Introduction to Topology

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 400

Subsection 1

Introduction

The Peano Axioms for the Natural Numbers

- The set of **positive integers** or **natural numbers** is a collection of objects $\mathbb{N}$ on which there is defined a function s, called the **successor function**, satisfying the conditions:
    1. For each x in $\mathbb{N}$, there is one and only one y in $\mathbb{N}$ such that $y = s(x)$;
    2. Given objects x and y in $\mathbb{N}$ such that $s(x) = s(y)$, then $x = y$;
    3. There is one and only one object in $\mathbb{N}$, denoted by 1, which is not the successor of an object in $\mathbb{N}$, i.e., $1 \neq s(x)$, for each x in $\mathbb{N}$;
    4. Given a collection T of objects in $\mathbb{N}$, such that:
        - 1 is in T and
        - for each x in T, $s(x)$ is also in T,

       then $T = \mathbb{N}$.
- The four conditions are the **Peano's axioms** for the natural numbers.
- The fourth is called the **Principle of Mathematical Induction**.

Commutative Fields

- A **commutative field** is a collection of objects $\mathbb{F}$ and two functions that associate to each pair a, b of objects from $\mathbb{F}$
  - an element $a + b$ of $\mathbb{F}$, called their **sum**;
  - an element $a \cdot b$ of $\mathbb{F}$, called their **product**,

  satisfying the conditions:
  1. For each a, b in $\mathbb{F}$, $a + b = b + a$;
  2. For each a, b, c in $\mathbb{F}$, $a + (b + c) = (a + b) + c$;
  3. There is a unique object in $\mathbb{F}$, denoted by 0, such that $a + 0 = 0 + a = a$, for each a in $\mathbb{F}$;
  4. For each a in $\mathbb{F}$, there is a unique object a' in $\mathbb{F}$, such that $a + a' = a' + a = 0$;
  5. For each a, b in $\mathbb{F}$, $a \cdot b = b \cdot a$;
  6. For each a, b, c in $\mathbb{F}$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;
  7. There is a unique object in $\mathbb{F}$, different from 0, denoted by 1, such that $a \cdot 1 = 1 \cdot a = a$ for each a in $\mathbb{F}$;
  8. For each a in $\mathbb{F}$, if a is different from 0, there is a unique object a^* in $\mathbb{F}$ such that $a \cdot a^* = a^* \cdot a = 1$;
  9. For each a, b, c in $\mathbb{F}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

Linearly Ordered and Complete Fields

- A field $\mathbb{F}$ is called **linearly ordered** if it has as additional structure a relation "$<$" which satisfies the conditions:
  1. For each pair of objects x, y in $\mathbb{F}$, one and only one of the three statements, $x < y$, $x = y$, $y < x$, is true;
  2. For each object z in $\mathbb{F}$, $x < y$ implies $x + z < y + z$;
  3. For each object z in $\mathbb{F}$ such that $0 < z$, $x < y$ implies $x \cdot z < y \cdot z$.
- Let T be a subcollection of objects from a linearly ordered field $\mathbb{F}$.
  - An object b in $\mathbb{F}$ is called an **upper bound** of T if for each x in T, either $x < b$ or $x = b$.
  - An object a in $\mathbb{F}$ is called a **least upper bound** of T, if a is an upper bound of T and if $a < b$, for any other upper bound b of T.
- A linearly ordered field $\mathbb{F}$ is called **complete** if every non-empty subcollection T of $\mathbb{F}$ that has an upper bound also has a least upper bound.

The Real Number System

- The **real number system** is a collection $\mathbb{R}$ of objects together with operations of addition and multiplication and a relation $<$ such that the collection $\mathbb{R}$, together with this structure, is a complete, linearly ordered, commutative field.
- Even though there are many real number systems, it is implicitly asserted that the conditions imposed on the collection $\mathbb{R}$ are categorical:

  Any two instances of the real number system are indistinguishable, apart from the names or notation used to denote the objects.

Subsection 2

Sets and Subsets

Objects, Sets and Membership

- We assume that the terms "object", "set" and the relation "is a member of" are familiar concepts.
- We use these concepts in a manner that is in agreement with the ordinary usage of these terms.
- If an object A belongs to a set S, we write $A \in S$ (read, "A in S").
- If an object A does not belong to a set S, we write $A \notin S$ (read, "A not in S").
- If $A_1, \ldots, A_n$ are objects, the set consisting of precisely these objects will be written $\{A_1, \ldots, A_n\}$.
- It is necessary to distinguish the set $\{A\}$, consisting of precisely one object A, from the object A itself.
  - $A \in \{A\}$ is a true statement;
  - $A = \{A\}$ is a false statement.
- We stipulate that there exists a set that has no members, the so-called **null** or **empty set**. The symbol for this set is $\emptyset$.

Subsets

- Let A and B be sets. If, for each object $x \in A$, it is true that $x \in B$, we say that A is a **subset** of B. In this event, we shall also say that A is **contained in** B, which we write $A \subseteq B$. Equivalently, B **contains** A, which we write $B \supseteq A$.
- In accordance with the definition of subset:
  - A set A is always a subset of itself: $A \subseteq A$;
  - The empty set is a subset of A: $\emptyset \subseteq A$.

