Introduction to Topology

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Subsection 1

Introduction

From Metric to Topological Spaces

- In the context of metric spaces, the various topological concepts such as continuity, neighborhood, and so on, may be characterized by means of open sets.
- Discarding the distance function and retaining the open sets of a metric space gives rise to a topological space.
- The topological concepts that we studied before must be reintroduced in the context of topological spaces.
- To formulate the definition of a term in a topological space, we find, in a metric space, the characterization of the term by means of open sets, using in most cases what is a theorem in a metric space as a definition in a topological space.
- There are other ways of introducing topological spaces.
  - E.g., if we discard the distance function of a metric space, but retain the systems of neighborhoods of the points, we obtain what we call a neighborhood space.

Subsection 2

Topological Spaces

Topological Spaces

Definition (Topological Space)

Let X be a non-empty set and $\mathcal{T}$ a collection of subsets of X such that:

O1. $X \in \mathcal{T}$.

O2. $\emptyset \in \mathcal{T}$.

O3. If $O_1, O_2, \ldots, O_n \in \mathcal{T}$, then $O_1 \cap O_2 \cap \cdots \cap O_n \in \mathcal{T}$.

O4. If for each $\alpha \in I$, $O_\alpha \in \mathcal{T}$, then $\bigcup_{\alpha \in I} O_\alpha \in \mathcal{T}$.

The pair of objects $(X, \mathcal{T})$ is called a **topological space**. The set X is called the **underlying set**, the collection $\mathcal{T}$ is called the **topology** on the set X, and the members of $\mathcal{T}$ are called **open sets**.

- If $\mathcal{T}$ is the collection of open sets of a metric space (X, d), then $(X, \mathcal{T})$ is a topological space. It is called the **topological space associated with the metric space** (X, d). The metric space (X, d) is said to **give rise to the topological space** $(X, \mathcal{T})$.

Examples of Topological Spaces I

- For each metric space, its associated topological space is an example of a topological space.
- On the other hand, any set X and collection $\mathcal{T}$ of subsets satisfying O1, O2, O3, O4 is an example of a topological space, and we shall see that not every such example arises from a metric space.

1. The **real line** is the topological space that arises from the metric space consisting of the real number system and the distance function $d(a, b) = |a - b|$.
2. The topological space that arises from the metric space $(\mathbb{R}^n, d)$. We shall call this topological space **Euclidean n-space with the usual topology**.
3. Let X be an arbitrary set. Let $\mathcal{T} = \{\emptyset, X\}$. Then $(X, \mathcal{T})$ is a topological space.

Examples of Topological Spaces II

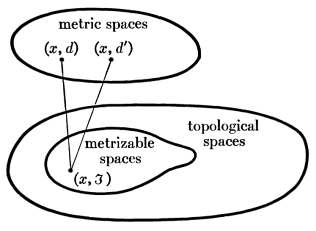
4. Let X be a set containing precisely two distinct elements a and b. Let $\mathcal{T}_1 = \{\emptyset, X\}$, $\mathcal{T}_2 = \{\emptyset, \{a\}, X\}$, $\mathcal{T}_3 = \{\emptyset, \{b\}, X\}$, $\mathcal{T}_4 = \{\emptyset, \{a\}, \{b\}, X\}$. Then $(X, \mathcal{T}_i)$, $i = 1, 2, 3, 4$, are four distinct topological spaces with the same underlying set.
5. Let X be an arbitrary set. Let $\mathcal{T}$ be the collection of all subsets of X, i.e., $\mathcal{T} = \mathcal{P}(X)$. Then $(X, \mathcal{T})$ is a topological space. Of all the various topologies that one may place on a set X, this one contains the largest number of elements. It is called the **discrete topology**.
6. Let X be an arbitrary set. Let $\mathcal{T}$ be the collection of all subsets of X whose complements are either finite or all of X. Then $(X, \mathcal{T})$ is a topological space.
7. Let Z be the set of positive integers. For each positive integer n, let $O_n = \{n, n+1, n+2, \ldots\}$. Let $\mathcal{T} = \{\emptyset, O_1, O_2, \ldots, O_n, \ldots\}$. Then $(Z, \mathcal{T})$ is a topological space.

Verifying the Topology Axioms

- To verify that $(X, \mathcal{T})$ is a topological space, one verifies that $\mathcal{T}$ is a topology, i.e., that it satisfies conditions O1, O2, O3, O4.
- Example: We show that, given an arbitrary subset X, and $\mathcal{T}$ the collection of all subsets of X whose complements are either finite or all of X, $\mathcal{T}$ is a topology.
  - O1. $X \in \mathcal{T}$, for its complement $\emptyset = C(X)$ is certainly finite.
  - O2. $\emptyset \in \mathcal{T}$, since $C(\emptyset) = X$.
  - O3. Let $O_1, O_2, \ldots, O_n$ be subsets of X, each of whose complements is finite or all of X. To show that $O_1 \cap O_2 \cap \cdots \cap O_n \in \mathcal{T}$, we must show that $C(O_1 \cap O_2 \cap \cdots \cap O_n)$ is either finite or all of X. But $C(O_1 \cap O_2 \cap \cdots \cap O_n) = C(O_1) \cup C(O_2) \cup \cdots \cup C(O_n)$.
    - Either this set is a union of finite sets and hence finite.
    - or for some i, $C(O_i) = X$ and the union is all of X.
  - O4. Finally, for each $\alpha \in I$, let $O_\alpha \in \mathcal{T}$, so that $C(O_\alpha)$ is either finite or X. Then $C(\bigcup_{\alpha \in I} O_\alpha) = \bigcap_{\alpha \in I} C(O_\alpha)$.
    - Either each of the sets $C(O_\alpha) = X$, in which case the intersection is X,
    - or at least one of them is finite, in which case the intersection is a subset of a finite set and hence finite.

Relation Between Metric and Topological Spaces

- The relationship between the totality of metric spaces and the totality of topological spaces is



- Two distinct metric spaces (X, d) and (X, d') may give rise to the same topological space $(X, \mathcal{T})$.
- Also there are topological spaces $(X, \mathcal{T})$, such as Example 7 above, which could not have arisen from a metric space.

- The subcollection of topological spaces that arise from metric spaces is called the collection of **metrizable topological spaces**.

  In passing from a metric space to its associated topological space, we may say that the "open" sets have been "preserved".

Neighborhoods

Definition (Neighborhood)

Given a topological space $(X, \mathcal{T})$, a subset N of X is called a **neighborhood** of a point $a \in X$ if N contains an open set that contains a.

- So a subset N of a metric space (X, d) is a neighborhood of a point $a \in X$ if and only if N is a neighborhood of a in the associated topological space.

  Thus, in passing from a metric space to a topological space, neighborhoods have also been "preserved".

Open Sets In Terms of Neighborhoods and Closed Sets

Corollary

Let $(X, \mathcal{T})$ be a topological space. A subset O of X is open if and only if O is a neighborhood of each of its points.

- First, suppose that O is open. Then, for each $x \in O$, O contains an open set containing x, namely, O itself.

  Conversely, suppose O is a neighborhood of each of its points. Then for each $x \in O$, there is an open set O_x, such that $x \in O_x \subseteq O$. Consequently, $O = \bigcup_{x \in O} O_x$ is a union of open sets and hence is open.

Definition (Closed Set)

Given a topological space $(X, \mathcal{T})$, a subset F of X is called a **closed set** if the complement, $C(F)$, is an open set.

Subsection 3

Neighborhoods and Neighborhood Spaces

Properties of Neighborhoods

Theorem

Let $(X, \mathcal{T})$ be a topological space.

N1. For each point $x \in X$, there is at least one neighborhood N of x.

N2. For each point $x \in X$ and each neighborhood N of x, $x \in N$.

N3. For each point $x \in X$, if N is a neighborhood of x and $N' \supseteq N$, then N' is a neighborhood of x.

N4. For each point $x \in X$ and each pair N, M of neighborhoods of x, $N \cap M$ is also a neighborhood of x.

