Introduction to Topology

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Subsection 1

Introduction

Metric Spaces and Closedness of Pairs of Points

- A metric space is a set of points and a prescribed quantitative measure of the degree of closeness of pairs of points in this space.
- The real number system and the coordinate plane of analytic geometry are familiar examples of metric spaces.
- Starting from the vague characterization of a continuous function as one that transforms nearby points into points that are themselves nearby, we can, in a metric space, formulate a precise definition of continuity.
- This definition may be stated in the so-called "ϵ, δ" terminology.
- Other, equivalent formulations available in a metric space include characterizations
    - in terms of the behavior of a function with respect to certain subsets called neighborhoods of a point;
    - with respect to certain subsets called open sets.

Subsection 2

Metric Spaces

Distance in $\mathbb{R}$

- Given two real numbers a and b, there is determined a non-negative real number $|a-b|$, called the **distance** between a and b.
- Since to each ordered pair (a,b) of real numbers there is associated the real number $|a-b|$, we may write this correspondence in functional notation by setting

$$d(a,b) = |a-b|.$$

  Thus, we have a function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.
- This function has four important properties. For all $x, y, z \in \mathbb{R}$,:
  1. $d(x,y) \geq 0$;
  2. $d(x,y) = 0$ if and only if $x = y$;
  3. $d(x,y) = d(y,x)$;
  4. $d(x,z) \leq d(x,y) + d(y,z)$
- For the purposes of discussing “continuity” of functions, these four properties of “distance” are sufficient.

Metric Spaces

- We can discuss "continuity" in a more general setting, in terms of any set of points with a "distance function", such as $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Definition (Metric Space)

A pair of objects (X, d) consisting of a non-empty set X and a function $d : X \times X \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, is called a **metric space** provided that:

1. $d(x, y) \geq 0$, for all $x, y \in X$;
2. $d(x, y) = 0$ if and only if $x = y$, for all $x, y \in X$;
3. $d(x, y) = d(y, x)$, for all $x, y \in X$;
4. $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

The function d is called a **distance function** or **metric** on X and the set X is called the **underlying set**.

Remarks

- A more precise notation that (X, d) for a metric space would be $(X, d : X \times X \to \mathbb{R})$ and for a distance function $d : X \times X \to \mathbb{R}$, but we frequently delete the sets and arrow in the symbol for a function, when, in a given context, it is clear which sets are involved.
- We may think of the distance function d as providing a quantitative measure of the degree of closeness of two points.
- The inequality $d(x, z) \leq d(x, y) + d(y, z)$, thus, asserts the transitivity of closeness:

  If x is close to y and y is close to z, then x is close to z.
- Let $a, b \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The verification that the function $d(a, b) = |a - b|$ satisfies the four properties in the definition establishes:

Theorem

$(\mathbb{R}, d)$ is a metric space, where d is the function defined by

$$d(a, b) = |a - b|, \quad \text{for all } a, b \in \mathbb{R}.$$

The Maximum Metric

- Given a finite collection $(X_1, d_1), (X_2, d_2), \ldots, (X_n, d_n)$ of metric spaces, there is a procedure for converting the set $X = \prod_{i=1}^{n} X_i$ into a metric space, i.e., for defining a distance function on X.

Theorem

Let $(X_1, d_1), (X_2, d_2), \ldots, (X_n, d_n)$ be metric spaces and set $X = \prod_{i=1}^{n} X_i$. For each pair of points $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X$, let $d : X \times X \to \mathbb{R}$ be the function defined by the correspondence $d(x, y) = \max_{1 \leq i \leq n} \{d_i(x_i, y_i)\}$. Then (X, d) is a metric space.

- With x and y as above, $d_i(x_i, y_i) \geq 0$, for $1 \leq i \leq n$, and therefore $d(x, y) \geq 0$.

  If $d(x, y) = 0$, then $d_i(x_i, y_i) = 0$, for $1 \leq i \leq n$, and therefore $x_i = y_i$, for each i. Consequently, $x = y$. Conversely, if $x = y$, then $d_i(x_i, y_i) = 0$, for each i, and $d(x, y) = 0$.

  Since $d_i(x_i, y_i) = d_i(y_i, x_i)$, for $1 \leq i \leq n$, $d(x, y) = d(y, x)$.

Showing the Triangle Inequality

- Finally, let $z = (z_1, \ldots, z_n) \in X$. Let j and k be integers such that $d(x, y) = d_j(x_j, y_j)$ and $d(y, z) = d_k(y_k, z_k)$. Then, for $1 \leq i \leq n$,

$$d_i(x_i, y_i) \leq d_j(x_j, y_j) \quad \text{and} \quad d_i(y_i, z_i) \leq d_k(y_k, z_k).$$

  Thus,

$$\begin{array}{rcl} d_i(x_i, z_i) & \leq & d_i(x_i, y_i) + d_i(y_i, z_i) \\ & \leq & d_j(x_j, y_j) + d_k(y_k, z_k) \\ & = & d(x, y) + d(y, z). \end{array}$$

  Therefore, $d(x, z) = \max_{1 \leq \leq n} \{d_i(x_i, z_i)\} \leq d(x, y) + d(y, z)$.
- As an immediate application of this theorem, we have:

Corollary

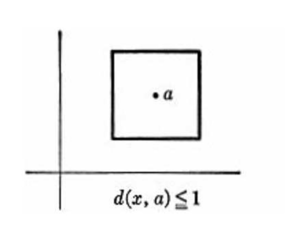
$(\mathbb{R}^n, d)$ is a metric space, where $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the function defined by $d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_{1 \leq i \leq n} \{|x_i - y_i|\}$, for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$.

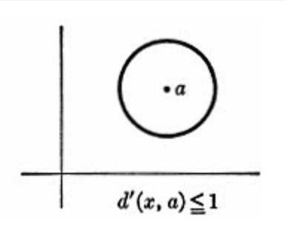
Two Metric Spaces on $\mathbb{R}^2$

- It is interesting to compare the metric space $(\mathbb{R}^2, d)$ with what might be considered a more natural model of the coordinate plane:
  - In $(\mathbb{R}^2, d)$, the distance from the point $(1, 2)$ to the point $(3, 1)$ is 2, since $\max\{|1-3|, |2-1|\} = 2$.
  - The distance function d' used in analytic geometry would yield
  $$d'((1,2),(3,1)) = \sqrt{(1-3)^2 + (2-1)^2} = \sqrt{5}.$$
- If, for each pair of points $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, we define
$$d'((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$
then we are constructing a new metric space $(\mathbb{R}^2, d')$ (provided, of course, that d' is a distance function), which must be distinguished from the metric space $(\mathbb{R}^2, d)$, where
$$d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

Illustration of the Metric Spaces

- In $(\mathbb{R}^2, d)$ the set M of points x, such that $d(x, a) \leq 1$ for a fixed point $a \in \mathbb{R}^2$ is a square of width 2 whose center is at a and whose sides are parallel to the coordinate axes:





- In $(\mathbb{R}^2, d')$ the set of points x, such that $d'(x, a) \leq 1$, for a fixed point $a \in \mathbb{R}^2$ is a circular disc whose center is a and whose radius is 1.

