H onto the subspace V_λ spanned by all those x_j for which $\lambda_j \le \lambda$.
- Thus $V_\lambda \subseteq V_\mu$, for $\lambda \le \mu$.
- As λ traverses $\mathbb{R}$ in the positive sense, E_λ grows from 0 to I.
  - The growth occurs at the eigenvalues of T;
  - E_λ remains unchanged for λ in any interval that is free of eigenvalues.
- Hence, E_λ has the following properties:
  - $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$, if $\lambda < \mu$;
  - $E_\lambda = 0$, if $\lambda < \lambda_1$;
  - $E_\lambda = I$, if $\lambda \ge \lambda_n$;
  - $E_{\lambda^+} = \lim_{\mu \to \lambda^+} E_\mu = E_\lambda$.

Spectral Family or Decomposition of Unity

Definition (Spectral Family or Decomposition of Unity)

A real **spectral family** (or real **decomposition of unity**) is a one-parameter family $\mathscr{E} = (E_\lambda)_{\lambda \in \mathbb{R}}$ of projections E_λ defined on a Hilbert space H (of any dimension) which depends on a real parameter λ and is such that:

- $E_\lambda \leq E_\mu$, hence $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$, $\lambda < \mu$;
- $\lim\limits_{\lambda \to -\infty} E_\lambda x = 0$, $\lim\limits_{\lambda \to +\infty} E_\lambda x = x$;
- $E_{\lambda^+} x = \lim\limits_{\mu \to \lambda^+} E_\mu x = E_\lambda x$, $x \in H$.

- Thus, a real spectral family can be regarded as a mapping $\mathbb{R} \to B(H,H)$; $\lambda \mapsto E_\lambda$.
- To each $\lambda \in \mathbb{R}$, it associates a projection $E_\lambda \in B(H,H)$, where $B(H,H)$ is the space of all bounded linear operators from H into H.

Spectral Family on an Interval

- $\mathscr{E}$ is called a **spectral family on an interval** $[a,b]$ if

$$E_\lambda = 0, \quad \lambda < a, \qquad E_\lambda = I, \quad \lambda \geq b.$$

- Such families are of particular interest, since the spectrum of a bounded self-adjoint linear operator lies in a finite interval on the real line.
- $\mu \to \lambda^+$ indicates that in this limit process we restrict to values $\mu > \lambda$.
- The condition $\lim_{\mu \to \lambda^+} E_\mu x = E_\lambda x$, $x \in H$, means that $\lambda \mapsto E_\lambda$ is strongly operator continuous from the right.
- We will see that with any given bounded self-adjoint linear operator T on any Hilbert space we can associate a spectral family which may be used for representing T by a Riemann-Stieltjes integral.
- This is known as a **spectral representation**.

The Spectral Representation

- Assume again, for simplicity, that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of T are all different, and $\lambda_1 < \lambda_2 < \cdots < \lambda_n$.
- Then we have:
  - $E_{\lambda_1} = P_1$;
  - $E_{\lambda_2} = P_1 + P_2$;

    $\vdots$
  - $E_{\lambda_n} = P_1 + \cdots + P_n$.
- Hence, conversely,

$$\begin{array}{rcl} P_1 & = & E_{\lambda_1}; \\ P_j & = & E_{\lambda_j} - E_{\lambda_{j-1}}, \quad j = 2, \ldots, n. \end{array}$$

- Note that E_λ remains the same for $\lambda \in [\lambda_{j-1}, \lambda_j)$.
- So we may write

$$P_j = E_{\lambda_j} - E_{\lambda_j^-}.$$

The Spectral Representation (Cont'd)

- Now we have
$$x = \sum_{j=1}^{n} P_j x = \sum_{j=1}^{n} (E_{\lambda_j} - E_{\lambda_j^-})x.$$
- Moreover,
$$Tx = \sum_{j=1}^{n} \lambda_j P_j x = \sum_{j=1}^{n} \lambda_j (E_{\lambda_j} - E_{\lambda_j^-})x.$$
- If we drop the x and write $\delta E_\lambda = E_\lambda - E_{\lambda^-}$, we get
$$T = \sum_{j=1}^{n} \lambda_j \delta E_{\lambda_j}.$$
- This is the **spectral representation** of the self-adjoint operator T with eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ on the n-dimensional Hilbert space H.

Spectral Representation as an Integral

- We obtained the spectral representation

$$T = \sum_{j=1}^{n} \lambda_j \delta E_{\lambda_j}$$

of the self-adjoint linear operator T with eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ on the n-dimensional Hilbert space H.

- The representation shows that for any $x, y \in H$,

$$\langle Tx, y \rangle = \sum_{j=1}^{n} \lambda_j \langle \delta E_{\lambda_j} x, y \rangle.$$

- We note that this may be written as a Riemann-Stieltjes integral

$$\langle Tx, y \rangle = \int_{-\infty}^{+\infty} \lambda dw(\lambda),$$

where $w(\lambda) = \langle E_\lambda x, y \rangle$.

Subsection 8

Spectral Family of a Bounded Self-Adjoint Operator

The Spectral Family of an Operator

- Let H be a complex Hilbert space.
- Let $T: H \to H$ be a bounded self-adjoint linear operator on H.
- With T we can associate a spectral family $\mathscr{E}$ that will be used for a spectral representation of T.
- To define $\mathscr{E}$ we need the following:
  - The operator
  $$T_\lambda = T - \lambda I;$$
  - The positive square root of T_λ^2,
  $$B_\lambda = (T_\lambda^2)^{1/2};$$
  - The operator
  $$T_\lambda^+ = \frac{1}{2}(B_\lambda + T_\lambda),$$
  called the **positive part** of T_λ.
- The **spectral family** $\mathscr{E}$ of T is defined by $\mathscr{E} = (E_\lambda)_{\lambda \in \mathbb{R}}$, where E_λ is the projection of H onto the null space $\mathscr{N}(T_\lambda^+)$ of T_λ^+.