N5. For each point $x \in X$ and each neighborhood N of x, there exists a neighborhood O of x, such that $O \subseteq N$ and O is a neighborhood of each of its points.

- For each point $x \in X$, X is a neighborhood of x.

Properties of Neighborhoods (Cont'd)

- N2 and N3 follow easily from the definition of neighborhood in a topological space.

  To verify N4, let N, M be neighborhoods of x. Then there are open sets O and O', such that $N \supseteq O$ and $M \supseteq O'$. Thus, $N \cap M$ contains the open set $O \cap O'$, which contains x, and, consequently, $N \cap M$ is a neighborhood of x.

  Finally, for a point $x \in X$, let N be a neighborhood of x. Then N contains an open set O containing x. In particular,
  - O is a neighborhood of x;
  - By the preceding corollary, O is a neighborhood of each of its points.

Complete System of Neighborhoods at a Point

Definition (Complete System of Neighborhoods at a Point)

For each point x in a topological space $(X, \mathcal{T})$, the collection $\mathfrak{N}_x$ of all neighborhoods of x is called a **complete system of neighborhoods at the point** x.

- One may paraphrase the properties N1-N5 of neighborhoods in terms of the complete system of neighborhoods $\mathfrak{N}_x$ at the points $x \in X$:
  - N1. For each $x \in X$, $\mathfrak{N}_x \neq \emptyset$;
  - N2. For each $x \in X$ and $N \in \mathfrak{N}_x$, $x \in N$;
  - N3. For each $x \in X$ and $N \in \mathfrak{N}_x$, if $N' \supseteq N$, then $N' \in \mathfrak{N}_x$;
  - N4. For each $x \in X$ and $N, M \in \mathfrak{N}_x$, $N \cap M \in \mathfrak{N}_x$;
  - N5. For each $x \in X$ and $N \in \mathfrak{N}_x$, there exists an $O \in \mathfrak{N}_x$, such that $O \subseteq N$ and $O \in \mathfrak{N}_y$ for each $y \in O$.

Differences Between Metric and Topological Spaces

- It is not always true that statements about neighborhoods that are true in a metric space are also true in a topological space:

  Example: Given two distinct points x and y in a metric space (X, d), there are neighborhoods N and M of x and y, respectively, such that $N \cap M = \emptyset$.

  This statement is false in many topological spaces.

  Let $Y = \{a, b\}$, $a \neq b$. Let $\mathcal{T} = \{\emptyset, \{a\}, Y\}$. Then $(Y, \mathcal{T})$ is a topological space.

  - The only neighborhood of b is Y.
  - Thus, for each neighborhood N of a and each neighborhood M of b, $N \cap M = N \cap Y = N \neq \emptyset$.

Hausdorff Spaces

Definition

A topological space $(X, \mathcal{T})$ is called a **Hausdorff space** or is said to satisfy the **Hausdorff axiom**, if for each pair a, b of distinct points of X, there are neighborhoods N and M of a and b respectively, such that $N \cap M = \emptyset$.

- Some authors use the term "separated space" instead of Hausdorff space.
- Many of the significant topological spaces are Hausdorff spaces.

  For this reason, certain authors require a topological space to be a Hausdorff space and use the two terms synonymously.

  I.e., they add to the list O1-O4 of properties of open sets in the definition of a topological space, the property:

  > For each pair x, y of distinct points there are open sets O_x and O_y containing x and y respectively, such that $O_x \cap O_y = \emptyset$.

Neighborhoods Spaces

- Suppose we have a metric space (X, d) and we discard the distance function, retaining only the neighborhoods of the points in X.
  Then for each point $x \in X$, we have a collection of subsets of X; namely the complete system of neighborhoods at x.
  We select some of the properties that neighborhoods satisfy and use them as a set of axioms for "neighborhood spaces".

Definition

Let X be a set. For each $x \in X$, let there be given a collection $\mathfrak{N}_x$ of subsets of X (called the **neighborhoods of** x), satisfying the conditions N1-N5 of the preceding theorem. This object is called a **neighborhood space**.

Definition

In a neighborhood space, a subset O is said to be **open** if it is a neighborhood of each of its points.

Open Sets in Neighborhood Spaces

Lemma

In a neighborhood space:

- the empty set and the whole space are open;
- a finite intersection of open sets is open;
- an arbitrary union of open sets is open.

- We may use only the properties N1-N5 of neighborhoods and the definition of open sets.
  - The empty set is open, for in order for it not to be open it would have to contain a point x of which it was not a neighborhood.
  - Given a point x, there is some neighborhood N of x. So, by N3, the whole space is a neighborhood of x. Thus, the whole space is a neighborhood of each of its points. Hence, it is open.

Open Sets in Neighborhood Spaces (Cont'd)

- If O and O' are open, then $O \cap O'$ is also open, for, by N4, given $x \in O \cap O'$, O and O' are neighborhoods of x, hence so is $O \cap O'$.

  Thus the intersection of two open sets is a neighborhood of each of its points.

  By induction, any finite intersection of open sets is open.
- Finally, suppose for each $\alpha \in I$, O_α is open. If $x \in \bigcup_{\alpha \in I} O_\alpha$, then $x \in O_\beta$ for some $\beta \in I$. But O_β is a neighborhood of x and $O_\beta \subseteq \bigcup_{\alpha \in I} O_\alpha$. Thus, by N3, $\bigcup_{\alpha \in I} O_\alpha$ is a neighborhood of x. It is therefore open.

Topological to Neighborhood to Topological

- If we start with a topological space and define neighborhoods, the underlying set and the complete systems of neighborhoods of the points of the set yield a neighborhood space.
- If we start with a neighborhood space and define open sets, we obtain a topological space.
- If we have a topological space $(X, \mathcal{T})$,
  - use the neighborhoods of $(X, \mathcal{T})$ to form a neighborhood space;
  - then use the open sets in this neighborhood space to create a topological space $(X, \mathcal{T}')$,

  we end up with our original topological space $(X, \mathcal{T})$.
- To prove this, we must show that $\mathcal{T} = \mathcal{T}'$.
  - If $O \in \mathcal{T}$, O is a neighborhood of each of its points, from which it follows that $O \in \mathcal{T}'$.
  - Conversely, if $O \in \mathcal{T}'$, then O is a neighborhood of each of its points. But the neighborhoods of the neighborhood space we have created are the neighborhoods of $(X, \mathcal{T})$, so that O is open in $(X, \mathcal{T})$ or $O \in \mathcal{T}$.

Neighborhoods in terms of Open Sets

Lemma

In a neighborhood space, a subset N is a neighborhood of a point x if and only if N contains an open set containing x.

- First, let N contain an open set O containing x. Then O is a neighborhood of x. By N3, N is a neighborhood of x.

  Conversely, if N is a neighborhood of x, then by N5, N contains a neighborhood O of x (by N2, O contains x), such that O is a neighborhood of each of its points.

Neighborhood to Topological to Neighborhood

- To denote a neighborhood space, let us use the symbol $(X, \mathfrak{N})$, where for each $x \in X$, $\mathfrak{N}_x$ is the collection of neighborhoods of x.
- Now suppose that we start with a neighborhood space $(X, \mathfrak{N})$.
  - We define open sets, thus obtaining a topological space $(X, \mathcal{T})$.
  - In the topological space $(X, \mathcal{T})$, we define neighborhood to obtain a neighborhood space $(X, \mathfrak{N}')$.
- If $N \in \mathfrak{N}_x$, by the lemma, N contains an open set O containing x, so that N is a neighborhood of x in $(X, \mathcal{T})$, or $N \in \mathfrak{N}'_x$.

  Conversely, if $N \in \mathfrak{N}'_x$, then N contains a set $O \in \mathcal{T}$, and $x \in O$. Since $O \in \mathcal{T}$, O is open in the neighborhood space $(X, \mathfrak{N})$ and so N is a neighborhood of x.