Two Metric Spaces on $\mathbb{R}^n$

- The formula used to define the function d' may be generalized to yield the **Euclidean distance** function for $\mathbb{R}^n$:

Theorem (Euclidean Metric on $\mathbb{R}^n$)

$(\mathbb{R}^n, d')$ is a metric space, where d' is the function defined by the correspondence

$$d'(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2},$$

for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

- The proof is postponed for later.
- The fact that we have metric spaces $(\mathbb{R}^n, d)$ and $(\mathbb{R}^n, d')$, with d and d' defined as above, serves to emphasize the fact that a metric space consists of two objects, a set and a distance function.

  Two metric spaces may be distinct even though the underlying sets of points of the two spaces are the same.

Subsection 3

Continuity

Towards a Formal Definition of Continuity

- A distance function for the real numbers $\mathbb{R}$ provides a measure of the degree of closeness of two numbers.
- To capture "the function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $a \in \mathbb{R}$", we must formalize the statement "a number $f(x)$ will be close to the number $f(a)$ whenever the number x is close to a".
- We require that, no matter what choice is made for the degree of closeness of $f(x)$ to $f(a)$, we can find a corresponding degree of closeness so that whenever x is within this corresponding degree of closeness to a, then $f(x)$ is within the prescribed degree of closeness to $f(a)$.
- We thus obtain that "the function $f : \mathbb{R} \to \mathbb{R}$ is continuous at the number $a \in \mathbb{R}$, if given a prescribed degree of closeness, $f(x)$ will be within this prescribed degree of closeness to $f(a)$, whenever x is within some corresponding degree of closeness to a".

Real Continuous Functions

- To put the statement in its final form, we substitute:
  - for "a prescribed degree of closeness" the symbol "ϵ";
  - for the phrase "some corresponding degree of closeness" the symbol "δ" and use the distance function to measure the degree of closeness.

Definition (Continuous Real Function)

Let $f : \mathbb{R} \to \mathbb{R}$. The function f is said to be **continuous at the point** $a \in \mathbb{R}$, if, given $\epsilon > 0$, there is a $\delta > 0$, such that

$$|f(x) - f(a)| < \epsilon, \text{ whenever } |x - a| < \delta.$$

The function f is **continuous** if it is continuous at each point of $\mathbb{R}$.

Continuity in Metric Spaces

- We may easily devise a definition of "continuity" applicable to metric spaces in general:

Definition

Let (X, d) and (Y, d') be metric spaces, and let $a \in X$. A function $f : X \to Y$ is said to be **continuous at the point** $a \in X$ if, given $\epsilon > 0$, there is a $\delta > 0$, such that

$$d'(f(x), (a)) < \epsilon \text{ whenever } x \in X \text{ and } d(x, a) < \delta.$$

The function $f : X \to Y$ is **continuous** if it is continuous at each point of X.

- Definitions are created to serve two purposes:
  - They are abbreviations. Thus, "given $\epsilon > 0$, there is ..." is replaced by the shorter statement, "$f : X \to Y$ is continuous at the point $a \in X$".
  - They are attempts to formulate precise characterizations of significant properties, e.g., the property of being continuous at a point.

Continuity of Constant and Identity Functions

Theorem (Continuity of Constant Functions)

Let (X, d) and (Y, d') be metric spaces. Let $f : X \to Y$ be a constant function. Then f is continuous.

- Let a point $a \in X$ and $\epsilon > 0$ be given. Choose any $\delta > 0$, say $\delta = 1$. Then, whenever $d(x, a) < \delta$, we have $d'(f(x), f(a)) = 0 < \epsilon$.

Theorem (Continuity of Identities)

Let (X, d) be a metric space. Then the identity function $i_X : X \to X$ is continuous.

- Suppose $a \in X$. Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. Then, whenever $d(x, a) < \delta$, we have $d(i_X(x), i_X(a)) = d(x, a) < \epsilon$.
- In the last proof, δ could be any positive number, provided only that $\delta \leq \epsilon$. The choice of δ need not be a very efficient choice as long as it "does the job".

Avoiding Ambiguities

- There is one situation we shall have to consider for which the notation $f : X \to Y$ that we have adopted for a function from a metric space (X, d) into a metric space (Y, d') is ambiguous.
- Consider metric spaces (X, d) and (X, d') with the same underlying set. If we simply write $f : X \to X$ for a function, it is impossible to tell which metric space is denoted by the first occurrence of X and which by the second.
- For this reason, when considering one set X with two different distance functions, we shall write

$$f : (X, d) \to (X, d')$$

if we intend to think of $f : X \to X$ as a function from the metric space (X, d) into the metric space (X, d').

A Representative Theorem

Theorem

Let $i : \mathbb{R}_n \to \mathbb{R}_n$ be the identity function. Then $i : (\mathbb{R}^n, d) \to (\mathbb{R}^n, d')$ and $i : (\mathbb{R}^n, d') \to (\mathbb{R}^n, d)$ are continuous, where the distance function d is the "max distance" and d' is the Euclidean distance.

- Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$.
  - We prove $i : (\mathbb{R}^n, d) \to (\mathbb{R}^n, d')$ is continuous. Let $\epsilon > 0$ be given. Choose $\delta = \frac{\epsilon}{\sqrt{n}}$. Suppose $x = (x_1, x_2, \ldots, x_n)$ is such that $d(x, a) < \delta$, i.e., $\max_{1 \leq i \leq n}\{|a_i - x_i|\} < \delta$. Then $d'(x, a) = \sqrt{\sum_{i=1}^n (a_i - x_i)^2}$ $< \sqrt{n\delta^2} = \sqrt{\epsilon^2} = \epsilon$. Therefore, given $\epsilon > 0$, there is a a $\delta > 0$, such that $d'(i(x), i(a)) < \epsilon$ whenever $d(x, a) < \delta$.
  - We prove $i : (\mathbb{R}^n, d') \to (\mathbb{R}^n, d)$ is continuous. Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. Let $x = (x_1, x_2, \ldots, x_n)$ be such that $d'(x, a) < \delta$. Then $\sum_{i=1}^n (a_i - x_i)^2 < \delta^2$ and, therefore, for each i, $(a_i - x_i)^2 < \delta^2$, or $|a_i - x_i| < \delta = \epsilon$. Consequently, $d(x, a) < \epsilon$. Thus, given $\epsilon > 0$, there is a $\delta > 0$, such that $d(i(x), i(a)) < \epsilon$ whenever $d'(x, a) < \delta$.

Composition of Continuous Functions

- The composition of two continuous functions is continuous.

Theorem

Let $(X,d),(Y,d'),(Z,d'')$ be metric spaces. Let $f:X\to Y$ be continuous at the point $a\in X$ and let $g:Y\to Z$ be continuous at the point $f(a)\in Y$. Then $gf:X\to Z$ is continuous at the point $a\in X$.

- Let $\epsilon>0$ be given. We must find a $\delta>0$, such that whenever $x\in X$ and $d(x,a)<\delta$, then $d''(g(f(x)),g(f(a)))<\epsilon$. Since g is continuous at $f(a)$, there is an $\eta>0$, such that whenever $y\in Y$ and $d'(y,f(a))<\eta$, then $d''(g(y),g(f(a)))<\epsilon$. Since f is continuous at a, given $\eta>0$, there is a $\delta>0$, such that $x\in X$ and $d(x,a)<\delta$ imply that $d'(f(x),f(a))<\eta$ and, hence, $d''(g(f(x)),g(f(a)))<\epsilon$.