Definition of Operators B, T^+, T^-

- Consider the operators

$$\begin{array}{ll} B = (T^2)^{1/2} & \text{(positive square root of } T^2\text{);} \\ T^+ = \frac{1}{2}(B+T) & \text{(positive part of } T\text{);} \\ T^- = \frac{1}{2}(B-T) & \text{(negative part of } T\text{).} \end{array}$$

- Let E be the projection of H onto the null space of T^+,

$$E : H \to Y = \mathscr{N}(T^+).$$

- By subtraction and addition we see that

$$\begin{array}{rcl} T & = & T^+ - T^-; \\ B & = & T^+ + T^-. \end{array}$$

Properties of the Operators

Lemma (Operators related to T)

The operators just defined have the following properties:

(a) B, T^+ and T^- are bounded and self-adjoint.

(b) B, T^+ and T^- commute with every bounded linear operator that T commutes with; in particular,

$$BT = TB, \quad T^+T = TT^+, \quad T^-T = TT^-, \quad T^+T^- = T^-T^+.$$

(c) E commutes with every bounded self-adjoint linear operator that T commutes with; in particular, $ET = TE$ and $EB = BE$.

(d) Furthermore,

$$\begin{array}{ll} T^+T^- = 0 & T^-T^+ = 0 \\ T^+E = ET^+ = 0 & T^-E = ET^- = T^- \\ TE = -T^- & T(I-E) = T^+ \\ T^+ \geq 0 & T^- \geq 0. \end{array}$$

Proof of Properties (a),(b)

(a) Clear, since T and B are bounded and self-adjoint.

(b) Suppose that $TS = ST$. Then

$$T^2S = TST = ST^2.$$

$BS = SB$ follows from a previous theorem.

Hence,

$$T^+S = \frac{1}{2}(BS + TS) = \frac{1}{2}(SB + ST) = ST^+.$$

The proof of $T^-S = ST^-$ is similar.

Proof of Property (c)

(c) For every $x \in H$, we have $y = Ex \in Y = \mathcal{N}(T^+)$.

Hence, $T^+ y = 0$. And, also, $ST^+ y = S0 = 0$.

From $TS = ST$ and Part (b) we have $ST^+ = T^+ S$ and

$$T^+ SEx = T^+ Sy = ST^+ y = 0.$$

Hence $SEx \in Y$.

But E projects H onto Y.

Thus, $ESEx = SEx$, for every $x \in H$.

That is, $ESE = SE$.

Since a projection is self-adjoint, by a previous result, and so is S,

$$ES = E^* S^* = (SE)^* = (ESE)^* = E^* S^* E^* = ESE = SE.$$

Proof of Properties (d)

(d) We prove all equalities in Part (d):

- From $B = (T^2)^{1/2}$, we have $B^2 = T^2$. Also $BT = TB$ by Part (b). Hence, again by Part (b),

$$T^+ T^- = T^- T^+ = \frac{1}{2}(B - T)\frac{1}{2}(B + T) = \frac{1}{4}(B^2 + BT - TB - T^2) = 0.$$

- By definition, $Ex \in \mathcal{N}(T^+)$. So $T^+ Ex = 0$, for all $x \in H$. Since T^+ is self-adjoint, by Parts (b) and (c),

$$ET^+ x = T^+ Ex = 0.$$

  That is, $ET^+ = T^+ E = 0$.
  By the previous subpart, $T^+ T^- x = 0$. So $T^- x \in \mathcal{N}(T^+)$.
  Hence, $ET^- x = T^- x$. Since T^- is self-adjoint, Part (c) yields

$$T^- Ex = ET^- x = T^- x, \quad x \in H.$$

  That is, $T^- E = ET^- = T^-$.

Proof of Properties (d) (Cont'd)

(d) We continue with the equalities in Part (d):

- From a previous subpart,

$$TE = (T^+ - T^-)E = -T^-.$$

From this,

$$T(I - E) = T - TE = T + T^- = T^+.$$

- Now note that:
  - E and B are self-adjoint and commute;
  - $E \geq 0$, by the Positivity Theorem, and $B \geq 0$, by definition.

So, by a preceding subpart and a preceding theorem,

$$T^- = ET^- + ET^+ = E(T^- + T^+) = EB \geq 0.$$

Similarly, since, by the Positivity Theorem, $I - E \geq 0$,

$$T^+ = B - T^- = B - EB = (I - E)B \geq 0.$$

Operators Related to T_λ

- Instead of T, we now consider $T_\lambda = T - \lambda I$.
- Instead of B, T^+, T^- and E we now have to take:
  - The positive square root of T_λ^2,
$$B_\lambda := (T_\lambda^2)^{1/2};$$
  - The positive part and negative part of T_λ, defined by
$$T_\lambda^+ = \frac{1}{2}(B_\lambda + T_\lambda) \quad \text{and} \quad T_\lambda^- = \frac{1}{2}(B_\lambda - T_\lambda);$$
  - The projection
$$E_\lambda : H \to Y_\lambda = \mathcal{N}(T_\lambda^+)$$
of H onto the null space $Y_\lambda = \mathcal{N}(T_\lambda^+)$ of T_λ^+.

Properties of the Operators Related to T_λ

Lemma (Operators Related to T_λ)

The previous lemma remains true if we replace T, B, T^+, T^-, E by $T_\lambda B_\lambda, T_\lambda^+, T_\lambda^-, E_\lambda$, respectively, where λ is real. Moreover, for any real $\kappa, \lambda, \mu, \nu, \tau$, the following operators all commute: $T_\kappa, B_\lambda, T_\mu^+, T_\nu^-, E_\tau$.

- The first statement is obvious. We turn to the second statement. Note that $IS = SI$ and

$$T_\lambda = T - \lambda I = T - \mu I + (\mu - \lambda) I = T_\mu + (\mu - \lambda) I.$$

Hence,

$$\begin{array}{rll} ST = TS & \text{implies} & ST_\mu = T_\mu S \\ & \text{implies} & ST_\lambda = T_\lambda S \\ & \text{implies} & SB_\lambda = B_\lambda S, SB_\mu = B_\mu S \\ & & \vdots \end{array}$$

For $S = T_\kappa$, we get $T_\kappa B_\lambda = B_\lambda T_\kappa, \ldots$.

Spectral Family Associated with an Operator

Theorem (Spectral Family Associated with an Operator)

Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Furthermore, let E_λ (λ real) be the projection of H onto the null space $Y_\lambda = \mathcal{N}(T_\lambda^+)$ of the positive part T_λ^+ of $T_\lambda = T - \lambda I$. Then $\mathscr{E} = (E_\lambda)_{\lambda \in \mathbb{R}}$ is a spectral family on the interval $[m, M] \subseteq \mathbb{R}$, where $m = \inf_{\|x\|=1} \langle Tx, x \rangle$ and $M = \sup_{\|x\|=1} \langle Tx, x \rangle$.