  These two subsets, A and $\emptyset$, of A are called **improper subsets**.

  Any other subset is called a **proper subset**.
- Example: For each pair of real numbers a, b with $a < b$,
  - the set of all real numbers x, such that $a \leq x \leq b$ is called the **closed interval** from a to b and is denoted by $[a, b]$;
  - the set of all real numbers x, such that $a < x < b$ is called the **open interval** from a to b and is denoted by (a, b).

  We thus have $(a, b) \subseteq [a, b] \subseteq \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

Equality and Powersets

- Two sets are **identical** if they have precisely the same members. Thus, if A and B are sets, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
- Sets may themselves be objects belonging to other sets.
  Example: $\{\{1, 3, 5, 7\}, \{2, 4, 6\}\}$ is a set to which there belong two objects, these two objects being
  - the set of odd positive integers less than 8 and
  - the set of even positive integers less than 8.
- If A is any set, the collection of subsets of A consists of objects that may be used to constitute a new set.
- In particular, for each set A, there is a set, denoted by $\mathcal{P}(A)$ or 2^A, called the **powerset** of A, whose members are the subsets of A.
  Thus, for each set A, we have

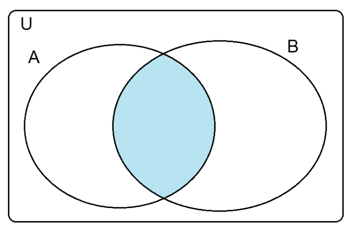
$$B \in \mathcal{P}(A) \quad \text{if and only if} \quad B \subseteq A.$$

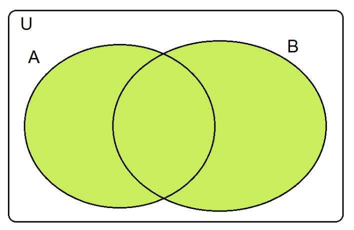
Subsection 3

Set Operations

Intersection and Union

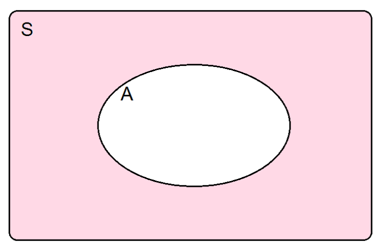
- If x is an object, A a set and $x \in A$, we shall say that x is an **element**, **member**, or **point** of A.
- Let A and B be sets. The **intersection** of the sets A and B is the set whose members are those objects x, such that $x \in A$ and $x \in B$. The intersection of A and B is denoted by $A \cap B$ (read, "A intersect B").
- The **union** of the sets A and B is the set whose members are those objects x, such that x belongs to at least one of the two sets A, B, i.e., $x \in A$ or $x \in B$. The union of A and B is denoted by $A \cup B$ (read, "A union B").





Complement

- Let $A \subseteq S$. The **complement of** A **in** S is the set of elements that belong to S but not to A. The complement of A in S is denoted by $C_S(A)$ or by $S - A$.



- The set S may be fixed throughout a given discussion, in which case the complement of A in S may simply be called the **complement of** A and denoted by $C(A)$.
- $C(A)$ is again a subset of S and one may take its complement. The complement of the complement of A is A, i.e., $C(C(A)) = A$.

DeMorgan's Laws

Theorem (DeMorgan's Laws)

Let $A \subseteq S$, $B \subseteq S$. Then

$$C(A \cup B) = C(A) \cap C(B) \quad \text{and} \quad C(A \cap B) = C(A) \cup C(B).$$

- Suppose $x \in C(A \cup B)$. Then $x \in S$ and $x \notin A \cup B$. Thus, $x \notin A$ and $x \notin B$, or $x \in C(A)$ and $x \in C(B)$. Therefore $x \in C(A) \cap C(B)$ and, consequently, $C(A \cup B) \subseteq C(A) \cap C(B)$

  Conversely, suppose $x \in C(A) \cap C(B)$. Then $x \in S$ and $x \in C(A)$ and $x \in C(B)$. Thus, $x \notin A$ and $x \notin B$, and, therefore, $x \notin A \cup B$. It follows that $x \in C(A \cup B)$ and, thus, $C(A) \cap C(B) \subseteq C(A \cup B)$.

  We have shown that $C(A) \cap C(B) = C(A \cup B)$.

  For the second identity, apply the preceding one to the two subsets $C(A)$ and $C(B)$ of S:
  $C(C(A) \cup C(B)) = C(C(A)) \cap C(C(B)) = A \cap B$. Taking complements, $C(A) \cup C(B) = C(C(C(A) \cup C(B))) = C(A \cap B)$.