Corollary

Let $(X,d),(Y,d'),(Z,d'')$ be metric spaces. Let $f:X\to Y$ and $g:Y\to Z$ be continuous. Then $gf:X\to Z$ is continuous.

Subsection 4

Open Balls and Neighborhoods

Open Balls in Metric Spaces

- Open balls will enable us to cast the definition of continuity in a more compact form.

Definition (Open Ball)

Let (X, d) be a metric space. Let $a \in X$ and $\delta > 0$ be given. The subset of X consisting of those points $x \in X$, such that $d(a, x) < \delta$, is called the **open ball about** a **of radius** δ and is denoted by $B(a; \delta)$.

- Thus, $x \in B(a; \delta)$ if and only if $x \in X$ and $d(x, a) < \delta$.
- Similarly, if (Y, d') is another metric space and $f : X \to Y$, we have $y \in B(f(a); \epsilon)$ if and only if $y \in Y$ and $d'(y, f(a)) < \epsilon$.

Continuity in terms of Open Balls

- Thus, we have the following:

Theorem

A function $f : (X, d) \to (Y, d')$ is continuous at a point $a \in X$ if and only if, given $\epsilon > 0$, there is a $\delta > 0$, such that $f(B(a; \delta)) \subseteq B(f(a); \epsilon)$.

- For a function $f : X \to Y$ we have $f(U) \subseteq V$ if and only if $U \subseteq f^{-1}(V)$, where U and V are subsets of X and Y, respectively.
- Thus, we also obtain

Theorem

A function $f : (X, d) \to (Y, d')$ is continuous at a point $a \in X$ if and only if, given $\epsilon > 0$, there is a $\delta > 0$, such that $B(a; \delta) \subseteq f^{-1}(B(f(a); \epsilon))$.

Neighborhoods of Points in Metric Spaces

- Given a point a in a metric space (X, d), the subset $B(a; \delta)$ of X, for each $\delta > 0$, is an example of the type of subset of X that is called a neighborhood of a:

Definition (Neighborhood)

Let (X, d) be a metric space and $a \in X$. A subset N of X is called a **neighborhood of** a if there is a $\delta > 0$, such that $B(a; \delta) \subseteq N$. The collection $\mathfrak{N}_a$ of all neighborhoods of a point $a \in X$ is called a **complete system of neighborhoods** of the point a.

- A neighborhood of a point $a \in X$ may be thought of as containing all the points of X that are sufficiently close to a or as "enclosing" a by virtue of the fact that it contains some open ball about a.
- In particular, for each $\delta > 0$, $B(a; \delta)$ is a neighborhood of a.

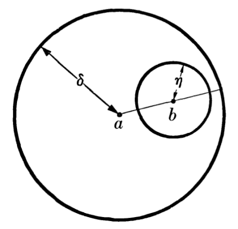
Open Balls Neighborhoods of All Their Points

- The open balls $B(a;\delta)$ have the property that they are neighborhoods of each of their points.

Lemma

Let (X,d) be a metric space and $a \in X$. For each $\delta > 0$, the open ball $B(a;\delta)$ is a neighborhood of each of its points.

- Let $b \in B(a;\delta)$.
In order to show that $B(a;\delta)$ is a neighborhood of b we must show that there is an $\eta > 0$, such that $B(b;\eta) \subseteq B(a;\delta)$. Since $b \in B(a;\delta)$, $d(a,b) < \delta$. Choose $\eta < \delta - d(a,b)$. If $x \in B(b;\eta)$, then we obtain



$$d(a,x) \leq d(a,b) + d(b,x) < d(a,b) + \eta < d(a,b) + \delta - d(a,b) = \delta.$$

Therefore, $x \in B(a;\delta)$. Thus, $B(b;\eta) \subseteq B(a;\delta)$ and $B(a;\delta)$ is a neighborhood of b.

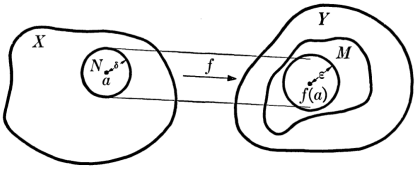
Continuity in Terms of Complete Neighborhood Systems

- The complete system of neighborhoods of a point may be used to characterize continuity of a function at a point.

Theorem

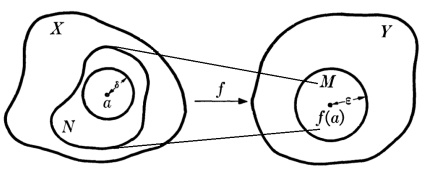
Let $f : (X, d) \to (Y, d')$. f is continuous at a point $a \in X$ if and only if for each neighborhood M of $f(a)$ there is a corresponding neighborhood N of a, such that $f(N) \subseteq M$ or equivalently, $N \subseteq f^{-1}(M)$.

- First suppose that f is continuous at the point $a \in X$. We must show that, given a neighborhood M of $f(a)$, we can find a neighborhood N of a such that $f(N) \subseteq M$.
  Since M is a neighborhood of $f(a)$, there is an $\epsilon > 0$, such that $B(f(a); \epsilon) \subseteq M$. Since f is continuous at a, there is a $\delta > 0$, such that $f(B(a; \delta)) \subseteq B(f(a); \epsilon)$. But $N = B(a; \delta)$ is a neighborhood of a, whence $f(N) = f(B(a; \delta)) \subseteq B(f(a); \epsilon) \subseteq M$.



Proof of the Converse

- Conversely, suppose that for each neighborhood M of $f(a)$, there is a corresponding neighborhood N of a, such that $f(N) \subseteq M$. Let $\epsilon > 0$ be given. To prove that f is continuous at a, we must show that there is a $\delta > 0$, such that $f(B(a;\delta)) \subseteq B(f(a);\epsilon)$. But $B(f(a);\epsilon) = M$ is a neighborhood of $f(a)$ whence there is a neighborhood N of a, such that $f(N) \subseteq M$. Since N is a neighborhood of a, there is a $\delta > 0$, such that $B(a;\delta) \subseteq N$. Therefore, $f(B(a;\delta)) \subseteq f(N) \subseteq M = B(f(a);\epsilon)$.



- If N is a neighborhood of a in (X, d) and $N \subseteq N'$, then N' is also a neighborhood of a. Therefore, we obtain:

Theorem

Let $f : (X, d) \to (Y, d')$. f is continuous at a point $a \in X$ if and only if for each neighborhood M of $f(a)$, $f^{-1}(M)$ is a neighborhood of a.

Properties of Neighborhoods

- The collections of neighborhoods of points in a metric space possess five important properties:

Theorem

Let (X, d) be a metric space.

N1. For each point $a \in X$, there exists at least one neighborhood of a.

N2. For each point $a \in X$ and each neighborhood N of a, $a \in N$.

N3. For each point $a \in X$, if N is a neighborhood of a and $N' \supseteq N$, then N' is a neighborhood of a.

N4. For each point $a \in X$ and each pair N, M of neighborhoods of a, $N \cap M$ is also a neighborhood of a.

N5. For each point $a \in X$ and each neighborhood N of a, there exists a neighborhood O of a, such that $O \subseteq N$ and O is a neighborhood of each of its points.