- $\mathscr{E} = (E_\lambda)$ is called the **spectral family associated with** T.
- We shall prove:
  - (i) $\lambda < \mu$ implies $E_\lambda \leq E_\mu$;
  - (ii) $\lambda < m$ implies $E_\lambda = 0$;
  - (iii) $\lambda \geq M$ implies $E_\lambda = I$;
  - (iv) $\lim\limits_{\mu \to \lambda^+} E_\mu x = E_\lambda x$.

Spectral Family Associated with an Operator (Proof)

- In the proof we use the following properties:
  (a) $T_\lambda E_\lambda = -T_\lambda^-$;
  (b) $T_\lambda(I - E_\lambda) = T_\lambda^+$;
  (c) $T_\lambda^+ \geq 0$;
  (d) $T_\lambda^- \geq 0$;
  (e) $T_\mu^+ T_\mu^- = 0$;
  (f) $T_\mu E_\mu = -T_\mu^-$;
  (g) $T_\mu^+ \geq 0$;
  (h) $T_\mu^- \geq 0$.

Proof of Property (i)

- Let $\lambda < \mu$. Since $-T_\lambda^- \leq 0$, we have $T_\lambda = T_\lambda^+ - T_\lambda^- \leq T_\lambda^+$. Hence,

$$T_\lambda^+ - T_\mu \geq T_\lambda - T_\mu = (\mu - \lambda)I \geq 0.$$

$T_\lambda^+ - T_\mu$ is self-adjoint and commutes with T_μ^+. Also $T_\mu^+ \geq 0$.
A previous theorem, thus, implies

$$T_\mu^+(T_\lambda^+ - T_\mu) = T_\mu^+(T_\lambda^+ - T_\mu^+ + T_\mu^-) \geq 0.$$

We have $T_\mu^+ T_\mu^- = 0$, by one of the preceding identities.
Hence, $T_\mu^+ T_\lambda^+ \geq T_\mu^{+2}$. I.e., for all $x \in H$,

$$\langle T_\mu^+ T_\lambda^+ x, x \rangle \geq \langle T_\mu^{+2} x, x \rangle = \| T_\mu^+ x \|^2 \geq 0.$$

This shows that $T_\lambda^+ x = 0$ implies $T_\mu^+ x = 0$.
Hence, $\mathcal{N}(T_\lambda^+) \subseteq \mathcal{N}(T_\mu^+)$.
So, by the Partial Order Theorem, $E_\lambda \leq E_\mu$.

Proof of Property (ii)

- Let $\lambda < m$ but that, nevertheless, $E_\lambda \neq 0$.

  Then $E_\lambda z \neq 0$, for some z.

  We set $x = E_\lambda z$. Then

  $$E_\lambda x = E_\lambda^2 z = E_\lambda z = x.$$

  So, without loss of generality, we assume $\|x\| = 1$.

  It follows that

  $$\begin{array}{rcl} \langle T_\lambda E_\lambda x, x \rangle & = & \langle T_\lambda x, x \rangle \\ & = & \langle Tx, x \rangle - \lambda \\ & \geq & \inf_{\|\widetilde{x}\|=1} \langle T\widetilde{x}, \widetilde{x} \rangle - \lambda \\ & = & m - \lambda > 0. \end{array}$$

  This contradicts $T_\lambda E_\lambda = -T_\lambda^- \leq 0$.

Proof of Property (iii)

- Suppose that $\lambda > M$, but $E_\lambda \neq I$.

  So $I - E_\lambda \neq 0$.

  Then, $(I - E_\lambda)x = x$, for some x of norm $\|x\| = 1$.

  Hence,

$$\begin{array}{rcl}\langle T_\lambda (I - E_\lambda)x, x\rangle & = & \langle T_\lambda x, x\rangle \\ & = & \langle Tx, x\rangle - \lambda \\ & \leq & \sup_{\|\widetilde{x}\|=1} \langle T\widetilde{x}, \widetilde{x}\rangle - \lambda \\ & = & M - \lambda < 0.\end{array}$$

  This contradicts $T_\lambda (I - E_\lambda) = T_\lambda^+ \geq 0$.

  Also $E_M = 1$, by the continuity from the right to be proved next.

Proof of Property (iv)

- With an interval $\Delta = (\lambda, \mu]$ we associate the operator $E(\Delta) = E_\mu - E_\lambda$.
  Since $\lambda < \mu$, we have $E_\lambda \leq E_\mu$. Hence, $E_\lambda(H) \subseteq E_\mu(H)$.
  This shows that $E(\Delta)$ is a projection. Also, $E(\Delta) \geq 0$.
  We also have

$$\begin{aligned} E_\mu E(\Delta) &= E_\mu^2 - E_\mu E_\lambda = E_\mu - E_\lambda = E(\Delta); \\ (I - E_\lambda)E(\Delta) &= E(\Delta) - E_\lambda(E_\mu - E_\lambda) = E(\Delta). \end{aligned}$$

  Now $E(\Delta), T_\mu^-$ and T_λ^+ are positive and commute.
  So the products $T_\mu^- E(\Delta)$ and $T_\lambda^+ E(\Delta)$T are positive. Hence

$$\begin{aligned} T_\mu E(\Delta) &= T_\mu E_\mu E(\Delta) = -T_\mu^- E(\Delta) \leq 0; \\ T_\lambda E(\Delta) &= T_\lambda (I - E_\lambda) E(\Delta) = T_\lambda^+ E(\Delta) \geq 0. \end{aligned}$$

  This implies $TE(\Delta) \leq \mu E(\Delta)$ and $TE(\Delta) \geq \lambda E(\Delta)$, respectively.
  Taken together, $\lambda E(\Delta) \leq TE(\Delta) \leq \mu E(\Delta)$.