Subsection 4

Indexed Families of Sets

Indexed Families of Sets

- Let I be a set. For each $\alpha \in I$, let A_α be a subset of a given set S. We call I an **indexing set** and the collection of subsets of S indexed by the elements of I is called an **indexed family** of subsets of S. We denote this indexed family of subsets of S by $(A_\alpha)_{\alpha \in I}$.
- Indexed families of subsets allow for a more general formation of unions and intersections of sets.
- Let $(A_\alpha)_{\alpha \in I}$ be an indexed family of subsets of a set S.
    - The **union** of this indexed family, written $\bigcup_{\alpha \in I} A_\alpha$, is the set of all elements $x \in S$, such that $x \in A_\beta$, for at least one index $\beta \in I$.
    - The **intersection** of this indexed family, written $\bigcap_{\alpha \in I} A_\alpha$, is the set of all elements $x \in S$, such that $x \in A_\beta$, for all $\beta \in I$.
- Note that $\bigcup_{\alpha \in I} A_\alpha = \bigcup_{\gamma \in I} A_\gamma$, for which reason the two occurrences of "α" in the expression $\bigcup_{\alpha \in I} A_\alpha$ are referred to as dummy indices.

Example and Special Cases

- Let A_1, A_2, A_3, A_4 be respectively the set of freshmen, sophomores, juniors, and seniors in some specified college.
  Here we have $I = \{1, 2, 3, 4\}$ as an indexing set.
  - $\bigcup_{\alpha \in I} A_\alpha$ is the set of undergraduates;
  - $\bigcap_{\alpha \in I} A_\alpha = \emptyset$.
- If the indexing set I contains precisely two distinct indices, then the union (intersection) over α in I of A_α is the same as the union (intersection) of two sets, i.e.,

$$\bigcup_{\alpha \in \{i,j\}} A_\alpha = A_i \cup A_j \quad \text{and} \quad \bigcap_{\alpha \in \{i,j\}} A_\alpha = A_i \cap A_j.$$

- In case $I = \emptyset$, we get

$$\bigcup_{\alpha \in \emptyset} A_\alpha = \emptyset \quad \text{and} \quad \bigcap_{\alpha \in \emptyset} A_n = S.$$

Generalized DeMorgan's Laws

Theorem

Let $(A_\alpha)_{\alpha \in I}$ be an indexed family of subsets of a set S. Then

$$C(\bigcup_{\alpha \in I} A_\alpha) = \bigcap_{\alpha \in I} C(A_\alpha) \quad \text{and} \quad C(\bigcap_{\alpha \in I} A_\alpha) = \bigcup_{\alpha \in I} C(A_\alpha).$$

- Suppose $x \in C(\bigcup_{\alpha \in I} A_\alpha)$. Then $x \notin \bigcup_{\alpha \in I} A_\alpha$, i.e., $x \notin A_\beta$, for each index $\beta \in I$. Thus $x \in C(A_\beta)$, for each index $\beta \in I$, and $x \in \bigcap_{\alpha \in I} C(A_\alpha)$. Therefore, $C(\bigcup_{\alpha \in I} A_\alpha) \subseteq \bigcap_{\alpha \in I} C(A_\alpha)$.

  Conversely, suppose that $x \in \bigcap_{\alpha \in I} C(A_\alpha)$. Then $x \in C(A_\beta)$, for each index $\beta \in I$. Thus $x \notin A_\beta$, for each index $\beta \in I$, i.e., $x \notin \bigcup_{\alpha \in I} A_\alpha$. Therefore, $x \in C(\bigcup_{\alpha \in I} A_\alpha)$ and $\bigcap_{\alpha \in I} C(A_\alpha) \subseteq C(\bigcup_{\alpha \in I} A_i)$.

  The second law can be proved similarly.

Unions and Intersections of Indexed Families

- Given any collection of subsets of a set S, the concept of indexed family of subsets allows us to define the union or intersection of these subsets by constructing some convenient indexing set.
- If the collection of subsets is finite, the finite set $\{1, 2, \ldots, n\}$ of integers is a convenient indexing set.
  - Given subsets $A_1, A_2, \ldots, A_n$ of S, we write $A_1 \cup A_2 \cup \cdots \cup A_n$ or $\bigcup_{i=1}^n A_i$ for $\bigcup_{\alpha \in \{1,2,\ldots,n\}} A_\alpha$.
  - Similarly, $A_1 \cap A_2 \cap \cdots \cap A_n$ or $\bigcap_{i=1}^n A_i$ are used in place of $\bigcap_{\alpha \in \{1,2,\ldots,n\}} A_\alpha$.

Subsection 5

Products of Sets

Ordered Pairs and Cartesian Products

- Let x and y be objects. The **ordered pair** (x, y) is a sequence of two objects,
  - the first object of the sequence being x;
  - the second object of the sequence being y.
- Let A and B be sets. The **Cartesian product** of A and B, written $A \times B$, (read "A cross B") is the set whose elements are all the ordered pairs (x, y), such that $x \in A$ and $y \in B$.

  Examples:

  1. The coordinate plane of analytical geometry is the Cartesian product of two lines.
  2. The possible outcomes of the throw of a pair of dice is the Cartesian product of two sets, each of which is comprised of the numbers $1, 2, 3, 4, 5, 6$.
- The two Cartesian products $A \times B$ and $B \times A$ are distinct unless $A = B$.