Topological Spaces and Neighborhood Spaces

- Collecting together the results on the correspondence between topological spaces and neighborhood spaces, we get:

Theorem

Let neighborhood in a topological space and open set in a neighborhood space be defined as before. Then:

- The neighborhoods of a topological space $(X, \mathcal{T})$ give rise to a neighborhood space $(X, \mathfrak{N}) = \mathfrak{A}(X, \mathcal{T})$.
- The open sets of a neighborhood space $(Y, \mathfrak{N}')$ give rise to a topological space $(Y, \mathcal{T}') = \mathfrak{A}'(Y, \mathfrak{N}')$.
- For each topological space $(X, \mathcal{T})$, $(X, \mathcal{T}) = \mathfrak{A}'(\mathfrak{A}(X, \mathcal{T}))$.
- For each neighborhood space $(X, \mathfrak{N})$, $(X, \mathfrak{N}) = \mathfrak{A}(\mathfrak{A}'(X, \mathfrak{N}))$.

This establishes a one-one correspondence between the collection of all topological spaces and the collection of all neighborhood spaces.

Illustration of Correspondence

- The preceding theorem justifies the specification of a topological space by defining for a given set X what subsets of X are to be the neighborhoods of a point, i.e., by specifying the corresponding neighborhood space.
- Example: Let X be the set of positive integers.

  Given a point $n \in X$, and a subset U of X, let us call U a neighborhood of n if for each integer $m \geq n$, $m \in U$.

  Verifying that these neighborhoods satisfy conditions N1-N5, we have a neighborhood space.

  Consequently, exploiting the preceding correspondence, we also have a topological space.

Subsection 4

Closure, Interior, Boundary

Closeness in Topological Spaces

Lemma

In a metric space (X, d), for a given point x and a given subset A, $d(x, A) = 0$ if and only if each neighborhood N of x contains a point of A.

- First, suppose that each neighborhood N of x contains a point of A. In particular, for each $\epsilon > 0$, there is a point of A in $B(x; \epsilon)$. Thus, $\text{g.l.b.}_{a \in A}\{d(x, a)\} < \epsilon$, for each $\epsilon > 0$. Consequently, $d(x, A) = \text{g.l.b.}_{a \in A}\{d(x, a)\} = 0$.

  Conversely, suppose that there is a neighborhood N of x that does not contain a point of A. Since N is a neighborhood of x in a metric space, there is an $\epsilon > 0$, such that $B(x; \epsilon) \subseteq N$. It follows that $a \in A$ implies that $d(x, a) \geq \epsilon$. Thus, $d(x, A) \geq \epsilon$.
- In a topological space, the points of a subset A are arbitrarily close to a given point x, if each neighborhood of x contains a point of A.

Closure of a Set

- Given a subset A, the collection of points that are arbitrarily close to A is called the closure of A.

Definition

Let A be a subset of a topological space. A point x is said to be **in the closure of** A if, for each neighborhood N of x, $N \cap A \neq \emptyset$. The closure of A is denoted by $\overline{A}$.

- A description of the closure of a subset in terms of closed sets:

Lemma

Given a subset A of a topological space and a closed set F containing A, $\overline{A} \subseteq F$.

- Suppose $x \notin F$, then x is in the open set $C(F)$. Also, $F \supseteq A$ implies $C(F) \subseteq C(A)$. Thus, $C(F) \cap A = \emptyset$. Since $C(F)$ is a neighborhood of x, $x \notin \overline{A}$. We have thus shown that $C(F) \subseteq C(\overline{A})$ or $\overline{A} \subseteq F$.

Closure and Closed Sets

Lemma

Given a subset A of a topological space and a point $x \notin \overline{A}$, then $x \notin F$, for some closed set F containing A.

- If $x \notin \overline{A}$, then there is a neighborhood and hence an open set O containing x, such that $O \cap A = \emptyset$. Let $F = C(O)$. Then F is closed and $F = C(O) \supseteq A$. But $x \in O$ and, therefore, $x \notin F$.
- Combining these two lemmas, we obtain:

Theorem

Given a subset A of a topological space, $\overline{A} = \bigcap_{\alpha \in I} F_\alpha$, where $\{F_\alpha\}_{\alpha \in I}$ is the family of all closed sets containing A.

- By the pre-preceding lemma, $\overline{A} \subseteq \bigcap_{\alpha \in I} F_\alpha$, since $\overline{A} \subseteq F_\alpha$, for each $\alpha \in I$. By the preceding lemma, $x \in F_\alpha$, for each $\alpha \in I$, implies that $x \in \overline{A}$, or $\bigcap_{\alpha \in I} F_\alpha \subseteq \overline{A}$. Thus, $\overline{A} = \bigcap_{\alpha \in I} F_\alpha$.

Closed Sets in terms of Closure

- Another possible description of the closure $\overline{A}$ of a subset A is the characterization of $\overline{A}$ as the smallest closed set containing A.

  $\overline{A}$ is contained in each closed set containing A. Moreover, $\overline{A}$, being the intersection of closed sets, is itself a closed set.
- The next theorem characterizes closed sets in terms of closure.

Theorem

A is closed if and only if $A = \overline{A}$.

- We have just seen that $\overline{A}$ is closed. So, if $A = \overline{A}$, then A is closed.

  Conversely, suppose A is closed. In this event A itself is a closed set containing A. Therefore, $\overline{A} \subseteq A$. On the other hand, for an arbitrary subset A, we have $A \subseteq \overline{A}$, for if $x \in A$, then each neighborhood N of x contains a point of A; namely x itself.

  Thus, if A is closed, $A = \overline{A}$.

Properties of Closure

- The act of taking the closure of a set associates to each subset A of a topological space a new subset $\overline{A}$.
- This operation satisfies the following five properties:

Theorem

In a topological space $(X, \mathcal{T})$,

CL1. $\overline{\emptyset} = \emptyset$;

CL2. $\overline{X} = X$;

CL3. For each subset A of X, $A \subseteq \overline{A}$;

CL4. For each pair of subsets A, B of X, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

CL5. For each subset A of X, $\overline{\overline{A}} = \overline{A}$.

- The property CL3 has already been established.

  CL2 follows from CL3.

Properties of Closure (Cont'd)

CL1 is true, for given a point $x \in X$ and a neighborhood N of x, $N \cap \emptyset = \emptyset$. Thus, there are no points in $\overline{\emptyset}$.

To prove CL5 we note that $\overline{A}$ is closed, so, $\overline{\overline{A}} = \overline{A}$.

It remains for us to prove CL4. Suppose $x \in \overline{A}$, then each neighborhood N of x contains points of A and hence points of $A \cup B$. Thus, $\overline{A} \subseteq \overline{A \cup B}$. Similarly, $\overline{B} \subseteq \overline{A \cup B}$, and, consequently, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. On the other hand, $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, so $A \cup B \subseteq \overline{A} \cup \overline{B}$. Thus, $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$, whence $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

- One may use the properties CL1-CL5 as a set of axioms for what we will call a **closure space**.

  Then one proves that there is a "natural" one-one correspondence between the collection of topological spaces and the collection of closure spaces.

The Interior of a Set

Definition (Interior)

Given a subset A of a topological space, a point x is said to be **in the interior of** A if A is a neighborhood of x. $\text{Int}(A)$ denotes the interior of A.

Lemma

Given a subset A of a topological space and open $O \subseteq A$, $O \subseteq \text{Int}(A)$.

- If $x \in O$, then A is a neighborhood of x, since O is open and $O \subseteq A$. Thus $x \in \text{Int}(A)$ and $O \subseteq \text{Int}(A)$.

Lemma

Given a subset A of a topological space, if $x \in \text{Int}(A)$, then $x \in O$, for some open set $O \subseteq A$.

- If $x \in \text{Int}(A)$, then A is a neighborhood of x, whence A contains an open set O containing x.

Interior and Closure

- The preceding two lemmas combine to yield:

Theorem

Given a subset A of a topological space, $\text{Int}(A) = \bigcup_{\alpha \in I} O_\alpha$, where $\{O_\alpha\}_{\alpha \in I}$ is the family of all open sets contained in A.