Proof of the Properties of Neighborhoods

- For $a \in X$, X is a neighborhood of a, thus N1 is true.
- N2 is trivial
- N3 has already been discussed.
- To prove N4, let N and M be neighborhoods of a. Then N and M contain open balls $B(a;\delta_1)$ and $B(a;\delta_2)$, respectively, and, therefore, $N \cap M$ contains the open ball $B(a;\delta)$, where $\delta = \min\{\delta_1, \delta_2\}$.
- To prove N5, let N be a neighborhood of a. Then N contains an open ball $B(a;\delta)$ and, hence, $O = B(a;\delta)$ is a neighborhood of each of its points.

Basis for Neighborhood System

- For a given point a in a metric space X, the collection of open balls with center a has been used to generate the complete system of neighborhoods at a, in the sense that the neighborhoods of a are precisely those subsets of X which contain one of these open balls.

Definition (Basis for Neighborhood System)

Let a be a point in a metric space X. A collection $\mathfrak{B}_a$ of neighborhoods of a is called a **basis for the neighborhood system at** a if every neighborhood N of a contains some element B of $\mathfrak{B}_a$.

- Example: If a is a point on the real line $\mathbb{R}$, a basis for the neighborhood system at a is the collection of open intervals containing a.

Subsection 5

Limits

Limit of a Sequence in $\mathbb{R}$

- Recall the concept of limit of a sequence of real numbers:

Definition (Limit of a Sequence in $\mathbb{R}$)

Let $a_1, a_2, \ldots$ be a sequence of real numbers. A real number a is said to be the **limit of the sequence** $a_1, a_2, \ldots$ if, given $\epsilon > 0$, there is a positive integer N, such that, whenever $n > N$, $|a - a_n| < \epsilon$. In this event we shall also say that the sequence $a_1, a_2, \ldots$ **converges to** a and write $\lim_n a_n = a$.

- Interpreting ϵ as an "arbitrary degree of closeness" and N as "sufficiently far out in the sequence", we see that we have defined $\lim_n a_n = a$ in the event that a_n may be made arbitrarily close to a by requiring that a_n be sufficiently far out in the sequence.

Limit of a Sequence in a Metric Space

- Suppose we have a metric space (X, d) and a sequence $a_1, a_2, \ldots$ of points of X. Given a point $a \in X$ we measure the distance from a to the successive points of the sequence, by the sequence of real numbers $d(a, a_1), d(a, a_2), \ldots$.
- It is natural to say that the limit of the sequence $a_1, a_2, \ldots$ of points of X is the point a if the limit of the sequence of real numbers $d(a, a_1), d(a, a_2), \ldots$ is the real number 0.

Definition (Limit of a Sequence in a Metric Space)

Let (X, d) be a metric space. Let $a_1, a_2, \ldots$ be a sequence of points of X. A point $a \in X$ is said to be the **limit of the sequence** $a_1, a_2, \ldots$ if $\lim_n d(a, a_n) = 0$. Again, in this event, we shall say that the sequence $a_1, a_2, \ldots$ **converges to** a and write $\lim_n a_n = a$.

Limit of a Sequence in Terms of Neighborhoods

Corollary

Let (X, d) be a metric space and $a_1, a_2, \ldots$ be a sequence of points of X. Then $\lim_n a_n = a$, for a point $a \in X$, if and only if, for each neighborhood V of a, there is an integer N, such that $a_n \in V$ whenever $n > N$.

- Let V be a neighborhood of a. For some $\epsilon > 0$, $a \in B(a; \epsilon) \subseteq V$. Thus, if $\lim_n a_n = a$, there is an integer N such that, whenever $n > N$, $d(a, a_n) < \epsilon$ and hence $a_n \in V$.
  Conversely, given $\epsilon > 0$, $B(a; \epsilon)$ is a neighborhood of a. If there is an integer N, such that for $n > N$, $a_n \in B(a; \epsilon)$, then $d(a, a_n) < \epsilon$ and $\lim_n a_n = a$.
- If S is a set of infinite points, and there is at most a finite number of elements of S for which a certain statement is false, then the statement is said to be true for **almost all** the elements of S.
- Thus, $\lim_n a_n = a$ if, for each neighborhood V of a almost all the points a_n are in V.

Continuity in Terms of Limits

- Continuity may be characterized in terms of limits of sequences:

Theorem

Let $(X,d),(Y,d')$ be metric spaces. A function $f:X\to Y$ is continuous at a point $a\in X$ if and only if, whenever $\lim_n a_n = a$ for a sequence $a_1,a_2,\ldots$ of points of X, $\lim_n f(a_n)=f(a)$.

- Suppose f is continuous at a and $\lim_n a_n = a$. Let V be a neighborhood of $f(a)$. Then $f^{-1}(V)$ is a neighborhood of a. By the preceding corollary, there is an integer N, such that $a_n\in f^{-1}(V)$, whenever $n>N$. Consequently, $f(a_n)\in V$, whenever $n>N$. Thus, for each neighborhood V of $f(a)$, there is an integer N, such that $f(a_n)\in V$, whenever $n>N$, and again, by the corollary, $\lim_n f(a_n)=f(a)$.

Continuity in Terms of Limits: The Converse

- To prove the "if" part, we show that if f is not continuous at a, then there is at least one sequence $a_1, a_2, \ldots$ of points of X, such that $\lim_n a_n = a$, but $\lim_n f(a_n) = f(a)$ is false. Since f is not continuous at a, there is a neighborhood V of $f(a)$, such that for each neighborhood U of a, $f(U) \nsubseteq V$. In particular, for each neighborhood $B(a; \frac{1}{n})$, $n = 1, 2, \ldots$, $f(B(a; \frac{1}{n})) \nsubseteq V$. Thus, for each positive integer n, there is a point a_n, with $a_n \in B(a; \frac{1}{n})$ and $f(a_n) \notin V$. Now we have:
  - $d(a, a_n) < \frac{1}{n}$ and, therefore, $\lim_n a_n = a$;
  - On the other hand, $\lim_n f(a_n) = f(a)$ is impossible, since $f(a_n) \notin V$, for all n.
- If $\lim_n a_n = a$, $\lim_n f(a_n) = f(a)$ can be written as

$$\lim_n f(a_n) = f(\lim_n a_n).$$

We may therefore describe a continuous function as one that commutes with the operation of taking limits.

Bounds and Completeness of $\mathbb{R}$

- Recall some facts about the real number system.

Definition

Let A be a set of real numbers. A number b is called an **upper bound** of A if $x \leq b$, for each $x \in A$. A number c is called a **lower bound** of A if $c \leq x$, for each $x \in A$. If A has both an upper and lower bound, A is said to be **bounded**.

An upper bound b^* of A is called a **least upper bound** (abbreviated **l.u.b.**) of A if for each upper bound b of A, $b^* \leq b$. A lower bound c^* of A is called a **greatest lower bound** (abbreviated **g.l.b.**) of A if for each lower bound c of A, $c \leq c^*$.

- Not every set of real numbers has an upper bound.
- One of the properties of the real number system, usually referred to as **the completeness postulate**, is that a non-empty set A of real numbers which has an upper bound necessarily has a l.u.b.

More on Bounds in $\mathbb{R}$

- Given a non-empty set B of real numbers which has a lower bound, the set of negatives of elements of B has an upper bound, hence a l.u.b. whose negative is a g.l.b. of B.
- It follows that every non-empty set B of real numbers which has a lower bound has a g.l.b.
- The g.l.b. of a set A in $\mathbb{R}$ may or may not be an element of A.
  Example: 0 is a g.l.b. of $[0,1]$ and $0 \in [0,1]$, whereas 0 is also a g.l.b. of $(0,1)$ but $0 \notin (0,1)$.
- The g.l.b. of a set in $\mathbb{R}$ must be arbitrarily close to that set.