Direct Product of a Sequence of Sets

- A generalization of the Cartesian product of two sets is the direct product of a sequence of sets.
- Let $A_1, A_2, \ldots, A_n$ be a finite sequence of sets, indexed by $\{1, 2, \ldots, n\}$. The **direct product** of $A_1, A_2, \ldots, A_n$, written

$$\prod_{i=1}^{n} A_i,$$

  is the set consisting of all sequences $(a_1, a_2, \ldots, a_n)$, such that $a_1 \in A_1$, $a_2 \in A_2$, ..., $a_n \in A_n$.
- As a particular case, $\prod_{i=1}^{2} A_i = A_1 \times A_2$.
- For this reason we often write $A_1 \times A_2 \times \cdots \times A_n$ for $\prod_{i=1}^{n} A_i$.

Direct Products of Infinite Sequences of Sets

- The concept of direct product may be extended to an infinite sequence $A_1, A_2, \ldots, A_n, \ldots$ of sets, indexed by the positive integers.
- The **direct product** of $A_1, A_2, \ldots, A_n, \ldots$, written $\prod_{i=1}^{\infty} A_i$ or $A_1 \times A_2 \times \cdots \times A_n \times \cdots$, is the set whose elements are all infinite sequences $(a_1, a_2, \ldots, a_n, \ldots)$, such that $a_i \in A_i$, for each positive integer i.
- Example: The set of points of Euclidean n-space yields an example of a direct product of sets. If for $i = 1, 2, \ldots, n$, we have $A_i = \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, then $\mathbb{R}^n = \prod_{i=1}^{n} A_i$ is the set of points of a Euclidean n-space. An element $x \in \mathbb{R}^n$ is a sequence $x = (x_1, x_2, \ldots, x_n)$ of real numbers.
- In general, if the sets $A_1, A_2, \ldots, A_n$ are all equal to the same set A, we write $A^n = \prod_{i=1}^{n} A_i$ and call an element $a = (a_1, a_2, \ldots, a_n) \in A^n$ an n-**tuple**.

Subsection 6

Functions

Functions and Graphs

Definition (Function)

Let A and B be sets. A correspondence that associates with each element $x \in A$ an element $f(x) \in B$ is called a **function from A to B**. We write $f : A \to B$ or $A \xrightarrow{f} B$ to denote the function.

Definition (Graph of a Function)

Let $f : A \to B$. The subset $\Gamma_f \subseteq A \times B$, which consists of all ordered pairs of the form $(a, f(a))$, is called the **graph** of $f : A \to B$.

- Let A and B be sets. Given a subset Γ of $A \times B$, there is a function $f : A \to B$, such that Γ is the graph of $f : A \to B$, if, for each $x \in A$, there is one and only one element of the form $(x, y) \in \Gamma$.

Image, Inverse Image, Domain and Range

Definition (Image and Inverse Image)

Let $f : A \to B$ be given. For each subset X of A, the subset of B whose elements are the points $f(x)$, such that $x \in X$, is denoted by $f(X)$. $f(X)$ is called the **image** of X.

For each subset Y of B, the subset of A whose elements are the points $x \in A$, such that $f(x) \in Y$ is denoted by $f^{-1}(Y)$. $f^{-1}(Y)$ is called the **inverse image** of Y or f **inverse** of Y.

Definition (Domain and Range)

Let $f : A \to B$ be given.

A is called the **domain** of f.

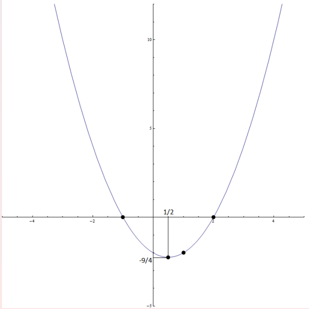
$f(A)$ is called the **range** of f.

An Example

- Let $f : \mathbb{R} \to \mathbb{R}$, $\mathbb{R}$ the set of real numbers, be the function such that, for each $x \in \mathbb{R}$,

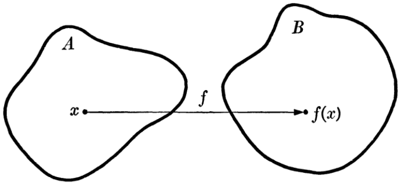
$$f(x) = x^2 - x - 2.$$

  - If X is the closed interval $[1,2]$, then $f(X) = [-2,0]$.
  - If Z is the open interval $(-1,1)$, then $f(Z) = [-\frac{9}{4}, 0)$.
  - $f^{-1}([-2,0]) = [1,2] \cup [-1,0]$.
  - $f^{-1}(\{0\}) = \{2,-1\}$ is the set of roots of the polynomial $x^2 - x - 2$.
  - $f^{-1}([-5,-4]) = \emptyset$.