Proof of Property (iv) (Cont'd)

- We keep λ fixed and let $\mu \to \lambda$ from the right in a monotone fashion. Then $E(\Delta)x \to P(\lambda)x$ by the analog of the Monotone Sequence Theorem for a decreasing sequence.
  Here $P(\lambda)$ is bounded and self-adjoint.
  Since $E(\Delta)$ is idempotent, so is $P(\lambda)$.
  Hence $P(\lambda)$ is a projection.
  Also $\lambda P(\lambda) = TP(\lambda)$. I.e., $T_\lambda P(\lambda) = 0$. From this,

$$T_\lambda^+ P(\lambda) = T_\lambda (I - E_\lambda) P(\lambda) = (I - E_\lambda) T_\lambda P(\lambda) = 0.$$

  Hence, $T_\lambda^+ P(\lambda)x = 0$, for all $x \in H$. Hence, $P(\lambda)x \in \mathcal{N}(T_\lambda^+)$.
  By definition, E_λ projects H onto $\mathcal{N}(T_\lambda^+)$.
  Consequently, we have $E_\lambda P(\lambda)x = P(\lambda)x$. I.e., $E_\lambda P(\lambda) = P(\lambda)$.
  On the other hand, if we let $\mu \to \lambda^+$, then $(I - E_\lambda)P(\lambda) = P(\lambda)$.
  Taken, together, $P(\lambda) = 0$. But we had $E(\Delta)x \to P(\lambda)x$.
  So $P(\lambda) = 0$ proves continuity of $\mathscr{E}$ from the right.

Subsection 9

Spectral Representation of Bounded Self-Adjoint Operators

Spectral Theorem for Bounded Self-Adjoint Linear Operators

Spectral Theorem for Bounded Self-Adjoint Linear Operators

Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Then:

(a) T has the spectral representation

$$T = \int_{m^-}^{M} \lambda dE_\lambda,$$

where $\mathscr{E} = (E_\lambda)$ is the spectral family associated with T.

The integral is to be understood in the sense of uniform operator convergence [convergence in the norm on $B(H,H)$], and for all $x, y \in H$,

$$\langle Tx, y \rangle = \int_{m^-}^{M} \lambda dw(\lambda), \quad w(\lambda) = \langle E_\lambda x, y \rangle,$$

where the integral is an ordinary Riemann-Stieltjes integral.

Spectral Theorem (Cont'd)

Spectral Theorem for Bounded Self-Adjoint Linear Operators

(b) More generally, let p is a polynomial in λ with real coefficients, say, $p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$.
Then the operator $p(T)$ defined by

$$p(T) = \alpha_n T^n + \alpha_{n-1} T^{n-1} + \cdots + \alpha_0 I$$

has the spectral representation

$$p(T) = \int_{m^-}^{M} p(\lambda) dE_\lambda.$$

Moreover, for all $x, y \in H$,

$$\langle p(T)x, y\rangle = \int_{m^-}^{M} p(\lambda) dw(\lambda), \quad w(\lambda) = \langle E_\lambda x, y\rangle.$$

Comments on the Spectral Theorem

- The notation m^- indicates that one must take into account a contribution at $\lambda = m$ which occurs if $E_m \neq 0$ (and $m \neq 0$).
- Thus, using any $a < m$, we can write

$$\int_a^M \lambda dE_\lambda = \int_{m^-}^M \lambda dE_\lambda = mE_m + \int_m^M \lambda dE_\lambda.$$

- Similarly,

$$\int_a^M p(\lambda) dE_\lambda = \int_{m^-}^M p(\lambda) dE_\lambda = p(m)E_m + \int_m^M p(\lambda) dE_\lambda.$$

Proof of the Spectral Theorem Part (a)

(a) Choose a sequence $(\mathscr{P}_n)$ of partitions of $(a,b]$, where $a<m$ and $M<b$. Here every $\mathscr{P}_n$ is a partition of $(a,b]$ into intervals $\Delta_{nj}=(\lambda_{nj},\mu_{nj}]$, $j=1,\ldots,n$, of length $\ell(\Delta_{nj})=\mu_{nj}-\lambda_{nj}$.
Note that $\mu_{nj}=\lambda_{n,j+1}$, for $j=1,\ldots,n-1$.
We assume $(\mathscr{P}_n)$ to be such that $\eta(\mathscr{P}_n)=\max_j \ell(\Delta_{nj}) \stackrel{n\to\infty}{\longrightarrow} 0$.
We have shown that $\lambda_{nj}E(\Delta_{nj})\leq TE(\Delta_{nj})\leq \mu_{nj}E(\Delta_{nj})$.
Summing over j, we get

$$\sum_{j=1}^n \lambda_{nj}E(\Delta_{nj})\leq \sum_{j=1}^n TE(\Delta_{nj})\leq \sum_{j=1}^n \mu_{nj}E(\Delta_{nj}).$$

Since $\mu_{nj}=\lambda_{n,j+1}$, for $j=1,\ldots,n-1$, we get

$$T\sum_{j=1}^n E(\Delta_{nj})=T\sum_{j=1}^n (E_{\mu_{nj}}-E_{\lambda_{nj}})=T(I-0)=T.$$

Proof of the Spectral Theorem Part (a) (Cont'd)

- For every $\varepsilon > 0$, there is an n, such that $\eta(\mathscr{P}_n) < \varepsilon$. Hence,

$$\sum_{j=1}^{n} \mu_{nj} E(\Delta_{nj}) - \sum_{j=1}^{n} \lambda_{nj} E(\Delta_{nj}) = \sum_{j=1}^{n} (\mu_{nj} - \lambda_{nj}) E(\Delta_{nj}) < \varepsilon I.$$

It follows that, given any $\varepsilon > 0$, there is an N, such that, for every $n > N$ and every choice of $\lambda_{nj} \in \Delta_{nj}$, we have

$$\left\| T - \sum_{j=1}^{n} \widehat{\lambda}_{nj} E(\Delta_{nj}) \right\| < \varepsilon.$$

Since E_λ is constant for $\lambda < m$ and for $\lambda \geq M$, the particular choice of an $a < m$ and a $b > M$ is immaterial.

Proof of the Spectral Theorem Part (b)

(b) We prove the theorem for polynomials, starting with $p(\lambda) = \lambda^r$, $r \in \mathbb{N}$.

For any $\kappa < \lambda \leq \mu < \nu$, we have

$$\begin{aligned}(E_\lambda - E_\kappa)(E_\mu - E_\nu) &= E_\lambda E_\mu - E_\lambda E_\nu - E_\kappa E_\mu + E_\kappa E_\nu \\ &= E_\lambda - E_\lambda - E_\kappa + E_\kappa = 0.\end{aligned}$$

This shows that $E(\Delta_{nj})E(\Delta_{nk}) = 0$, for $j \neq k$.