Lemma

Let b be a greatest lower bound of the non-empty subset A. Then, for each $\epsilon > 0$, there is an element $x \in A$, such that $x - b < \epsilon$.

- Suppose there were an $\epsilon > 0$, such that $x - b \geq \epsilon$, for each $x \in A$. Then $b + \epsilon \leq x$, for each $x \in A$ and $b + \epsilon$ would be a lower bound of A. Since b is a g.l.b. of A, we obtain the contradiction.

Distance Between Point and Set

Corollary

Let b be a greatest lower bound of the non-empty subset A of real numbers. Then there is a sequence $a_1, a_2, \ldots$ of real numbers such that $a_n \in A$, for each n, and $\lim_n a_n = b$.

- For $\epsilon = \frac{1}{n}$, we obtain an element $a_n \in A$, such that $a_n - b < \frac{1}{n}$. Since b is a lower bound of A, $0 \leq a_n - b$. Therefore, $\lim_n a_n = b$.

Definition

Let (X, d) be a metric space. Let $a \in X$ and $A \neq \emptyset$ a subset of X. The greatest lower bound of the set of numbers of the form $d(a, x)$ for $x \in A$ is called the **distance between** a **and** A and is denoted by $d(a, A)$.

- From the preceding corollary we obtain:

Corollary

Let (X, d) be a metric space, $a \in X$, and $A \neq \emptyset$ a subset of X. Then there is a sequence $a_1, a_2, \ldots$ of points of A such that $\lim_n d(a, a_n) = d(a, A)$.

Subsection 6

Open Sets and Closed Sets

Open Sets in Metric Spaces

Definition (Open Set)

A subset O of a metric space is said to be **open** if O is a neighborhood of each of its points.

- Open sets may be characterized in terms of open balls.

Theorem

A subset O of a metric space (X, d) is an open set if and only if it is a union of open balls.

- Suppose O is open. Then for each $a \in O$, there is an open ball $B(a;\delta_a) \subseteq O$. Therefore $O = \bigcup_{a \in O} B(a;\delta_a)$ is a union of open balls. Conversely, if O is a union of open balls, then using the centers of these balls as the elements of an indexing set we can write $O = \bigcup_{a \in I} B(a;\delta_a)$. If $x \in O$, then $x \in B(a;\delta_a)$, for some $a \in I$. $B(a;\delta_a)$ is a neighborhood of x and, since $B(a;\delta_a) \subseteq O$, by N3, O is a neighborhood of x. Thus O is a neighborhood of each of its points.

Continuity in terms of Open Sets

- Open sets provide a simple characterization of continuity.

Theorem

Let $f : (X, d) \to (Y, d')$. Then f is continuous if and only if for each open set O of Y, the subset $f^{-1}(O)$ is an open subset of X.

- First, suppose f is continuous. Let $O \subseteq Y$ be open. We must show that $f^{-1}(O)$ is open, i.e., $f^{-1}(O)$ is a neighborhood of each of its points. To this end, let $a \in f^{-1}(O)$, then $f(a) \in O$ and O is a neighborhood of $f(a)$. Since f is continuous at a, $f^{-1}(O)$ is a neighborhood of a.

  Conversely, suppose for each open set $O \subseteq Y$, $f^{-1}(O)$ is open. Let $a \in X$ and let M be a neighborhood of $f(a)$. Then, there is an $\epsilon > 0$, such that $B(f(a); \epsilon) \subseteq M$. But $B(f(a); \epsilon)$ is open and therefore $f^{-1}(B(f(a); \epsilon))$ is open. Since $a \in f^{-1}(B(f(a); \epsilon))$, this subset is a neighborhood of a. Therefore $f^{-1}(M)$ contains a neighborhood of a and f is continuous at a. Since a was arbitrary, f is continuous.

Properties of Open Sets in Metric Spaces

- The collection of open sets in a metric space satisfy some important properties.

Theorem

Let (X, d) be a metric space.

O1. The empty set is open.

O2. X is open.

O3. If $O_1, O_2, \ldots, O_n$ are open, then $O_1 \cap O_2 \cap \cdots \cap O_n$ is open.

O4. If for each $\alpha \in I$, O_α is an open set, then $\bigcup_{\alpha \in I} O_\alpha$ is open.

O1. The empty set is open, for in order for it not to be open there would have to be a point $x \in \emptyset$.

O2. Given a point $a \in X$, for any $\delta > 0$, $B(a; \delta) \subseteq X$, and, therefore, X is a neighborhood of each of its points, i.e., X is open.

Proofs of O3 and O4

- O3. To prove O3, let $a \in O_1 \cap \cdots \cap O_n$, where for $i = 1, \ldots, n$, O_i is open. Then each O_i is a neighborhood of a. By N4, the intersection of two neighborhoods of a is again a neighborhood of a, and, hence, by induction, the intersection of a finite number of neighborhoods of a is again a neighborhood of a. Therefore $O_1 \cap \cdots \cap O_n$ is a neighborhood of each of its points.
- O4. Finally, to prove O4, let $a \in \bigcup_{\alpha \in I} O_\alpha$, where for each $\alpha \in I$, O_α is open. Then $\alpha \in O_\beta$, for some $\beta \in I$, and O_β is a neighborhood of a. Since $O_\beta \subseteq O$, by N3, O is a neighborhood of a. Therefore O is a neighborhood of each of its points.

Closed Sets in Metric Spaces

Definition (Closed Set)

A subset F of a metric space is said to be **closed** if its complement, $C(F)$, is open.

- In the real number system, a closed interval $[a, b]$ is a closed set, for its complement is the union of the two open sets O_1 and O_2, where
  - O_1 is the set of real numbers x, such that $x < a$;
  - O_2 is the set of real numbers x, such that $x > b$.
- A set can be both open and closed:

  In any metric space (X, d), the two sets $\emptyset$ and X are open, and therefore their complements X and $\emptyset$ are closed. Thus, X and also $\emptyset$ are both open and both closed.
- A set may also be neither open nor closed.

Limit Points

Definition (Limit Point)

Let A be a subset of a metric space X. A point $b \in X$ is called a **limit point** of A if every neighborhood of b contains a point of A different from b.

- If b is a limit point of A, then each of the open balls $B(b; \frac{1}{n})$ contains a point$a_n \in A$ and $lim_n a_n = b$.

  Thus a limit point of a set is the limit of a convergent sequence of points of A.
- The converse is false, for the point b may be a point of A while for some δ, $B(b; \delta)$ contains no point of A other than b. Thus b is not a limit point of A although the sequence $b, b, \ldots$ converges to b.

  In this latter case b is called an **isolated point** of A.

Limit Points and Closed Sets

Theorem

In a metric space X, a set $F \subseteq X$ is closed if and only if F contains all its limit points.

- Let F' denote the set of limit points of F.

  First suppose F is closed and consequently $C(F)$ is open. Choose a point $b \notin F$. Since $C(F)$ is open, there is a $\delta > 0$, such that $B(b;\delta) \subseteq C(F)$ or $B(b;\delta) \cap F = \emptyset$. Hence $b \notin F'$ and $F' \subseteq F$.