- Thus, $\text{Int}(A)$, being the union of open sets, is itself open, and is the largest open set contained in A.
- If $\{O_\alpha\}_{\alpha \in I}$ is the family of open sets contained in a given set A, then $\{C(O_\alpha)\}_{\alpha \in I}$ is the family of closed sets containing $C(A)$:

Theorem

$C(\text{Int}(A)) = \overline{C(A)}$.

Corollary

$\text{Int}(A) = C(\overline{C(A)})$ and $C(\overline{A}) = \text{Int}(C(A))$.

Boundary of a Set

- For a given subset A, the set of points that are arbitrarily close to both A and $C(A)$ is called the "boundary" of A.

Definition (Boundary)

Given a subset A of a topological space, a point x is said to be **in the boundary of** A if x is in both the closure of A and the closure of the complement of A. The boundary of A is denoted by $\text{Bdry}(A)$.

- Thus, $\text{Bdry}(A) = \overline{A} \cap \overline{C(A)}$.
- Note $\text{Bdry}(C(A)) = \overline{C(A)} \cap \overline{C(C(A))} = \overline{C(A)} \cap \overline{A} = \text{Bdry}(A)$.
- A point x is in the boundary of a set A if and only if each neighborhood N of x contains both points of A and points of the complement of A.

Corollary

For each subset A, $\text{Bdry}(A)$ is closed.

- The boundary of A is the intersection of two closed sets.

Subsection 5

Functions, Continuity, Homeomorphism

Continuous Functions

Definition (Function Between Topological Spaces)

A function f from a topological space $(X, \mathcal{T})$ to a topological space $(Y, \mathcal{T}')$ is a function $f : X \to Y$.

- If f is a function from a topological space $(X, \mathcal{T})$ to a topological space $(Y, \mathcal{T}')$ we shall write $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$.

  If the topologies on X and Y need not be explicitly mentioned, we may abbreviate this notation by $f : X \to Y$ or simply f.

Definition (Continuous Function)

A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is said to be **continuous at a point** $a \in X$ if for each neighborhood N of $f(a)$, $f^{-1}(N)$ is a neighborhood of a. f is said to be **continuous** if f is continuous at each point of X.

Functions Between Topological and Metric Spaces

- Let (X, d) and (Y, d') be metric spaces and let their associated topological spaces be $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$, respectively.
- Given a function f from the first metric space to the second, we also have a function, which we still denote by f, from the first topological space to the second.
- For each point $a \in X$, a function $f : (X, d) \to (Y, d')$ is continuous at a if and only if $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous at a.

Theorem

A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous if and only if for each open subset O of Y, $f^{-1}(O)$ is an open subset of X.

Proof of the Theorem

Theorem

A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous if and only if for each open subset O of Y, $f^{-1}(O)$ is an open subset of X.

- First, suppose that f is continuous. Let O is an open subset of Y. Suppose $a \in f^{-1}(O)$. Then O is a neighborhood of $f(a)$. So $f^{-1}(O)$ is a neighborhood of a. Thus, $f^{-1}(O)$ is a neighborhood of each of its points. Hence $f^{-1}(O)$ is an open subset of X.

  Conversely, suppose that for each open subset O of Y, $f^{-1}(O)$ is an open subset of X. Let $a \in X$ and a neighborhood N of $f(a)$ be given. N contains an open set O containing $f(a)$, so, by our hypothesis, $f^{-1}(N)$ contains the open set $f^{-1}(O)$ containing a. Thus, $f^{-1}(N)$ is a neighborhood of a. We conclude that f is continuous at a. Since a was arbitrary, f is continuous.

Continuity In Terms of Closed Sets

- For any set X, given a collection $\boldsymbol{E}$ of subsets of X, let $C'(\boldsymbol{E})$ denote the collection of subsets of X that are complements of members of $\boldsymbol{E}$.
- Given $f : X \to Y$ and a collection $\boldsymbol{E}$ of subsets of Y, let $f^{-1}(\boldsymbol{E})$ be the collection of subsets of X of the form $f^{-1}(E)$ for some $E \in \boldsymbol{E}$.
- The theorem states that $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous if and only if $f^{-1}(\mathcal{T}') \subseteq \mathcal{T}$. Let $\mathcal{F} = C'(\mathcal{T})$ and $\mathcal{F}' = C'(\mathcal{T}')$ be the closed subsets of X and Y, respectively.

  If $F \in \mathcal{F}'$, $f^{-1}(C(F)) = C(f^{-1}(F))$. so $f^{-1}(\mathcal{F}') = C'(f^{-1}(\mathcal{T}'))$. Thus, $f^{-1}(\mathcal{T}') \subseteq \mathcal{T}$ is equivalent to $f^{-1}(\mathcal{F}') \subseteq \mathcal{F}$:

Theorem

A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous if and only if, for each closed subset F of Y, $f^{-1}(F)$ is a closed subset of X.

Continuous versus Open Mappings

- It is important to remember that the theorem says that a function f is continuous if and only if the inverse image of each open set is open.
- This should not be confused with another property that a function may or may not possess, the property that the image of each open set is an open set (such functions are called **open mappings**).
- There are many situations in which a function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ has the property that for each open subset A of X, the set $f(A)$ is an open subset of Y, and yet f is not continuous.

  Example: Let Y be a set containing two distinct elements a and b and let each subset of Y be an open set. Let $\mathbb{R}$ be the real line and define $f : \mathbb{R} \to Y$ by $f(x) = a$, for $x \geq 0$ and $f(x) = b$ for $x < 0$. Every subset of Y is open, so, in particular, for each open subset U of $\mathbb{R}$, $f(U)$ is an open subset of Y. On the other hand $\{a\}$ is an open subset of Y but $f^{-1}(\{a\})$, the set of non-negative real numbers, is not an open subset of the reals.

Continuity and Closure

Theorem

$f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous if and only if for each subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$.

- First suppose that f is continuous. Given a subset A of X, $f(A) \subseteq \overline{f(A)}$, whence $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)})$. The set $f^{-1}(\overline{f(A)})$ is closed. So $\overline{A} \subseteq f^{-1}(\overline{f(A)})$. Thus $f(\overline{A}) \subseteq \overline{f(A)}$.

  Conversely, suppose that for each subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$. Let F be a closed subset of Y. Then $f(\overline{f^{-1}(F)}) \subseteq \overline{f(f^{-1}(F))} \subseteq \overline{F} = F$. Thus $\overline{f^{-1}(F)} \subseteq f^{-1}(F)$. Since it is always the case that $f^{-1}(F) \subseteq \overline{f^{-1}(F)}$, we have $f^{-1}(F) = \overline{f^{-1}(F)}$. consequently, $f^{-1}(F)$ is closed. So f is continuous.

Continuity of Composition

Theorem

Let $f:(X,\mathcal{T})\to(Y,\mathcal{T}')$ be continuous at a point $a\in X$ and let $g:(Y,\mathcal{T}')\to(Z,\mathcal{T}'')$ be continuous at $f(a)$. Then the composite function $gf:(X,\mathcal{T})\to(Z,\mathcal{T}'')$ is continuous at a.

- Let N be a neighborhood of $(gf)(a)=g(f(a))$. Then $(gf)^{-1}(N)=f^{-1}(g^{-1}(N))$. But $g^{-1}(N)$ is a neighborhood of $f(a)$, since g is continuous at $f(a)$, and, therefore, $f^{-1}(g^{-1}(N))$ is a neighborhood of a, since f is continuous at a.

Homeomorphism

Definition (Homeomorphism)

Topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are called **homeomorphic** if there exist inverse functions $f : X \to Y$ and $g : Y \to X$, such that f and g are continuous. In this event the functions f and g are said to be **homeomorphisms** and we say that f and g define a **homeomorphism between** $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$.

- Homeomorphism is the translation from metric spaces to topological spaces of the concept of topological equivalence.