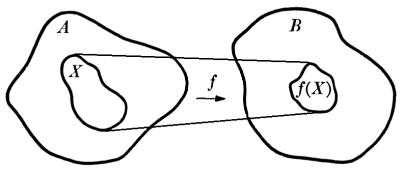


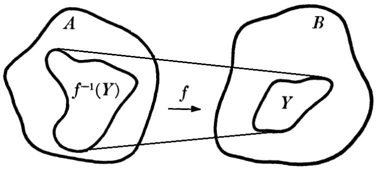
A Mapping or Transformation in Pictures

- A function $f : A \to B$ is also called a **mapping** or **transformation** of A into B. We may think of such a function as carrying each point $x \in A$ into its corresponding point $f(x) \in B$:



- $f : A \to B$ carries each subset X of A onto the subset $f(X)$ of B
- f^{-1} of a subset Y of B is the set of all $x \in A$ that are carried into points of Y.





One-to-One and Onto Functions

Definition (One-to-One Function)

A function $f : A \to B$ is called **one-one** if whenever $f(a) = f(a')$, for $a, a' \in A$, then $a = a'$.

- Thus, $f : A \to B$ is one-one if, for each $b \in f(A)$, there is only one $a \in A$, such that $f(a) = b$.
- Equivalently, by contraposition, $f : A \to B$ is one-one if, for all $a, a' \in A$, if $a \neq a'$ then $f(a) \neq f(a')$.

Definition (Onto Function)

A function $f : A \to B$ is called **onto** if $B = f(A)$.

Constant and Identity Functions

- Certain particular types of functions are frequently considered:

Definition (Constant Function)

A function $f : A \to B$ is called a **constant function** if there is a point $b \in B$, such that $f(x) = b$, for all $x \in A$.

Definition (Identity Function)

A function $f : A \to A$ is called the **identity function (on** A**)** if $f(x) = x$, for all $x \in A$.

Subsection 7

Relations

Relations

- A function may be viewed as a special case of what is called a relation.

  E.g., to say that the number 2 is less than the number 3, or $2 < 3$, is to say that $(2, 3)$ is one of the number pairs (x, y) for which the relation "less than" is true.

Definition (Relation)

A **relation** R **from** the elements of a set A **to** the elements of a set B is a subset of $A \times B$.
A **relation** R **on** a set E is a subset of $E \times E$.

- If $(x, y) \in R \subseteq A \times B$, one frequently writes $a \ R \ b$.

Reflexivity, Symmetry and Transitivity

- We define certain properties that a relation on a set E may or may not have:

Definition (Reflexivity, Symmetry and Transitivity)

A relation R on a set E is called

- **reflexive** if $a\ R\ a$ is true for all $a \in E$;
- **symmetric** if, whenever $a\ R\ b$, also $b\ R\ a$;
- **transitive** if, whenever $a\ R\ b$ and $b\ R\ c$, then $a\ R\ c$.

- Example:
  - Let $<$ be the pairs of real numbers (x, y), such that $x < y$. Then $<$ is a transitive relation on the set E of real numbers, but $<$ is not reflexive and not symmetric.
  - Let R be the pairs of real numbers (x, y), such that $|x - y| < 1$. Then R is reflexive and symmetric, but not transitive.
  - Let Λ be the pairs of real numbers (x, y), such that $x - y$ is an integer. Then Λ is reflexive, symmetric, and transitive.

Equivalence Relations and Equivalence Classes

Definition (Equivalence Relation)

A relation R on a set E which is reflexive, symmetric, and transitive is called an **equivalence relation**.

Definition (Equivalence Class)

Let R be an equivalence relation on a set E. For each $a \in E$, the **equivalence class** of a, denoted by $\pi(a)$, is the subset of E consisting of all x, such that $a\ R\ x$.

- Two equivalence classes are either disjoint or identical.

Lemma

Let R be an equivalence relation on a set E and let $\pi(a) \cap \pi(b) \neq \emptyset$, for $a, b \in E$. Then $\pi(a) = \pi(b)$.

Quotients and Projections

Lemma

Let R be an equivalence relation on a set E and let $\pi(a) \cap \pi(b) \neq \emptyset$, for $a, b \in E$. Then $\pi(a) = \pi(b)$.

- Let $c \in \pi(a) \cap \pi(b)$. Then $a \ R \ c$ and $b \ R \ c$. Suppose $x \in \pi(a)$ so that $a \ R \ x$. $c \ R \ a$ by symmetry, so $c \ R \ x$ by transitivity. Another application of transitivity yields $b \ R \ x$, so $x \in \pi(b)$. Thus $\pi(a) \subseteq \pi(b)$. Similarly, $\pi(b) \subseteq \pi(a)$.
- By the reflexive property, $a \in \pi(a)$ is always true.
- So the equivalence classes are non-empty and disjoint.
- Let E/R be the set of equivalence classes. Then $\pi : E \to E/R$ is an onto function. E/R is sometimes called the **quotient of E by the relation R**, and π is called the **projection**.