  Conversely, suppose $F' \subseteq F$, or equivalently, $C(F) \subseteq C(F')$. If $b \in C(F)$, then $b \notin F'$. It follows that for some $\delta > 0$, $B(b;\delta) \cap F = \emptyset$, or $B(b;\delta) \subseteq C(F)$. Hence $C(F)$ is open and F is closed.

Closed Sets in terms of Limit Points

Theorem

In a metric space (X, d), a set $F \subseteq X$ is closed if and only if for each sequence $a_1, a_2, \ldots$ of points of F that converges to a point $a \in X$, we have $a \in F$.

- First, let F be closed. Suppose $\lim_n a_n = a$ and $a_n \in F$, for $n = 1, 2, \ldots$.
  - If the set of points $\{a_1, a_2, \ldots\}$ is infinite then every neighborhood of a contains infinitely many points of F, a is a limit point of F, whence, by the preceding theorem, $a \in F$.
  - If this set of points is finite, then for some integer N, $a_n = a_m$, whenever $n, m > N$. Since $\lim_n a_n = a$, $d(a_n, a) = 0$, for $n > N$ or $a_n = a$, whence $a \in F$.

  Conversely, suppose that F is a set such that for each sequence with $\lim_n a_n = a$ and $a_n \in F$, for all n, we have $a \in F$. If b is a limit point of F, then b is the limit of a convergent sequence of points of F and $b \in F$. Thus, by the preceding theorem, F is closed.

Closed Sets in terms of Distance

- We characterize closed sets in terms of distance from a point to a set.

Theorem

A subset F of a metric space (X, d) is closed if and only if for each point $x \in X$, $d(x, F) = 0$ implies $x \in F$.

- First, suppose F is closed. Let $x \in X$ be such that $d(x, F) = 0$. Then, there is a sequence of points of F, such that $\lim_n d(x, a_n) = 0$. Thus, every neighborhood of x contains points of F.
  - If some $a_n = x$, x is in F.
  - Otherwise, each a_n is different from x, so that x is a limit point of the sequence and hence of F. Thus, by the preceding theorem, $x \in F$.

  Conversely, suppose that F is such that $d(x, F) = 0$ implies $x \in F$. If x is a limit point of F then $d(x, F) = 0$. Thus, in this case F contains all its limit points and is closed.

Continuity in terms of Closed Sets

- Continuity may be characterized by means of closed sets.

Theorem

Let $(X, d), (Y, d')$ be metric spaces. A function $f : X \to Y$ is continuous if and only if for each closed subset A of Y, the set $f^{-1}(A)$ is a closed subset of X.

- For $A \subseteq Y$, we have $C(f^{-1}(A)) = f^{-1}(C(A))$. But f is continuous if and only if the inverse image of each open set is an open set, and this is true if and only if the inverse image of each closed set is a closed set.

Properties of Closed Sets

- We record the following facts about closed sets.

Theorem

Let (X, d) be a metric space.

C1. X is closed.

C2. $\emptyset$ is closed.

C3. The union of a finite collection of closed sets is closed.

C4. The intersection of a family of closed sets is closed.

- C1 and C2 have already been discussed.
  C3 and C4 follow from the application of DeMorgans formulas to the corresponding properties O3 and O4 of open sets.
- The union of closed sets need not, in general, be a closed set.
  Example: For each positive integer n let F_n be the closed interval $[\frac{1}{n}, 1]$. Then $\bigcup_{n=1}^{\infty} F_n = (0, 1]$, where $(0, 1]$ is the set of real numbers x, such that $0 < x \leq 1$. The set $(0, 1]$ is not closed, for 0 is a limit point of the set but is not in the set.

Subsection 7

Subspaces and Equivalence of Metric Spaces

Subspaces

- Let (X, d) be a metric space.

  Given a non-empty subset Y of X, we may convert Y into a metric space by restricting the distance function d to $Y \times Y$.

  In this manner each non-empty subset Y of X gives rise to a new metric space $(Y, d|_{Y \times Y})$.
- On the other hand, we may be given two metric spaces (X, d) and (Y, d'). If $Y \subseteq X$, it makes sense to ask whether or not d' is the restriction of d.

Definition (Subspace)

Let (X, d) and (Y, d') be metric spaces. We say that (Y, d') is a **subspace** of (X, d) if:

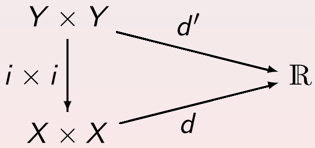
1. $Y \subseteq X$;
2. $d' = d|_{Y \times Y}$.

Subspaces in terms of Diagrams

- Let $Y \subseteq X$ and $i : Y \to X$ be an inclusion mapping.
- Denote by $i \times i : Y \times Y \to X \times X$ the inclusion mapping defined by

$$(i \times i)(y_1, y_2) = (y_1, y_2).$$

- Then (Y, d') is a subspace of (X, d) if the diagram



  is commutative.
- There are as many subspaces of a metric space (X, d) as there are non-empty subsets of X.

Examples I

- Let $\mathbb{Q}$ be the set of rational numbers. Define $d_{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ by
$$d_{\mathbb{Q}}(a, b) = |a - b|.$$
Then $(\mathbb{Q}, d_{\mathbb{Q}})$ is a subspace of $(\mathbb{R}, d)$.
- Let $\mathbb{I}^n$ (the unit n-cube) be the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers such that $0 \leq x_i \leq 1$, for $i = 1, 2, \ldots, n$. Define $d_c : \mathbb{I}^n \times \mathbb{I}^n \to \mathbb{R}$ by
$$d_c((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \max_{1 \leq i \leq n} \{|x_i - y_i|\}.$$
Then $(\mathbb{I}^n, d_c)$ is a subspace of $(\mathbb{R}^n, d)$.

Examples II

- Let S^n (the n-sphere) be the set of all $(n+1)$-tuples $(x_1, x_2, \ldots, x_{n+1})$ of real numbers such that $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$. Define $d_S : S^n \times S^n \to \mathbb{R}$ by
$$d_S((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) = \sqrt{\sum_{i=1}^{n+1} (x_i - y_i)^2}.$$
Then (S^n, d_S) is a subspace of the Euclidean space $(\mathbb{R}^{n+1}, d')$.
- Let A be the set of all $(n+1)$-tuples $(x_1, x_2, \ldots, x_{n+1})$ of real numbers such that $x_{n+1} = 0$. Define $d_A : A \times A \to \mathbb{R}$ by
$$d_A((x_1, \ldots, x_n, 0), (y_1, \ldots, y_n, 0)) = \max_{1 \leq i \leq n} \{|x_i - y_i|\}.$$
Then (A, d_A) is a subspace of $(\mathbb{R}^{n+1}, d)$.

Continuity of Subspace Inclusion Map

Theorem

Let (Y, d') be a subspace of (X, d). Then the inclusion mapping $i : Y \to X$ is continuous.

- Given $a \in Y$ and $\epsilon > 0$, choose $\delta = \epsilon$. If $d'(a, y) < \delta$, then $d(i(a), i(y)) = d(a, y) = d'(a, y) < \delta = \epsilon$.

Metric Equivalences or Isometries

- The metric space (A, d_A) of the last example is in most respects a copy of the metric space $(\mathbb{R}^n, d)$, the only distinction being that a point of $\mathbb{R}^n$ is an n-tuple of real numbers, whereas a point of A is an $(n+1)$-tuple of real numbers of which the last one is zero.
- This relationship is called "metric equivalence" or "isometry".