Corollary

Let (X, d) and (Y, d') be metric spaces. Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be the topological spaces associated with (X, d) and (Y, d'), respectively. Then the metric spaces (X, d) and (Y, d') are topologically equivalent if and only if the topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are homeomorphic.

Characterization of Homeomorphism

Theorem

A necessary and sufficient condition that two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be homeomorphic is that there exist a function $f : X \to Y$, such that:

1. f is one-one;
2. f is onto;
3. A subset O of X is open if and only if $f(O)$ is open.

- Suppose that $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are homeomorphic. Let the homeomorphism be defined by inverse functions $f : X \to Y$ and $g : Y \to X$. f is invertible and consequently one-one and onto. Furthermore, given an open set O in X, the set $f(O) = g^{-1}(O)$ is open in Y, since g is continuous. On the other hand, if $f(O) = O'$ is an open subset of Y, then $O = f^{-1}(O')$ is open in X.

Characterization of Homeomorphism: The Converse

- Now, suppose that a function $f : X \to Y$ with the prescribed properties exists. Then f is invertible, Define $g : Y \to X$ by

$$g(b) = a \quad \text{if} \quad f(a) = b.$$

Then f and g are inverse functions.
If O is an open subset of X, then $f(O) = g^{-1}(O)$ is open in Y. So g is continuous.
Also, if O' is an open subset of Y, then $f^{-1}(O') = O$ is an open subset of X. Hence f is continuous.

Subsection 6

Subspaces

Subspaces

Definition (Subspace)

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces. The topological space Y is called a **subspace** of the topological space X if $Y \subseteq X$ and if the open subsets of Y are precisely the subsets O' of the form $O' = O \cap Y$, for some open subset O of X.

- In the event that Y is a subspace of X, we may say that each open subset O' of Y is the restriction to Y of an open subset O of X.
- A subset O' that is open in Y is often called **relatively open in** Y or simply **relatively open**.
- A subset O of X that is open in X and is contained in Y is necessarily relatively open in Y, but the relatively open subsets of Y are in general not open in X.

Nonempty Subsets and Subspaces

- There are as many subspaces of a topological space X as there are non-empty subsets Y of X.

Proposition

Let $(X, \mathcal{T})$ be a topological space and let Y be a subset of X. Define the collection $\mathcal{T}'$ of subsets of Y as the collection of subsets O' of Y of the form $O' = O \cap Y$, where $O \in \mathcal{T}$. Then $(Y, \mathcal{T}')$ is a topological space and therefore a subspace of $(X, \mathcal{T})$ provided $Y \neq \emptyset$.

- We must prove that $\mathcal{T}'$ is a topology.
  - $\emptyset = \emptyset \cap Y$ and $Y = X \cap Y$. So $\emptyset, Y \in \mathcal{T}'$.
  - Suppose $O_1', O_2', \ldots, O_n' \in \mathcal{T}'$, so that, for $i = 1, 2, \ldots, n$, $O_i' = O_i \cap Y$, for some $O_i \in \mathcal{T}$. Then $O_1' \cap O_2' \cap \cdots \cap O_n' = (O_1 \cap O_2 \cap \cdots \cap O_n) \cap Y$ is in $\mathcal{T}'$, since $O_1 \cap O_2 \cap \cdots \cap O_n$ is open in X.
  - Finally, suppose that for each $\alpha \in I$, $O_\alpha' \in \mathcal{T}'$. Thus, for each $\alpha \in I$, $O_\alpha' = O_\alpha \cap Y$, for some $O_\alpha \in \mathcal{T}$. But $\bigcup_{\alpha \in I} O_\alpha' = \bigcup_{\alpha \in I}(O_\alpha \cap Y) = (\bigcup_{\alpha \in I} O_\alpha) \cap Y$ is in $\mathcal{T}'$, since $\bigcup_{\alpha \in I} O_\alpha$ is open in X.

Relative Neighborhoods

- Given a subset Y of a topological space $(X, \mathcal{T})$, the preceding topology $\mathcal{T}'$ of Y is said to be **induced** by the topology $\mathcal{T}$ on X and is called the **relative topology** on Y. The neighborhoods in $\mathcal{T}'$ are called **neighborhoods in** Y or **relative neighborhoods**.

Theorem

Let Y be a subspace of a topological space X and let $a \in Y$. Then a subset N' of Y is a relative neighborhood of a if and only if $N' = N \cap Y$, where N is a neighborhood of a in X.

- If N' is a relative neighborhood of a, N' contains a relatively open set O', which contains a. Let $O' = O \cap Y$, where O is an open subset of X. Then $N = N' \cap O$ is a neighborhood of a in X and
$N \cap Y = (N' \cup O) \cap Y = N' \cup (O \cap Y) = N'$.
Conversely, if $N' = N \cap Y$, where N is a neighborhood of a in X, N contains an open set O containing a. So N' contains the relatively open set $O' = O \cap Y$ containing a. So N' is a relative nbhd of a.

Example I: Closed Interval $[a, b]$

- The closed interval $[a, b]$ of the real line with induced topology is a subspace of the real line.
- A relative neighborhood of the point a is any subset N of $[a, b]$ that contains a half-open interval $[a, c)$, where $a < c$.
- Similarly, a relative neighborhood of the point b is any subset M of $[a, b]$ that contains a half-open interval $(c, b]$, where $c < b$.
- If d is such that $a < d < b$, then a relative neighborhood of d is any subset U of $[a, b]$ that is a neighborhood of d in the real line $\mathbb{R}$.

Metric and Topological Subspaces

- The relationship of subspace is "preserved" in passing from metric spaces to topological spaces.

Lemma

Let (X, d) be a metric space and let (Y, d') be a subspace of (X, d). If $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are the topological spaces associated with (X, d) and (Y, d'), respectively, then $(Y, \mathcal{T}')$ is a subspace of $(X, \mathcal{T})$.

- Since d' is the restriction of d, an open ball in (Y, d') is the restriction of an open ball in (X, d) to Y. Consequently, a subset O' of Y is open in Y if and only if, for each $y \in O'$, there is an $\epsilon_y > 0$, such that $B(y; \epsilon_y) \cap Y \subseteq O'$. Let $O = \bigcup_{y \in O'} B(y; \epsilon_y)$. Then O is open in X and $O' = O \cap Y$. Thus, $O' \in \mathcal{T}'$.
  Conversely, if $O' \in \mathcal{T}'$, then $O' = O \cap Y$, for some $O \in \mathcal{T}$. For each $y \in O'$, we have $y \in O$, and O is open. So there is an ϵ_y such that $B(y; \epsilon_y) \subseteq O$. It follows that $B(y; \epsilon_y) \cap Y \subseteq O'$, and, hence, O' is open in (Y, d').

Example II: A Subset of $\mathbb{R}^{n+1}$

- Let A be the subset of $\mathbb{R}^{n+1}$ consisting of all points $x = (x_1, x_2, \ldots, x_{n+1})$, such that $x_{n+1} = 0$.
- Let $\mathbb{R}^{n+1}$ have the usual topology and let A have the induced topology so that A is a subspace of $\mathbb{R}^{n+1}$.

  Claim: The topological space A is homeomorphic to $\mathbb{R}^n$.

  To prove this, we use the fact that the relationship of subspace is "preserved" in passing from metric spaces to topological spaces.

  Define $f : \mathbb{R}^n \to A$ by setting $f(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, 0)$.
  - f is one-one, onto. Its inverse is the function $g : A \to \mathbb{R}^n$ defined by $g(x_1, x_2, \ldots, x_n, 0) = (x_1, x_2, \ldots, x_n)$.
  - $f : (\mathbb{R}^n, d) \to (A, d')$ is continuous.
  - $g : (A, d') \to (\mathbb{R}^n, d)$ is also continuous.

  So f and g are continuous functions defined on the topological spaces $\mathbb{R}^n$ and A, where A is considered as a subspace of $\mathbb{R}^{n+1}$, and define a homeomorphism.