Subsection 8

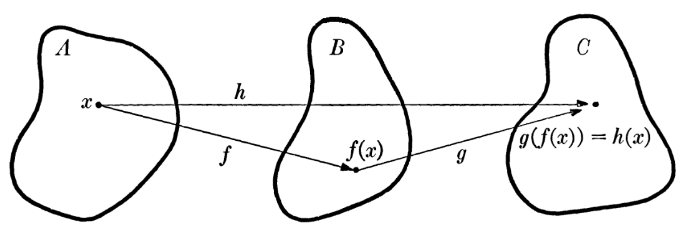
Composition and Diagrams

Composition of Functions

Definition (Composition)

Let $f : A \to B$ and $g : B \to C$ be given. The composition of $f : A \to B$ and $g : B \to C$ is the correspondence that associates with each element $a \in A$, the element $g(f(a)) \in C$. This function is written $gf : A \to C$, or $A \stackrel{gf}{\to} C$

- A function $h : A \to C$ is, therefore, the composition of $f : A \to B$ and $g : B \to C$, abbreviated $h = gf$, if for each $a \in A$, $h(a) = g(f(a))$. I.e., $h = gf$ when these functions behave as follows:



Composition of a Finite Number of Functions

Definition

Let $f_1 : A_1 \to A_2, f_2 : A_2 \to A_3, \ldots, f_n : A_n \to A_{n+1}$ be given. The **composition** of $f_1 : A_1 \to A_2, f_2 : A_2 \to A_3, \ldots, f_n : A_n \to A_{n+1}$ is the correspondence that associates with each element $x \in A_1$ the element $f_n(\cdots f_2(f_1(x))\cdots) \in A_{n+1}$. We write $f_n \cdots f_2 f_1 : A_1 \to A_{n+1}$ or $A_1 \xrightarrow{f_n \cdots f_2 f_1} A_{n+1}$ for this function.

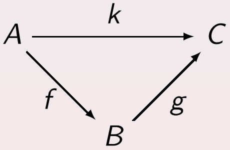
- Given $f : A \to B$, $g : B \to C$ and $h : C \to D$, consider:
  - $hgf : A \to D$;
  - $gf : A \to C$ composed with $h : C \to D$: $h(gf) : A \to D$.
  - Similarly, $(hg)f : A \to D$.

  We compute:
  - $(hgf)(x) = h(g(f(x)))$;
  - $(h(gf))(x) = h((gf)(x)) = h(g(f(x)))$;
  - $((hg)f)(x) = (hg)(f(x)) = h(g(f(x)))$.

  Since the three functions are equal, parenthesis may be dropped.

Triangles

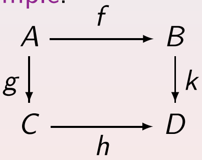
- Suppose we are given three functions $f : A \to B$, $g : B \to C$ and $k : A \to C$.
- The existence of these three functions may be indicated by a **diagram**:



- The letters A, B, C stand for the various sets, and an arrow leading from one set to another indicates a function from the first set to the second.
- The fact that we may form the composition of two functions (such as $gf : A \to C$ in the above diagram) is represented by a path in the direction of the arrows that goes from one set to a second and from the second set to a third.

Diagrams and Functions

- By a **diagram** we shall mean a figure consisting of several symbols denoting sets and arrows leading from one symbol to another, each arrow leading from a set X to a set Y having an associated symbol t, the arrow and its symbol representing a given function $t : X \to Y$.
- Example:

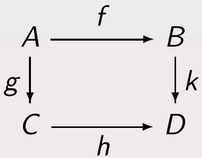


This diagram indicates the existence of given functions $f : A \to B$, $g : A \to C$, $k : B \to D$, $h : C \to D$. The diagram shows that by composing functions we may obtain two functions from A to D $kf, hg : A \to D$.

- In any diagram, a path from X to Y consisting of a sequence of arrows leading from X to Y indicates the existence of a function from X to Y obtained by composing the functions represented by these arrows in the order of their occurrence, starting at X and terminating at Y.

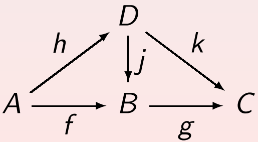
Commutative Diagrams

- In the diagram



it may or may not be true that $kf = hg$. In the event that $kf = hg$ we say that the **diagram** (or the **rectangle**) **commutes** or **is commutative**.

- In general, a diagram is said to **commute** or to **be commutative** if for each X and Y in the diagram that represent sets, and for any two paths in the diagram beginning at X and ending at Y, the two functions from X to Y so represented are equal.
- Example: Consider he diagram



The statement that "this diagram is commutative" means that: $f = jh$; $k = gj$; $kh = gjh = gf$.

It is worth noting that the first two equalities imply the third.