Definition (Isometry)

Two metric spaces (A, d_A) and (B, d_B) are said to be **metrically equivalent** or **isometric** if there are inverse functions $f : A \to B$ and $g : B \to A$, such that:

- for each $x, y \in A$, $d_B(f(x), f(y)) = d_A(x, y)$;
- for each $u, v \in B$, $d_A(g(u), g(v)) = d_B(u, v)$.

In this event we shall say that the **metric equivalence** or **isometry is defined by** f **and** g.

Characterization of Metric Equivalence

Theorem

A necessary and sufficient condition that two metric spaces (A, d_A) and (B, d_B) be metrically equivalent is that there exist a function $f : A \to B$, such that:

1. f is one-one;
2. f is onto;
3. for each $x, y \in A$, $d_B(f(x), f(y)) = d_A(x, y)$.

- The stated conditions are necessary, for if (A, d_A) and (B, d_B) are metrically equivalent, there are inverse functions $f : A \to B$ and $g : B \to A$, and therefore f is one-one and onto.

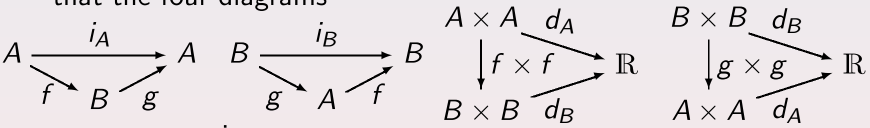
  Conversely, suppose $f : A \to B$ with the stated properties exists. Then f is invertible and the function $g : B \to A$, such that f and g are inverse functions is determined by setting $g(b) = a$ if $f(a) = b$. For $u, v \in B$, let $x = g(u)$, $y = g(v)$. Then $d_A(g(u), g(v)) = d_A(x, y) = d_B(f(x), f(y)) = d_B(u, v)$.

Distance Preservation in terms of Diagrams

- Given metric spaces (A, d_A) and (B, d_B) and functions $f : A \to B$ and $g : B \to A$, let us denote by $f \times f : A \times A \to B \times B$ the function defined by setting
$$(f \times f)(x, y) = (f(x), f(y)), \quad \text{for } x, y \in A.$$
Similarly, let $g \times g : B \times B \to A \times A$ be defined by setting
$$(g \times g)(u, v) = (g(u), g(v)), \quad \text{for } u, v \in B.$$
The statement that $d_B(f(x), f(y)) = d_A(x, y)$, for $x, y \in A$, is equivalent to the statement that the diagram
$$\begin{array}{ccc} A \times A & \xrightarrow{d_A} & \\ {\scriptstyle f \times f}\downarrow & & \mathbb{R} \\ B \times B & \xrightarrow{d_B} & \end{array}$$
is commutative. One may also describe this relation by saying that the function $f : A \to B$ is "distance preserving".

Metric Equivalence in terms of Diagrams

- The statement that (A, d_A) and (B, d_B) are metrically equivalent is the statement that there exist functions $f : A \to B$, $g : B \to A$, such that the four diagrams



  are commutative.
  - The first two diagrams say that f and g are inverse functions.
  - The last two diagrams say that f and g "preserve distances".
- Since the distance between x and y in A is the same as the distance between $f(x)$ and $f(y)$ in B, f is continuous and, similarly, g is continuous:

Lemma

Let a metric equivalence between (A, d_A) and (B, d_B) be defined by inverse functions $f : A \to B$ and $g : B \to A$. Then both f and g are continuous.

Topological Equivalence

- From the point of view of continuity, the relationship of metric equivalence is too narrow.
- We define a broader concept of equivalence in which we drop the requirement of "preservation of distance" and only require that the first two diagrams be commutative and the functions in these diagrams be continuous.

Definition (Topological Equivalence)

Two metric spaces (A, d_A) and (B, d_B) are said to be **topologically equivalent** if there are inverse functions $f : A \to B$ and $g : B \to A$, such that f and g are continuous. In this event we say that the **topological equivalence is defined by** f **and** g.

Metric versus Topological Equivalence

- As a corollary to the preceding lemma we obtain:

Corollary

Two metric spaces that are metrically equivalent are topologically equivalent.

- The converse of this corollary is false, i.e., there are metric spaces that are topologically equivalent, but are not metrically equivalent.

  Example: A circle of radius 1 is topologically equivalent to a circle of radius 2 (considered as subspaces of $(\mathbb{R}^2, d)$), but the two are not metrically equivalent.

Topological Equivalence of Two Metric over X

Lemma

Let (X, d_1) and (X, d_2) be two metric spaces. If there exists a number $K > 0$, such that for each $x, y \in X$, $d_2(x, y) \leq K d_1(x, y)$, then the identity mapping $i : (X, d_1) \to (X, d_2)$ is continuous.

- Given $\epsilon > 0$ and $a \in X$, set $\delta = \frac{\epsilon}{K}$. If $d_1(x, a) < \delta$, then $d_2(i(x), i(a)) = d_2(x, a) \leq K \cdot d_1(x, a) < K\delta = \epsilon$.

Corollary

Let (X, d) and (X, d') be two metric spaces with the same underlying set. If there exist positive numbers K and K', such that for each $x, y \in X$,

$$d'(x, y) \leq K \cdot d(x, y) \quad \text{and} \quad d(x, y) \leq K' \cdot d'(x, y),$$

then the identity mappings define a topological equivalence between (X, d) and (X, d').

An Example of Topological Equivalence

- Consider the two metric spaces $(\mathbb{R}^n, d)$ and $(\mathbb{R}^n, d')$, where:
  - the distance function d is the maximum distance between coordinates;
  - the distance function d' is the Euclidean distance function.

  For each pair of points $x, y \in \mathbb{R}^n$, the following inequality holds:

  $$d(x, y) \leq d'(x, y) \leq \sqrt{n} d(x, y).$$

  It therefore follows from the corollary that the metric spaces $(\mathbb{R}^n, d)$ and $(\mathbb{R}^n, d')$ are topologically equivalent.

Continuity of Inverse Functions

Theorem

Let (X, d) and (Y, d') be two metric spaces. Let $f : X \to Y$ and $g : Y \to X$ be inverse functions. Then the following four statements are equivalent:

1. f and g are continuous;
2. A subset O of X is open if and only if $f(O)$ is an open subset of Y;
3. A subset F of X is closed if and only if $f(F)$ is a closed subset of Y;
4. For each and subset N of X, N is a neighborhood of a if and only if $f(N)$ is a neighborhood of $f(a)$.

1⇒2 Let O be an open subset of X. Then $f(O) = g^{-1}(O)$ is open since g is continuous.

Conversely, if $f(O)$ is an open subset of Y, then $f^{-1}(f(O)) = O$ is open, since f is continuous.

Continuity of Inverse Functions (Cont'd)

2⇒4 For each $a \in X$ and $N \subseteq X$, N is a neighborhood of a if and only if N contains an open set O containing a if and only if $f(N)$ contains an open set $O' = f(O)$ containing $f(a)$ if and only if $f(N)$ is a neighborhood of $f(a)$.