Relatively Closed Sets

- Given a subspace $(Y, \mathcal{T}')$ of a topological space $(X, \mathcal{T})$, the closed subsets of the topological space $(Y, \mathcal{T}')$ are called **relatively closed in** Y or simply **relatively closed**.
- Again, the relatively closed subsets are the restriction to Y of the closed subsets of X.

Theorem

Let $(Y, \mathcal{T}')$ be a subspace of the topological space $(X, \mathcal{T})$. A subset F' of Y is relatively closed in Y if and only if $F' = F \cap Y$, for some closed subset F of X.

- Let F' be relatively closed. Then $C_Y(F')$ is relatively open. Thus, $C_Y(F') = O \cap Y$, where O is open in X. But then $F' = C_Y(O \cap Y)$ $= C_Y(O) = C_X(O) \cap Y$, where $C_X(O)$ is a closed subset of X.

  Conversely, suppose $F' = F \cap Y$, where F is a closed subset of X. Then, $C_Y(F') = C_X(F) \cap Y$. Hence $C_Y(F')$ is relatively open in Y. Therefore F' is relatively closed.

Example: A Relatively Open and Relatively Closed Subset

- Let $a < b < c < d$. Let $Y = [a, b] \cup (c, d)$ be considered as a subspace of the real line. Then the subset $[a, b]$ of Y is both relatively open and relatively closed.
  - Note that $[a, b] = [a, b] \cap Y$ so that $[a, b]$ is relatively closed.
  - On the other hand, for $0 < \epsilon < c - b$, $[a, b] = (a - \epsilon, b + \epsilon) \cap Y$ so that $[a, b]$ is relatively open.

  Since (c, d) is the complement in Y of a relatively open and relatively closed subset of Y, (c, d) is also relatively open and relatively closed in Y.

Inclusion Mappings

Theorem

Let the topological space Y be a subspace of the topological space X. Then the inclusion mapping $i : Y \to X$ is continuous.

- For each subset A of X, $i^{-1}(A) = A \cap Y$. Thus, if O is an open subset of X, $i^{-1}(O) = O \cap Y$ is a relatively open subset of Y.

Definition (Weaker Topology)

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set Y. The topology $\mathcal{T}_1$ is said to be **weaker** than $\mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

- If Y is a subset of a topological space $(X, \mathcal{T})$, then the relative topology $\mathcal{T}'$ on Y is the weakest topology such that the inclusion map $i : Y \to X$ is continuous:
  Suppose $\mathcal{T}_1$ is another topology on Y, such that $i : (Y, \mathcal{T}_1) \to (X, \mathcal{T})$ is continuous. Let $O' \in \mathcal{T}'$. Then $O' = i^{-1}(O)$, with $O \in \mathcal{T}$. Thus $O' \in \mathcal{T}_1$. We conclude that $\mathcal{T}' \subseteq \mathcal{T}_1$.

Restricting the Codomain of a Continuous Function

- Let X and Y be topological spaces and $f : Y \to X$ be a function which is not necessarily continuous. The function f induces a function $f' : Y \to f(Y)$ which agrees with f and is onto. Viewing $f(Y)$ as a subspace of X we have:

Lemma

$f : Y \to X$ is continuous if and only if $f' : Y \to f(Y)$ is continuous.

- The inclusion map $i : f(Y) \to X$ is continuous. Thus, the continuity of f' yields the continuity of $f = if'$.

  Conversely, if O' is a relatively open set in $f(Y)$, then $O' = O \cap f(Y)$, where O is open in X. If f is continuous, then $f^{-1}(O) = f'^{-1}(O')$ is open in Y. Therefore, f' is continuous.

Subsection 7

Products

Endowing a Product with a Topology

- Throughout this section let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2), \ldots, (X_n, \mathcal{T}_n)$ be topological spaces and let $X = \prod_{i=1}^{n} X_i$.
- We wish to define a topology on X that may be regarded as the product of the topologies on the factors of X.
- Our guide is the corresponding situation in metric spaces.
  - If these topological spaces were metrizable, then there is a standard procedure for converting the product of the corresponding metric spaces into a metric space.
    In this resulting metric space, the open subsets of X are the unions of sets of the form $O_1 \times O_2 \times \cdots \times O_n$, where each O_i is an open subset of X_i.
  - In the general situation, where the topological spaces may not be metrizable, one can show that the unions of the products of open sets will constitute a topology.

The Basis Lemma

Lemma

Let $\mathcal{B}$ be a collection of subsets of a set X with the property that $\emptyset \in \mathcal{B}$, $X \in \mathcal{B}$ and a finite intersection of elements of $\mathcal{B}$ is again in $\mathcal{B}$. Then the collection $\mathcal{T}$ of all subsets of X which are unions of elements of $\mathcal{B}$ is a topology.

- We verify the topology axioms:
  - Clearly $\emptyset$ and X are in $\mathcal{T}$.
  - Suppose O and O' are in $\mathcal{T}$. Then $O = \bigcup_{\alpha \in I} B_\alpha$, $O' = \bigcup_{\beta \in J} B_\beta$, where $B_\alpha \in \mathcal{B}$, for $\alpha \in I$, and $B_\beta \in \mathcal{B}$, for $\beta \in J$. Thus, for $(\alpha, \beta) \in I \times J$, $B_\alpha \cap B_\beta \in \mathcal{B}$. It follows that $O \cap O' = \bigcup_{(\alpha,\beta) \in I \times J} (B_\alpha \cap B_\beta)$ is in $\mathcal{T}$.
  - Finally a union of sets each of which is a union of sets of $\mathcal{B}$ is again a union of sets of $\mathcal{B}$.

    We conclude that $\mathcal{T}$ is a topology.

Product Topological Space

- Since in the product set X the collection of subsets of X that are unions of sets of the form $O_1 \times O_2 \times \cdots \times O_n$, where each O_i an open subset of X_i, satisfies the conditions of this lemma we may state:

Definition (Product Space)

The topological space $(X, \mathcal{T})$, where $\mathcal{T}$ is the collection of subsets of X that are unions of sets of the form $O_1 \times O_2 \times \cdots \times O_n$, where each O_i an open subset of X_i, is called the **product** of the topological spaces $(X_i, \mathcal{T}_i)$, $i = 1, 2, \ldots, n$.

- We often denote a topological space $(X, \mathcal{T})$ simply by X.
- When we say "let $X_1, X_2, \ldots, X_n$ be topological spaces and $X = \prod_{i=1}^n X_i$", we mean that X is considered as the product of the topological spaces.

Basis of a Topological Space

- The sets of the form $O_1 \times O_2 \times \cdots \times O_n$, O_i open in X_i, have been used as a "basis" for the open sets of X.

Definition (Basis)

Let X be a topological space and $\{O_\alpha\}_{\alpha \in I}$ a collection of open sets in X. $\{O_\alpha\}_{\alpha \in I}$ is called a **basis** for the open sets of X if each open set is a union of members of $\{O_\alpha\}_{\alpha \in I}$.

- The next proposition characterizes the neighborhoods in the product space.

Proposition

In a topological space $X = \prod_{i=1}^{n} X_i$, a subset N is a neighborhood of a point $a = (a_1, a_2, \ldots, a_n) \in N$ if and only if N contains a subset of the form $N_1 \times N_2 \times \cdots \times N_n$, where each N_i is a neighborhood of a_i.

Proof of the Proposition

- First suppose that $N_1 \times N_2 \times \cdots \times N_n \subseteq N$, where each N_i is a neighborhood of a_i. By the definition of neighborhood in a topological space, each N_i contains an open set O_i containing a_i, hence, N contains the open set $O_1 \times O_2 \times \cdots \times O_n$ containing a, and, therefore, N is a neighborhood of a.