Since $E(\Delta_{nj})$ is a projection, $E(\Delta_{nj})^s = E(\Delta_{nj})$, for every $s = 1, 2, \ldots$.

Consequently, we obtain

$$\left[\sum_{j=1}^{n} \widehat{\lambda}_{nj} E(\Delta_{nj})\right]^r = \sum_{j=1}^{n} \widehat{\lambda}_{nj}^r E(\Delta_{nj}).$$

Proof of the Spectral Theorem Part (b) (Cont'd)

- We have
$$\left[\sum_{j=1}^{n} \widehat{\lambda}_{nj} E(\Delta_{nj})\right]^r = \sum_{j=1}^{n} \widehat{\lambda}_{nj}^r E(\Delta_{nj}).$$
Suppose the sum on the left is close to T.
Then the expression on the left is close to T^r because multiplication (composition) of bounded linear operators is continuous.
Hence, given $\varepsilon > 0$, there is an N, such that, for all $n > N$,
$$\left\| T^r - \sum_{j=1}^{n} \widehat{\lambda}_{nj}^r E(\Delta_{nj}) \right\| < \varepsilon.$$
This proves the result for $p(\lambda) = \lambda^r$.
The formulas for an arbitrary polynomial with real coefficients follow from this case.

Properties of $p(T)$

Theorem (Properties of $p(T)$)

Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Let p, p_1 and p_2 be polynomials with real coefficients. Then:

(a) $p(T)$ is self-adjoint.

(b) If $p(\lambda) = \alpha p_1(\lambda) + \beta p_2(\lambda)$, then $p(T) = \alpha p_1(T) + \beta p_2(T)$.

(c) If $p(\lambda) = p_1(\lambda) p_2(\lambda)$, then $p(T) = p_1(T) p_2(T)$.

(d) If $p(\lambda) \geq 0$, for all $\lambda \in [m, M]$, then $p(T) \geq 0$.

(e) If $p_1(\lambda) \leq p_2(\lambda)$, for all $\lambda \in [m, M]$, then $p_1(T) \leq p_2(T)$.

(f) $\|p(T)\| \leq \max_{\lambda \in J} |p(\lambda)|$, where $J = [m, M]$.

(g) If a bounded linear operator commutes with T, it also commutes with $p(T)$.

Properties of $p(T)$ Parts (a)-(d)

(a) T is self-adjoint and p has real coefficients.
So we get $(\alpha_j T^j)^* = \overline{\alpha_j}(T^*)^j = \alpha_j T^j$.

(b) This is obvious from the definition.

(c) This is obvious from the definition.

(d) Note that p has real coefficients.
So complex zeros must occur in conjugate pairs if they occur at all.
We observe that:
- p changes sign if λ passes through a zero of odd multiplicity;
- $p(\lambda) \geq 0$ on $[m, M]$.

So zeros of p in (m, M) must be of even multiplicity.
Hence, we can write

$$p(\lambda) = \alpha \prod_j (\lambda - \beta_j) \prod_k (\gamma_k - \lambda) \prod_\ell [(\lambda - \mu_\ell)^2 + \nu_\ell^2],$$

where $\beta_j \leq m$, $\gamma_k \geq M$ and the quadratic factors correspond to complex conjugate zeros and to real zeros in (m, M).

Properties of $p(T)$ Part (d)

- We have $p(\lambda) = \alpha \prod_j (\lambda - \beta_j) \prod_k (\gamma_k - \lambda) \prod_\ell [(\lambda - \mu_\ell)^2 + \nu_\ell^2]$.
  We show that $\alpha > 0$ if $p \neq 0$.
  For all sufficiently large λ, say, for all $\lambda \geq \lambda_0$, we have
  $$\text{sgn} p(\lambda) = \text{sgn} \alpha_n \lambda^n = \text{sgn} \alpha_n,$$
  where n is the degree of p.
  - Suppose $\alpha_n > 0$. Then:
    - $p(\lambda_0) > 0$;
    - The number of the γ_k's (each counted according to its multiplicity) must be even, to make $p(\lambda) \geq 0$ in (m, M).

    Then all three products are positive at λ_0.
    Hence, we must have $\alpha > 0$ in order that $p(\lambda_0) > 0$.
  - Suppose $\alpha_n < 0$. Then:
    - $p(\lambda_0) < 0$;
    - The number of the γ_k's is odd, to make $p(\lambda) \geq 0$ on (m, M).

    It follows that the second product is negative at λ_0.
    Hence, $\alpha > 0$, as before.

Properties of $p(T)$ Part (d) (Cont'd)

- We replace λ by T.

  Then each of the factors above is a positive operator.

  Consider $x \neq 0$. Set $v = \frac{1}{\|x\|}x$. Then $x = \|x\| v$.

  Since $-\beta_j \geq -m$,

$$\begin{array}{rcl} \langle (T-\beta_j I)x, x\rangle & = & \langle Tx, x\rangle - \beta_j \langle x, x\rangle \\ & \geq & \|x\|^2 \langle Tv, v\rangle - m\|x\|^2 \\ & \geq & \|x\|^2 \inf_{\|\widetilde{v}\|=1} \langle T\widetilde{v}, \widetilde{v}\rangle - m\|x\|^2 \\ & = & 0. \end{array}$$

  That is, $T - \beta_j I \geq 0$. Similarly, $\gamma_k I - T \geq 0$.

  Now, $T - \mu_\ell I$ is self-adjoint. So its square is positive.

  It follows that $(T - \mu_\ell I)^2 + \nu_\ell^2 I \geq 0$.

  Since all those operators commute, their product is positive.

  So, since $\alpha > 0$, $p(T) \geq 0$.

Properties of $p(T)$ Parts (e)-(g)

(e) This follows immediately from Part (d).

(f) Let k denote the maximum of $|p(\lambda)|$ on J.

Then $0 \leq p(\lambda)^2 \leq k^2$, for $\lambda \in J$.

Hence Part (e) yields $p(T)^2 \leq k^2 I$.

Since $p(T)$ is self-adjoint, for all x,

$$\langle p(T)x, p(T)x \rangle = \langle p(T)^2 x, x \rangle \leq k^2 \langle x, x \rangle.$$

Now we get $\|p(T)x\| \leq k\|x\|$.