4⇒1 Let $a \in X$ and let U be a neighborhood of $f(a)$. Then $f^{-1}(U)$ is a neighborhood of a, for $U = f(f^{-1}(U))$ is a neighborhood of $f(a)$. Thus f is continuous. Similarly, let $b \in Y$ and let V be a neighborhood of $g(b)$. Then $g^{-1}(V) = f(V)$ is a neighborhood of $f(g(b)) = b$, and g is continuous.

We showed Statements 1, 2, and 4 equivalent. We leave equivalence of Statements 2 and 3 as an exercise.

Comments on Topological Equivalence

- Statement 1 in the preceding theorem is the statement that the metric spaces (X, d) and (Y, d') are topologically equivalent.

  Thus, the theorem asserts that two metric spaces are topologically equivalent if and only if there exist inverse functions that establish
  - a one-one correspondence between the open sets of the two spaces, or
  - a one-one correspondence between the closed sets of the two spaces, or
  - a one-one correspondence between the complete systems of neighborhoods of the two spaces.
- Both metrically equivalent and topologically equivalent are equivalence relations defined on a collection of metric spaces.
  - Since metric equivalence implies topological equivalence, each equivalence class of metrically equivalent metric spaces is contained in an equivalence class of topologically equivalent metric spaces.
  - Distinguishing which topologically equivalent equivalence class a metric space belongs to is a coarser, but consequently more fundamental, distinction.

Subsection 8

An Infinite Dimensional Euclidean Space

The Hilbert Space

- In this section we shall define a metric space H, called **Hilbert space**, which contains as subspaces isometric copies of the various Euclidean spaces $(\mathbb{R}^n, d')$.
- A point u of H is a sequence $u_1, u_2, \ldots$ of real numbers such that the series $\sum_{i=1}^{\infty} u_i^2$ is convergent.
- Let $u = (u_1, u_2, \ldots)$ and $v = (v_1, v_2, \ldots)$ be in H. The intention is to define a metric on H by setting

$$d(u, v) = \left[\sum_{i=1}^{\infty} (u_i - v_i)^2\right]^{1/2}.$$

  To do this, we must first know that the series in brackets converges.
- We make use of the following result, known as **Schwarz's Lemma** or **Cauchy's Inequality**.

The Cauchy-Schwarz Inequality

Lemma (Cauchy-Schwarz Inequality)

Let $(u_1, u_2, \ldots, u_n), (v_1, v_2, \ldots, v_n)$ be n-tuples of real numbers. Then

$$\sum_{i=1}^{n} u_i v_i \leq \left[\sum_{i=1}^{n} u_i^2\right]^{1/2} \left[\sum_{i=1}^{n} v_i^2\right]^{1/2}.$$

- It suffices to prove that $(\sum_{i=1}^n u_i v_i)^2 \leq (\sum_{i=1}^n u_i^2)(\sum_{i=1}^n v_i^2)$. Consider, for an arbitrary $\lambda \in \mathbb{R}$, the expression $\sum_{i=1}^n (u_i + \lambda v_i)^2$. We have $0 \leq \sum_{i=1}^n (u_i + \lambda v_i)^2 = \sum_{i=1}^n u_i^2 + 2\lambda \sum_{i=1}^n u_i v_i + \lambda^2 \sum_{i=1}^n v_i^2$. Therefore, the quadratic equation in λ

$$0 = \sum_{i=1}^{n} u_i^2 + 2\lambda \sum_{i=1}^{n} u_i v_i + \lambda^2 \sum_{i=1}^{n} v_i^2$$

can have at most one real solution. Hence, its discriminant is nonpositive: $(\sum_{i=1}^n u_i v_i)^2 - (\sum_{i=1}^n u_i^2)(\sum_{i=1}^n v_i^2) \leq 0$, or, equivalently, $(\sum_{i=1}^n u_i v_i)^2 \leq (\sum_{i=1}^n u_i^2)(\sum_{i=1}^n v_i^2)$.

The Series of Products of Coordinates

Corollary

Let $u = (u_1, u_2, \ldots)$, $v = (v_1, v_2, \ldots)$ be in H with $U = \sum_{i=1}^{\infty} u_i^2$, $V = \sum_{i=1}^{\infty} v_i^2$. Than the series $\sum_{i=1}^{\infty} u_i v_i$ is absolutely convergent and

$$\sum_{i=1}^{\infty} |u_i v_i| \leq U^{1/2} V^{1/2}.$$

- For each positive integer n,

$$\begin{array}{rcl} \sum_{i=1}^{n} |u_i v_i| & = & \sum_{i=1}^{n} |u_i||v_i| \\ & \leq & \left[\sum_{i=1}^{n} |u_i|^2\right]^{1/2} \left[\sum_{i=1}^{n} |v_i|^2\right]^{1/2} \\ & \leq & U^{1/2} V^{1/2}. \end{array}$$

  Thus the partial sums of this series of positive terms are bounded and the series converges to a limit not greater than $U^{1/2} V^{1/2}$.

Linear Combinations in H

- If $\alpha, \beta \in \mathbb{R}$ and we set $\alpha u + \beta v = (\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2, \ldots)$, then $\alpha u + \beta v$ is also in H for $\sum_{i=1}^{\infty} (\alpha u_i + \beta v_i)^2$ is the sum of three absolutely convergent series.

  In particular $u + v \in H$ and

$$\begin{array}{rcl} \sum_{i=1}^{\infty} (u_i + v_i)^2 & = & \sum_{i=1}^{\infty} |u_i^2 + 2u_i v_i + v_i^2| \\ & \leq & \sum_{i=1}^{\infty} u_i^2 + 2 \sum_{i=1}^{\infty} |u_i v_i| + \sum_{i=1}^{\infty} v_i^2 \\ & \leq & U + 2U^{1/2} V^{1/2} + V = (U^{1/2} + V^{1/2})^2. \end{array}$$

  Taking square roots we obtain

Corollary

$$\left[\sum_{i=1}^{\infty} (u_i + v_i)^2\right]^{1/2} \leq U^{1/2} + V^{1/2}.$$

The Space (H, d) and the Space $(\mathbb{R}^n, d')$

Theorem

(H, d) is a metric space, where $d(u, v) = \left[\sum_{i=1}^{\infty}(u_i - v_i)^2\right]^{1/2}$.

- It is obvious that d satisfies all the properties of a distance except $d(a, b) \leq d(a, c) + d(c, b)$, for $a, b, c \in H$. Let $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots)$, $c = (c_1, c_2, \ldots)$. Set $u = a - c$, $v = c - b$, so that $u_i = a_i - c_i$, $v_i = c_i - b_i$. Then $u_i + v_i = a_i - b_i$ and the preceding corollary yields the desired inequality.
- Let E^n be the collection of points $u = (u_1, u_2, \ldots) \in H$, such that $u_j = 0$, for $j > n$. To each point $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, we can associate the point $h(a) = (a_1, \ldots, a_n, 0, 0, \ldots) \in E^n$
  - h is a one-one mapping of $\mathbb{R}^n$ onto the subspace E^n of H.
  - Using $d'(a, b) = \left[\sum_{i=1}^{n}(a_i - b_i)^2\right]^{1/2}$ in $\mathbb{R}^n$, $d'(a, b) = d(h(a), h(b))$.

  Since E^n is a metric space, $(\mathbb{R}^n, d')$ is a metric space and h is an isometry of $(\mathbb{R}^n, d')$ with $(E^n, d|_{E^n})$.