  Conversely, suppose N is a neighborhood of a. Then N contains an open set O containing a. Since O is an open subset of the product space $X = \prod_{i=1}^{n} X_i$, we may write $O = \bigcup_{\alpha \in I} O_{\alpha,1} \times O_{\alpha,2} \times \cdots \times O_{\alpha,n}$, where for each i and each $\alpha \in I$, $O_{\alpha,i}$ is an open subset of X_i. Since $a \in O$, $a \in O_{\beta,1} \times O_{\beta,2} \times \cdots \times O_{\beta,n}$, for some $\beta \in I$, hence $a_i \in O_{\beta,i}$, for $i = 1, 2, \ldots, n$. But $O_{\beta,i}$ is open. Thus, if we set $N_i = O_{\beta,i}$, $i = 1, 2, \ldots, n$, N_i is a neighborhood of a_i and $N_1 \times N_2 \times \cdots \times N_n \subseteq O \subseteq N$.

Basis for the Neighborhoods at a Point

Definition (Basis for the Neighborhoods at a Point)

Let X be a topological space and $a \in X$. A collection $\mathfrak{N}_a$ of neighborhoods of a is called a **basis for the neighborhoods at** a if each neighborhood N of a contains a member of $\mathfrak{N}_a$.

- Thus, if $a = (a_1, a_2, \ldots, a_n) \in X = \prod_{i=1}^n X_i$, a basis for the neighborhoods at a is the collection consisting of all subsets of the form $N_1 \times N_2 \times \cdots \times N_n$, where each N_i is a neighborhood of a_i.
- In a product space the ith projection $p_i : X \to X_i$ is the function such that $p_i(a) = a_i$. If $O_i \in \mathcal{T}_i$, then $p_i^{-1}(O_i) = X_1 \times \cdots \times X_{i-1} \times O_i \times X_{i+1} \times \cdots \times X_n$. Since this set is an open subset of X the projection maps are continuous.
- A subset $O_1 \times O_2 \times \cdots \times O_n$ of X can be written as $p_1^{-1}(O_1) \cap \cdots \cap p_n^{-1}(O_n)$, so that we have a guide to the appropriate topology on an arbitrary product of topological spaces.

Arbitrary Topological Products

Definition (Topological Product)

Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ be an indexed family of topological spaces. The topological product of this family is the set $X = \prod_{\alpha \in A} X_\alpha$, with the topology $\mathcal{T}$ consisting of all unions of sets of the form $p_{\alpha_1}^{-1}(O_{\alpha_1}) \cap \cdots \cap p_{\alpha_k}^{-1}(O_{\alpha_k})$, where $O_{\alpha_i} \in \mathcal{T}_{\alpha_i}$, $i = 1, \ldots, k$.

- This collection is a topology that makes the projections continuous.
- Since any topology on X which makes the projection maps continuous must contain the sets of this form, the product topology is the weakest topology consistent with the continuity of the projections.
- A basis for the neighborhoods at a point x is the collection of sets of the form $p_{\alpha_1}^{-1}(N_{\alpha_1}) \cap \cdots \cap p_{\alpha_k}^{-1}(N_{\alpha_k})$, where N_{α_i} is a neighborhood of $p_{\alpha_i}(x) = x(\alpha_i) \in X_{\alpha_i}$, for $i = 1, \ldots, k$.
- In the product X, a point y is in a given neighborhood of x (close to x) if there is finite $\{\alpha_1, \ldots, \alpha_k\}$, such that $y(\alpha_i)$ is close to $x(\alpha_i)$.

Subsection 8

Identification Topologies

Identifications

- Let $\mathbb{R}$ be the real line and S the unit circle defined by $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
- The function $p : \mathbb{R} \to S$, defined by $p(t) = (\cos 2\pi t, \sin 2\pi t)$ maps $\mathbb{R}$ continuously onto S so that $p(t) = p(t')$, provided $t - t'$ is an integer.
- One may think of p as wrapping the real line around the circle so that the points which differ by an integer are identified or superimposed on each other.
- Furthermore, we shall see that the topology of S may be obtained from the topology of $\mathbb{R}$ in such a way as to make the mapping p an identification.

Definition (Identification)

Let $p : E \to B$ be a continuous function mapping the topological space E onto the topological space B. p is called an **identification** if for each subset U of B, $p^{-1}(U)$ open in E implies that U is open in B.

Factoring Through an Identification

- If $p : E \to B$ is an identification and $g : B \to Y$ is continuous on B, then g induces a continuous function $gp : E \to Y$.
- It turns out that frequently the reverse is true, that is, a continuous function $G : E \to Y$ will induce a continuous function $g : B \to Y$.

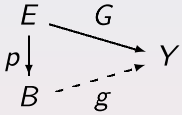
Theorem

Let $p : E \to B$ be an identification and let $G : E \to Y$ be a continuous function such that for each $x, x' \in E$, with $p(x) = p(x')$, we also have $G(x) = G(x')$. Then, for each $b \in B$, we may choose any $x \in p^{-1}(\{b\})$, define $g(b) = G(x)$, and the resulting function g is continuous.

- First, $g(b)$ does not depend on the choice of $x \in p^{-1}(\{b\})$: If $x' \in p^{-1}(\{b\})$, then $p(x) = p(x')$ and $G(x) = G(x')$. g is defined so that $gp = G$. Hence $G^{-1} = p^{-1}g^{-1}$. If O is an open subset of Y, then $G^{-1}(O)$ is open in E. But $G^{-1}(O) = p^{-1}(g^{-1}(O))$. Since p is an identification, $g^{-1}(O)$ is open in B. Therefore, g is continuous.

Identification Topology Determined by a Function

- The hypothesis on the function G is that Gp^{-1} be well-defined. The conclusion is then that the function g may be inserted in the following diagram and that commutativity will hold:



- One may use an onto function $p : X \to Y$ from a topological space X to a set Y (without a topology) to construct a topology for Y so that p becomes an identification.

Definition (Identification Topology Determined by a Function)

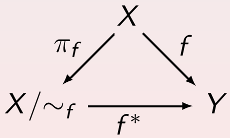
Let $p : X \to Y$ be a function from a topological space X onto a set Y. The **identification topology on Y determined by** p consists of those sets U such that $p^{-1}(U)$ is open in X.

- We can verify that this collection of sets is a topology.
- Once Y has been given the identification topology determined by p, p is an identification.

Factoring Through a Quotient

- Let $f : X \to Y$ be a function from a set X to a set Y. Let $\sim_f$ be the relation defined on X by $x \sim_f x'$ if $f(x) = f(x')$. $\sim_f$ is an equivalence relation. Let $X/{\sim_f}$ be the collection of equivalence sets under this relation and let $\pi_f : X \to X/{\sim_f}$ be the function which maps each $x \in X$ into its equivalence class. π_f is an onto function. Now suppose that X is a topological space and give $X/{\sim_f}$ the identification topology determined by π_f.
  Let Y also be a topological space.

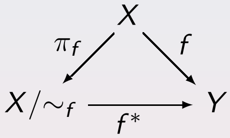
  Since $\pi_f(x) = \pi_f(x')$ if and only if $f(x) = f(x')$, f induces a continuous function $f^* : X/{\sim_f} \to Y$, such that $f = f^*\pi_f$.



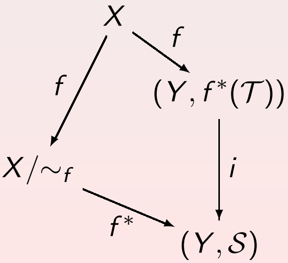
  Furthermore, f^* is one-one: If $f^*(u) = f^*(u')$, with $u, u' \in X/{\sim_f}$, then for $x \in \pi_f^{-1}(\{u\}), x' \in \pi_f^{-1}(\{u'\})$, $f(x) = f(x')$. Thus $x \sim_f x'$ or $u = \pi_f(x) = \pi_f(x') = u'$.

Topology Induced by Quotient

- Let $\mathcal{T}$ be the topology on $X/{\sim_f}$ and let $\mathcal{S}$ be the topology on Y. Since f^* is continuous, $f^{*-1}(\mathcal{S}) \subseteq \mathcal{T}$, or, equivalently, since f^* is one-one, $\mathcal{S} \subseteq f^*(\mathcal{T})$.