Taking the supremum over all x of norm 1,

$$\|p(T)\| \leq \max_{\lambda \in J} |p(\lambda)|.$$

(g) This follows immediately from the definition of $p(T)$.

Subsection 10

Extension of the Spectral Theorem to Continuous Functions

Extension to Continuous Functions

- The theorem holds for $p(T)$, where T is a bounded self-adjoint linear operator and p is a polynomial with real coefficients.
- We want to extend the theorem to operators $f(T)$, where T is as before and f is a continuous real-valued function.
- Let H be a complex Hilbert space.
- Let $T: H \to H$ be a bounded self-adjoint linear operator on H.
- Let f be a continuous real-valued function on $[m, M]$, where:
  - $m = \inf_{\|x\|=1} \langle Tx, x \rangle$;
  - $M = \sup_{\|x\|=1} \langle Tx, x \rangle$.
- By the Weierstraß approximation theorem, there is a sequence of polynomials (p_n), with real coefficients, such that $p_n(\lambda) \to f(\lambda)$ uniformly on $[m, M]$.

The Definition of $f(T)$

- Corresponding to the sequence of polynomials (p_n), we have a sequence of bounded self-adjoint linear operators $p_n(T)$.
- By the preceding theorem, for $J = [m, M]$,

$$\|p_n(T) - p_r(T)\| \le \max_{\lambda \in J} |p_n(\lambda) - p_r(\lambda)|.$$

- Since $p_n(\lambda) \to f(\lambda)$, given any $\varepsilon > 0$, there is an N, such that, for all $n, r > N$,

$$\max_{\lambda \in J} |p_n(\lambda) - p_r(\lambda)| < \varepsilon.$$

- Hence, $(p_n(T))$ is Cauchy.
- So, since $B(H, H)$ is complete, $(p_n(T))$ has a limit in $B(H, H)$.
- We define $f(T)$ to be that limit: $p_n(T) \to f(T)$.

$f(T)$ is Well-Defined

- Claim: $f(T)$ depends only on f (and T, of course), but not on the particular choice of a sequence of polynomials converging to f uniformly.

  Let $(\widetilde{p}_n)$ be another sequence of polynomials with real coefficients such that $\widetilde{p}_n(\lambda) \to f(\lambda)$ uniformly on $[m, M]$. Then $\widetilde{p}_n(T) \to \widetilde{f}(T)$ by the previous argument. So it suffices to show that $\widetilde{f}(T) = f(T)$.

  Clearly, $\widetilde{p}_n(\lambda) - p_n(\lambda) \to 0$. Hence, $\widetilde{p}_n(T) - p_n(T) \to 0$.

  Consequently, given $\varepsilon > 0$, there is an N, such that for $n > N$,

$$\|\widetilde{f}(T) - \widetilde{p}_n(T)\| < \frac{\varepsilon}{3}, \ \|\widetilde{p}_n(T) - p_n(T)\| < \frac{\varepsilon}{3}, \ \|p_n(T) - f(T)\| < \frac{\varepsilon}{3}.$$

  By the triangle inequality it follows that

$$\|\widetilde{f}(T) - f(T)\| \leq \|\widetilde{f}(T) - \widetilde{p}_n(T)\| + \|\widetilde{p}_n(T) - p_n(T)\| + \|p_n(T) - f(T)\| < \varepsilon.$$

  Since $\varepsilon > 0$ was arbitrary, $\widetilde{f}(T) - f(T) = 0$. Thus, $\widetilde{f}(T) = f(T)$.

Spectral Theorem

Spectral Theorem

Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H and f a continuous real-valued function on $[m, M]$. Then $f(T)$ has the spectral representation

$$f(T) = \int_{m^-}^{M} f(\lambda) dE_\lambda,$$

where $\mathscr{E} = (E_\lambda)$ is the spectral family associated with T.
The integral is to be understood in the sense of uniform operator convergence, and, for all $x, y \in H$,

$$\langle f(T)x, y\rangle = \int_{m^-}^{M} f(\lambda) dw(\lambda), \quad w(\lambda) = \langle E_\lambda x, y\rangle,$$

where the integral is an ordinary Riemann-Stieltjes integral.

Spectral Theorem (Proof)

- For every $\varepsilon > 0$, there is a polynomial p, with real coefficients, such that, for all $\lambda \in [m, M]$,

$$-\frac{\varepsilon}{3} \leq f(\lambda) - p(\lambda) \leq \frac{\varepsilon}{3}.$$

Hence, $\|f(T) - p(T)\| \leq \frac{\varepsilon}{3}$.

Note that $\sum E(\Delta_{nj}) = I$.

Using the preceding inequality, we get, for any partition,

$$-\frac{\varepsilon}{3} I \leq \sum_{j=1}^{n} [f(\widehat{\lambda}_{nj}) - p(\widehat{\lambda}_{nj})] E(\Delta_{nj}) \leq \frac{\varepsilon}{3} I.$$

It follows that

$$\left\| \sum_{j=1}^{n} [f(\widehat{\lambda}_{nj}) - p(\widehat{\lambda}_{nj})] E(\Delta_{nj}) \right\| \leq \frac{\varepsilon}{3}.$$

Spectral Theorem (Cont'd)

- Recall that $p(T)$ is represented by $p(T)=\int_{m^-}^{M}p(\lambda)dE_\lambda$.
  So there is an N, such that, for every $n>N$,
  $$\left\|\sum_{j=1}^{n}p(\widehat{\lambda}_{nj})E(\Delta_{nj})-p(T)\right\|\leq\frac{\varepsilon}{3}.$$
  We now estimate the norm of the difference between $f(T)$ and the Riemann-Stieltjes sums corresponding to the integral.
  For $n>N$, we obtain, by means of the triangle inequality,
  $$\begin{aligned}\|\textstyle\sum_{j=1}^{n}f(\widehat{\lambda}_{nj})E(\Delta_{nj})-f(T)\|&\leq\|\textstyle\sum_{j=1}^{n}[f(\widehat{\lambda}_{nj})-p(\widehat{\lambda}_{nj})]E(\Delta_{nj})\|\\&+\|\textstyle\sum_{j=1}^{n}p(\widehat{\lambda}_{nj})E(\Delta_{nj})-p(T)\|+\|p(T)-f(T)\|\leq\varepsilon.\end{aligned}$$
  Since $\varepsilon>0$ was arbitrary, this establishes the statement.