Diagrams with Multiple Occurrences of a Set

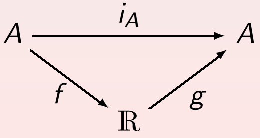
- A given set A may occur more than once in a diagram.
- Example: Let A be the set of positive real numbers and $\mathbb{R}$ the set of real numbers. Let $f : A \to \mathbb{R}$ be defined by the correspondence

$$f(x) = \ln x, \quad x \in A,$$

and let $g : \mathbb{R} \to A$ be defined by the correspondence

$$g(x) = e^x, \quad x \in \mathbb{R}.$$

Let $i_A : A \to A$ be the identity function. Then, since $(gf)(x) = e^{\ln x} = x = i_A(x)$, the following diagram is commutative:



Subsection 9

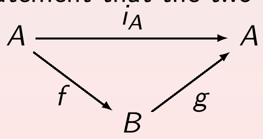
Inverse Functions, Extensions and Restrictions

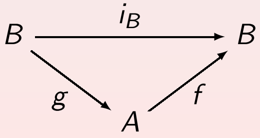
Inverse Functions

Definition (Inverse Functions)

Let $f : A \to B$ and $g : B \to A$ be given. The function $f : A \to B$ is called the **inverse** of $g : B \to A$ and the function $g : B \to A$ is called the **inverse** of $f : A \to B$ if $g(f(a)) = a$, for each $a \in A$, and $f(g(b)) = b$, for each $b \in B$.
In this event we also say that $f : A \to B$ and $g : B \to A$ are **inverse functions** and that each of them is **invertible**.

- Let $i_A : A \to A$ and $i_B : B \to B$ be identity functions. The statement that $f : A \to B$ and $g : B \to A$ are inverse functions is equivalent to the statement that the two diagrams





are commutative.

Invertibility Implies Bijectivity

Theorem

If $f : A \to B$ and $g : B \to A$ are inverse functions, then both functions are one-one and onto.

- Suppose $f(x) = f(y)$. Then $x = g(f(x)) = g(f(y)) = y$. Therefore, f is one-one.

  To show that f is onto, let $b \in B$. We have $f(g(b)) = b$. Therefore, if we set $a = g(b)$, we have $b = f(a)$ and f is onto.

  The roles of the two functions may be interchanged, since the definition of inverse functions imposes conditions symmetrical with regard to the two functions.

  Therefore, $g : B \to A$ is also one-one and onto.

Bijectivity Implies Invertibility

- We have shown that, given a function $h : X \to Y$, a necessary condition that this function be invertible is that the function be one-one and onto. This condition is also sufficient.

Theorem

Let $f : A \to B$ be one-one and onto. Then there exists a function $g : B \to A$, such that these two functions are inverse functions.

- We shall first define $g : B \to A$. Given $b \in B$, we may write $b = f(a)$, for some $a \in A$, since f is onto. Furthermore, since f is one-one, there is only one element such that $f(a) = b$. We define $g(b) = a$. The correspondence that associates with each $b \in B$ the element $a \in A$, as defined above, is a function $g : B \to A$.
  We have $f(g(b)) = b$, for each $b \in B$, by the definition of $g : B \to A$. Given $a \in A$, let $a' = g(f(a))$. Then $f(a') = f(g(f(a))) = f(a)$, by the remark just made. Since $f : A \to B$ is one-one, $a = a' = g(f(a))$. Thus, $f : A \to B$ and $g : B \to A$ are inverse functions.

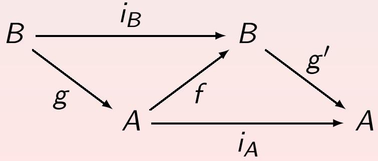
Uniqueness of Inverses

- If a function $f : A \to B$ has an inverse $g : B \to A$, the function $g : B \to A$ is uniquely determined.

Theorem

Let $f : A \to B$, $g : B \to A$ be inverse functions and let $f : A \to B$ and $g' : B \to A$ be inverse functions. Then $g : B \to A$ and $g' : B \to A$ are equal.

- We show $g(b) = g'(b)$, for each $b \in B$. We know $b = f(g(b))$. Thus, $g'(b) = g'(f(g(b))) = g(b)$.
- The proof of this last theorem may also be viewed as a direct consequence of the commutativity of the diagram



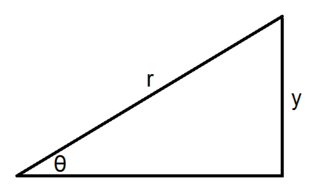
It yields $g'(b) = g'(i_B(b)) = g'(f(g(b))) = i_A(g(b)) = g(b)$.

Extensions and Restrictions

Definition (Extensions and Restrictions)

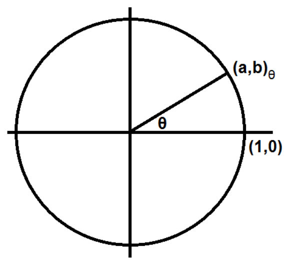
Let $A \subseteq X$. Let $f : A \to Y$ and $F : X \to Y$. If for each $x \in A$, $f(x) = F(x)$, we say that F is an **extension of f to X** or that f is a **restriction of F to A**. In this event we shall write $f = F|_A$.

- Example: Let A be the open interval $(0, \frac{\pi}{2})$. For each $\theta \in A$, let Δ_θ be a right triangle one of whose acute angles is θ radians, and let $f(\theta) = \frac{y}{r}$ be the ratio of the length y of the side of this triangle opposite the angle of magnitude θ to the length r of the hypotenuse of Δ_θ. Thus, $f : A \to \mathbb{R}$.