Uniqueness of the Spectral Representation

- Uniqueness Property: $\mathscr{E}=(E_\lambda)$ is the only spectral family on $[m,M]$ that yields the representations

$$\begin{aligned} f(T) &= \int_{m^-}^M f(\lambda)dE_\lambda; \\ \langle f(T)x,y\rangle &= \int_{m^-}^M f(\lambda)dw(\lambda), \quad w(\lambda)=\langle E_\lambda x,y\rangle. \end{aligned}$$

- The plausibility is indicated by the following:
  - The second equality holds for every continuous real-valued function f on $[m,M]$;
  - Its left hand side is defined in a way which does not depend on $\mathscr{E}$.
- A rigorous proof follows from a uniqueness theorem for Stieltjes integrals.

Uniqueness of the Spectral Representation (Cont'd)

- A uniqueness theorem for Stieltjes integrals states that, for any fixed x and y, the expression

$$w(\lambda) = \langle E_\lambda x, y\rangle$$

is determined, up to an additive constant, by

$$\langle f(T)x, y\rangle = \int_{m^-}^{M} f(\lambda) dw(\lambda), \quad w(\lambda) = \langle E_\lambda x, y\rangle,$$

at its points of continuity and at m^- and M.

Now we have:

  - $\langle E_M x, y\rangle = \langle x, y\rangle$, since $E_M = I$;
  - (E_λ) is continuous from the right.

It follows $w(\lambda)$ is uniquely determined everywhere.

Properties of $f(T)$

- The properties of $p(T)$, listed in a previous theorem, extend to $f(T)$.

Theorem (Properties of $f(T)$)

Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Let f, f_1 and f_2 be continuous real-valued functions on $[m, M]$. Then:

(a) $f(T)$ is self-adjoint.

(b) If $f(\lambda) = \alpha f_1(\lambda) + \beta f_2(\lambda)$, then $f(T) = \alpha f_1(T) + \beta f_2(T)$.

(c) If $f(\lambda) = f_1(\lambda) f_2(\lambda)$, then $f(T) = f_1(T) f_2(T)$.

(d) If $f(\lambda) \geq 0$, for all $\lambda \in [m, M]$, then $f(T) \geq 0$.

(e) If $f_1(\lambda) \leq f_2(\lambda)$, for all $\lambda \in [m, M]$, then $f_1(T) \leq f_2(T)$.

(f) $\|f(T)\| \leq \max_{\lambda \in J} |f(\lambda)|$, where $J = [m, M]$.

(g) If a bounded linear operator commutes with T, it also commutes with $f(T)$.

Subsection 11

Properties of Spectral Family of a Bounded Self-Adjoint Operator

Eigenvalues

Theorem (Eigenvalues)

Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H and $\mathscr{E} = (E_\lambda)$ the corresponding spectral family. Then $\lambda \mapsto E_\lambda$ has a discontinuity at any $\lambda = \lambda_0$ (that is, $E_{\lambda_0} \neq E_{\lambda_0^-}$) if and only if λ_0 is an eigenvalue of T. In this case, the corresponding eigenspace is

$$\mathscr{N}(T - \lambda_0 I) = (E_{\lambda_0} - E_{\lambda_0^-})(H).$$

- λ_0 is an eigenvalue of T if and only if $\mathscr{N}(T - \lambda_0 I) \neq \{0\}$.

  So the first statement follows from the displayed equation.

  Hence, it suffices to prove this equation.

  We set $F_0 = E_{\lambda_0} - E_{\lambda_0^-}$. We must show that:
  - $F_0(H) \subseteq \mathscr{N}(T - \lambda_0 I)$;
  - $F_0(H) \supseteq \mathscr{N}(T - \lambda_0 I)$.

Eigenvalues $F_0(H) \subseteq \mathcal{N}(T - \lambda_0 I)$

- Since $\lambda_0 - \frac{1}{n} < \lambda_0$, setting $\Delta_0 = (\lambda_0 - \frac{1}{n}, \lambda_0]$, we have

$$(\lambda_0 - \frac{1}{n})E(\Delta_0) \leq TE(\Delta_0) \leq \lambda_0 E(\Delta_0).$$

Now let $n \to \infty$. Then $E(\Delta_0) \to F_0$.

So the preceding inequalities yield

$$\lambda_0 F_0 \leq TF_0 \leq \lambda_0 F_0.$$

Hence, $TF_0 = \lambda_0 F_0$. That is, $(T - \lambda_0 I)F_0 = 0$.

Eigenvalues $F_0(H) \supseteq \mathcal{N}(T - \lambda_0 I)$

- Let $x \in \mathcal{N}(T - \lambda_0 I)$. We show that then $x \in F_0(H)$.

  Since F_0 is a projection, this amounts to $F_0 x = x$.

  Suppose $\lambda_0 \notin [m, M]$. Then $\lambda_0 \in \rho(T)$.

  Since $F_0(H)$ is a vector space, $\mathcal{N}(T - \lambda_0 I) = \{0\} \subseteq F_0(H)$.

  Suppose $\lambda_0 \in [m, M]$. By assumption, $(T - \lambda_0 I)x = 0$.

  This implies $(T - \lambda_0 I)^2 x = 0$.

  By the Spectral Representation Theorem, for $a < m$ and $b > M$,

$$\int_a^b (\lambda - \lambda_0)^2 dw(\lambda) = 0, \quad w(\lambda) = \langle E_\lambda x, x \rangle.$$

  Here $(\lambda - \lambda_0)^2 \geq 0$ and $\lambda \mapsto \langle E_\lambda x, x \rangle$ is monotone increasing.

  Hence, the integral over any subinterval of positive length must be zero.