Example of an Extension (Cont'd)

- For each $\theta \in \mathbb{R}$, let $(a, b)_\theta$ be the point of the plane $\mathbb{R}^2$ whose distance from the origin is 1 and such that the rotation about the origin of the line segment whose end points are the origin and $(1, 0)$ to the position of the line segment whose end points are the origin and $(a, b)_\theta$ represents an angle of magnitude θ radians.



Define $F(\theta) = b$. Then $F : \mathbb{R} \to \mathbb{R}$. F is an extension of f to $\mathbb{R}$ as is easily seen if one recognizes:

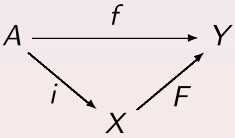
- $f : A \to \mathbb{R}$ as the sine function, defined for acute angles by means of right triangles;
- $F : \mathbb{R} \to \mathbb{R}$ as the sine function defined for angles of arbitrary magnitude by means of the unit circle.

Inclusion Mappings

Definition (Inclusion Mapping)

Let $A \subseteq X$. The function $i : A \to X$, defined by the correspondence $i(x) = x$, for each $x \in A$, is called an **inclusion mapping** or **function**.

- Let $A \subseteq X$ and $F : X \to Y$. Then F is an extension of f if and only if the diagram



  is commutative, where $i : A \to X$ is an inclusion mapping.
- Given $F : X \to Y$, there are as many restrictions of $F : X \to Y$ as there are subsets of X. Given a subset $A \subseteq X$, we may obtain the restriction of F to A by forming the composition of the inclusion mapping $i : A \to X$ and $F : X \to Y$. Thus, we have $F|_A = Fi$.

Subsection 10

Arbitrary Products

Points in Product Spaces Viewed as Functions

- Let $X_1, \ldots, X_n$ be sets.
  We have defined a point $x = (x_1, \ldots, x_n) \in \prod_{i=1}^n X_i$ as an ordered sequence such that $x_i \in X_i$.
  - Given such a point, by setting $x(i) = x_i$ we obtain a function x which associates to each integer $i, 1 \leq i \leq n$, the element $x(i) \in X_i$.
  - Conversely, given a function x which associates to each integer $i, 1 \leq i \leq n$, an element $x(i) \in X_i$, we obtain the point $(x(1), \ldots, x(n)) \in \prod_{i=1}^n X_i$.

  It is easily seen that this correspondence between points of $\prod_{i=1}^n X_i$ and functions of the above type is one-one and onto.
- Thus, a point of $\prod_{i=1}^n X_i$ may also be defined as a function x which associates to each integer $i, 1 \leq i \leq n$, a point $x(i) \in X_i$.
- The advantage of this second point of view is that it allows us to define the product of an arbitrary family of sets.

Product of Indexed Family of Sets

Definition (Product of an Indexed Family of Sets)

Let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of sets. The **product** of the sets $\{X_\alpha\}_{\alpha \in I}$, written $\prod_{\alpha \in I} X_\alpha$, consists of all functions x, with domain the indexing set I, having the property that for each $\alpha \in I$, $x(\alpha) \in X_\alpha$.

- Given a point $x \in \prod_{\alpha \in I} X_\alpha$, one may refer to $x(\alpha)$ as the α**th coordinate** of x.
- Unless the indexing set has been ordered in some fashion, there is no first coordinate, second coordinate, and so on.

Definition (Projections)

Let $x \in \prod_{\alpha \in I} X_\alpha$. The function $p_\alpha : \prod_{\alpha \in I} X_\alpha \to X_\alpha$, defined by $p_\alpha(x) = x(\alpha)$, is called the α**th projection**.

- Clearly two points $x, x' \in \prod_{\alpha \in I} X_\alpha$ are identical if and only if, for each $\alpha \in I$, $p_\alpha(x) = p_\alpha(x')$, i.e., $x(\alpha) = x'(\alpha)$.

Axiom of Choice and Surjectivity of Projections

- In dealing with product spaces, we use the Axiom of Choice:

  Axiom of Choice: If, for all $\alpha \in I$, we can choose $x_\alpha \in X_\alpha$, then we may construct a point (function) $x \in \prod_{\alpha \in I} X_\alpha$ by setting $x(\alpha) = x_\alpha$.
- This is equivalent to the statement:

  The product of non-empty sets is non-empty.
- Using the axiom of choice we may prove:

Proposition (Projections of Nonempty Products are Onto)

If for each $\alpha \in I$, X_α is non-empty, then each of the projection maps $p_\alpha : \prod_{\alpha \in I} X_\alpha \to X_\alpha$ is onto.

- Let $x_\alpha \in X_\alpha$ be given. Set $x(\alpha) = x_\alpha$. Suppose $\beta \in I, \beta \neq \alpha$. Since X_β is non-empty, choose a point $x(\beta) \in X_\beta$. Then $x \in \prod_{\alpha \in I} X_\alpha$ and $p_\alpha(x) = x(\alpha) = x_\alpha$. Hence p_α is onto.
- If $B \subseteq X_\alpha$, then $x \in p_\alpha^{-1}(B)$ means that the αth coordinate of x lies in B with all other coordinates unrestricted.