Eigenvalues $F_0(H) \supseteq \mathcal{N}(T-\lambda_0 I)$ (Cont'd)

- In particular, for every $\varepsilon > 0$, we must have

$$0 = \int_a^{\lambda_0-\varepsilon}(\lambda-\lambda_0)^2 dw(\lambda) \geq \varepsilon^2 \int_a^{\lambda_0-\varepsilon} dw(\lambda) = \varepsilon^2 \langle E_{\lambda_0-\varepsilon}x, x\rangle;$$
$$0 = \int_{\lambda_0+\varepsilon}^{b}(\lambda-\lambda_0)^2 dw(\lambda) \geq \varepsilon^2 \int_{\lambda_0+\varepsilon}^{b} dw(\lambda) = \varepsilon^2 \langle Ix, x\rangle - \varepsilon^2 \langle E_{\lambda_0+\varepsilon}x, x\rangle.$$

  Since $\varepsilon > 0$, by the Positivity Theorem,

$$\langle E_{\lambda_0-\varepsilon}x, x\rangle = 0 \quad \text{implies} \quad E_{\lambda_0-\varepsilon}x = 0;$$
$$\langle x - E_{\lambda_0+\varepsilon}x, x\rangle = 0 \quad \text{implies} \quad x - E_{\lambda_0+\varepsilon}x = 0.$$

  We may thus write $x = (E_{\lambda_0+\varepsilon} - E_{\lambda_0-\varepsilon})x$.

  But $\lambda \mapsto E_\lambda$ is continuous from the right.

  So, letting $\varepsilon \mapsto 0$, we obtain $x = F_0 x$.

Resolvent Set

Theorem (Resolvent Set)

Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H and $\mathscr{E} = (E_\lambda)$ the corresponding spectral family. Then a real λ_0 belongs to the resolvent set $\rho(T)$ of T if and only if there is a $\gamma > 0$, such that $\mathscr{E} = (E_\lambda)$ is constant on the interval $[\lambda_0 - \gamma, \lambda_0 + \gamma]$.

- We prove that:
  - (a) The given condition is sufficient for $\lambda_0 \in \rho(T)$;
  - (b) The given condition is necessary for $\lambda_0 \in \rho(T)$.
- We use the previously shown fact that $\lambda_0 \in \rho(T)$ if and only if there exists a $\gamma > 0$, such that

$$\|(T - \lambda_0 I)x\| \geq \gamma \|x\|, \quad \text{for all } x \in H.$$

Resolvent Set (Sufficiency)

(a) Suppose that λ_0 is real, such that, for some $\gamma > 0$, $\mathscr{E}$ is constant on $J = [\lambda_0 - \gamma, \lambda_0 + \gamma]$.

By a previous result,

$$\|(T - \lambda_0 I)x\|^2 = \langle (T - \lambda_0 I)^2, x\rangle = \int_{m^-}^{M} (\lambda - \lambda_0)^2 d\langle E_\lambda x, x\rangle.$$

Since $\mathscr{E}$ is constant on J, integration over J yields the value zero.

Moreover, for $\lambda \notin J$, we have $(\lambda - \lambda_0)^2 \geq \gamma^2$.

Thus, the previous equation implies

$$\|(T - \lambda_0 I)x\|^2 \geq \gamma^2 \int_{m^-}^{M} d\langle E_\lambda x, x\rangle = \gamma^2 \langle x, x\rangle.$$

Taking square roots, we obtain $\|(T - \lambda_0 I)x\| \geq \gamma \|x\|$.

Hence, $\lambda_0 \in \rho(T)$.

Resolvent Set (Necessity)

(b) Conversely, suppose that $\lambda_0 \in \rho(T)$.
Then, for some $\gamma > 0$,

$$\|(T - \lambda_0 I)x\| \geq \gamma \|x\|, \quad \text{for all } x \in H.$$

So, by the equation above,

$$\int_{m^-}^{M} (\lambda - \lambda_0)^2 d\langle E_\lambda x, x\rangle \geq \gamma^2 \int_{m^-}^{M} d\langle E_\lambda x, x\rangle.$$

Suppose that $\mathscr{E}$ is not constant on the interval $[\lambda_0 - \gamma, \lambda_0 + \gamma]$.
Since $E_\lambda \leq E_\mu$, for $\lambda < \mu$, we can find a positive $\eta < \gamma$, such that

$$E_{\lambda_0 + \eta} - E_{\lambda_0 - \eta} \neq 0.$$

Hence, there is a $y \in H$, such that $x = (E_{\lambda_0 + \eta} - E_{\lambda_0 - \eta})y \neq 0$.
Using this x, we get

$$E_\lambda x = E_\lambda (E_{\lambda_0 + \eta} - E_{\lambda_0 - \eta})y.$$

Resolvent Set (Necessity Cont'd)

- Now $E_\lambda x = E_\lambda(E_{\lambda_0+\eta} - E_{\lambda_0-\eta})y$ is:
  - $(E_\lambda - E_\lambda)y = 0$, when $\lambda < \lambda_0 - \eta$;
  - $(E_{\lambda_0+\eta} - E_{\lambda_0-\eta})y$, when $\lambda > \lambda_0 + \eta$.

  So it is independent of λ. Thus, we may take $K = [\lambda_0 - \eta, \lambda_0 + \eta]$ as the interval of integration in the integral above.

  If $\lambda \in K$, by straightforward calculation,

$$\langle E_\lambda x, x\rangle = \langle (E_\lambda - E_{\lambda_0-\eta})y, y\rangle.$$

  Hence, the inequality gives

$$\int_{\lambda_0-\eta}^{\lambda_0+\eta} (\lambda - \lambda_0)^2 d\langle E_\lambda y, y\rangle \geq \gamma^2 \int_{\lambda_0-\eta}^{\lambda_0+\eta} d\langle E_\lambda y, y\rangle.$$

  This is impossible because the integral on the right is positive and, when $\lambda \in K$, $(\lambda - \lambda_0)^2 \leq \eta^2 < \gamma^2$.

  Thus, $\mathscr{E}$ must be constant on $[\lambda_0 - \gamma, \lambda_0 + \gamma]$.

Continuous Spectrum

Theorem (Continuous Spectrum)

Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H and $\mathscr{E} = (E_\lambda)$ the corresponding spectral family. Then a real λ_0 belongs to the continuous spectrum $\sigma_c(T)$ of T if and only if $\mathscr{E}$ is:

- Continuous at λ_0 (thus, $E_{\lambda_0} = E_{\lambda_0^-}$);
- Not constant in any neighborhood of λ_0 on $\mathbb{R}$.

- The preceding theorem shows that $\lambda_0 \in \sigma(T)$ if and only if $\mathscr{E}$ is not constant in any neighborhood of λ_0 on $\mathbb{R}$.

  Moreover, we have:
  - $\sigma_r(T) = \emptyset$;
  - Points of $\sigma_p(T)$ correspond to discontinuities of $\mathscr{E}$.

  These yield the conclusion of the theorem.