- If $\mathcal{S}'$ were some other topology on Y so that f were continuous we would again have $\mathcal{S}' \subseteq f^*(\mathcal{T})$. Thus, the topology carried over to Y by f^* is the weakest or smallest topology such that f is continuous.
- Introducing the topologies into the preceding diagram we obtain the one on the right in which the inclusion map $i : (Y, f^*(\mathcal{T})) \to (Y, \mathcal{S})$ is continuous.



The Covering of the Circle by the Real Line

- Let $p(t) = (\cos 2\pi t, \sin 2\pi t)$ so that $p : \mathbb{R} \to S$ is a continuous mapping of the real line onto the circle.

  Claim: p is an identification mapping, i.e., if $U \subseteq S$ is such that $p^{-1}(U)$ is open, then U is open.

  Let $x \in p^{-1}(U)$ and $s = p(x)$.

  x is the center of an open interval $O \subseteq p^{-1}(U)$ of length $2\epsilon < 1$.

  Under p, O is mapped into an arc of S centered at s of length $4\pi\epsilon$ and contained in U.

  This arc is an open ball in S with center s.

  Hence U is open.

The Covering of the Circle (Cont'd)

- The function $g(t) = (\cos 2\pi t, \sin 2\pi t, t)$ is a homeomorphism of the real line with a helix H in $\mathbb{R}^3$.

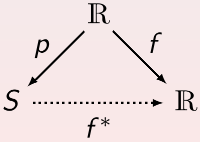
  Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 0\}$.

  The projection of H onto S defined by

  $$(\cos 2\pi t, \sin 2\pi t, t) \mapsto (\cos 2\pi t, \sin 2\pi t, 0)$$

  is also an identification.

- Let f be continuous on $\mathbb{R}$. f is called **periodic** of period 1 if $f(t+1) = f(t)$ for all $t \in \mathbb{R}$. It follows that $f(t) = f(t')$, provided $t - t'$ is an integer.



  Hence f induces a continuous function f^*, defined on the circle S, such that $f^*(p(t)) = f(t)$.

Shrinking a Subset to a Point

- Let X be a topological space and A a non-empty subset of X.
- Define a new untopologized set X/A as the union of $X - A$ and a new point a^*.
- Define a function $f : X \to X/A$ by

$$f(x) = \begin{cases} x, & \text{for } x \in X - A \\ a^*, & \text{for } x \in A \end{cases}.$$

- Now give X/A the identification topology determined by f.
- This space is the space obtained by shrinking A to a point.

Example of Shrinking a Subset to a Point

- Let $\overset{\circ}{I} = \{0, 1\}$ be the boundary of the unit interval $I = [0, 1]$.

  Claim: $I/\overset{\circ}{I}$ is homeomorphic to a circle.

  The function
  $$p(t) = (\cos 2\pi t, \sin 2\pi t), \quad t \in I,$$
  must induce a continuous function $p^* : I/\overset{\circ}{I} \to S$.

  p^* is one-one.

  Moreover, a basis for the open sets containing a^* is the totality of images of sets of the form $[0, \epsilon) \cup (1 - \epsilon, 1]$.
- Shrinking the boundary of I to a point amounts to pasting the two end points together to make the single point a^* out of the boundary.

Attaching a Space X to a Space Y

- Let X and Y be topological spaces and let A be a non-empty closed subset of X. Assume that X and Y are disjoint and that a continuous function $f : A \to Y$ is given.
- Form the set $(X - A) \cup Y$ and define a function $\varphi : X \cup Y \to (X - A) \cup Y$ by

$$\varphi(x) = \begin{cases} f(x), & \text{if } x \in A \\ x, & \text{if } x \in (X - A) \cup Y \end{cases}.$$

- Give $X \cup Y$ the topology in which a set is open (or closed) if and only if its intersections with both X and Y are open (or closed).
- φ is onto.
- Let $X \cup_f Y$ be the set $(X - A) \cup Y$ with the identification topology determined by φ.

Attaching a Space X to a Space Y: Special Case

- If Y is a single point a^*, then attaching X to a^* by a function $f : A \to a^*$ is the same as shrinking A to a point.
- Let I^2 be the unit square in $\mathbb{R}^2$.

  Let A be the union of its two vertical edges so that

  $$A = \{(x, y) \in \mathbb{R}^2 : x = 0, 0 \leq y \leq 1 \text{ or } x = 1, 0 \leq y \leq 1\}.$$

  Let $Y = [0, 1]$ be the unit interval.

  Define $f : A \to Y$ by $f(x, y) = y$.

  Then $I^2 \cup_f Y$ is a cylinder formed by identifying the two vertical edges of I^2.

Subsection 9

Categories and Functors

Categories

- When considering a collection of topological spaces and collections of continuous mappings between these spaces the following abstract structure is involved:

Definition (Category)

A **category** C is a collection of objects A whose members are called the **objects** of the category and, for each ordered pair (X, Y) of objects of the category, a set $H(X, Y)$, called the **maps** of X into Y, together with a rule of **composition** which associates to each $f \in H(X, Y)$, $g \in H(Y, Z)$ a map $gf \in H(X, Z)$. This composition is:

- associative, that is, if $f \in H(X, Y), g \in H(Y, Z)$ and $h \in H(Z, W)$, then $h(gf) = (hg)f$;
- identities exist, that is, for each object $X \in A$, there is an element $1_X \in H(X, X)$, such that for all $g \in H(X, Y)$, $g1_X = g$ and, for all $h \in H(W, X)$, $1_X h = h$.

Category of Sets and Subcategories

- We know the category C_S of sets and functions:
  - A_S is the class of all sets;
  - for $X, Y \in A_S$, $H(X, Y)$ is the set of all functions from X to Y.

  For $X \in A_S$, 1_X is the identity mapping of X onto itself.

  Composition is the ordinary composition of functions.
- One may obtain **subcategories** C' of C_S by taking:
  - as objects A' some specified collection of sets;
  - for $X, Y \in A'$, $H'(X, Y)$ to be some specified set of functions from X to Y provided that:
    - we always include the identity mapping 1_X in $H(X, X)$ for each $X \in A'$;
    - for each ordered pair (X, Y) of A' include in $H'(X, Y)$ all functions f which can be written in the form hg for $g \in H'(X, W)$ and $h \in H'(W, Y)$.

Examples

- A' might be all finite sets and $H'(X, Y)$ all functions from X to Y.
- In particular A' could contain a single set X and H'(X, X) could be all invertible functions.
- Another category is the category C_M of all metric spaces and continuous functions.
- Another is the category C_T of all topological spaces and continuous mappings.

Groups and Group Homomorphisms

Definition (Group)

A **group** G is a set G together with a function which associates to each ordered pair g_1, g_2 of elements of G an element $g_1 g_2 \in G$, such that:

(i) $g_1(g_2 g_3) = (g_1 g_2) g_3$ for $g_1, g_2, g_3 \in G$;

(ii) there is an element $e \in G$, called the **identity** such that, for all $g \in G$, $eg = ge = g$;

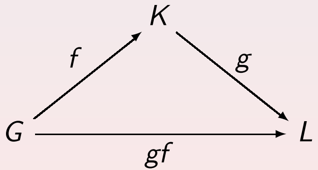
(iii) for each $g \in G$, there is an element $g^{-1} \in G$, called the **inverse** of g, such that $gg^{-1} = g^{-1}g = e$.

A **homomorphism** f from a group G to a group K is a function $f : G \to K$, such that:

- $f(e) = e'$ if e and e' are identities in G and K, respectively;
- for all $g, g' \in G$, $f(gg') = f(g)f(g')$.

The Category of Groups

- Let $\mathcal{G}$ be a collection of groups and for $G, K \in \mathcal{G}$, let $H(G, K)$ be the set of all homomorphisms of G into K.
- Use the ordinary composition of functions to define for $f \in H(G, K)$ and $g \in H(K, L)$, an element $gf \in H(G, L)$.



- It is easily verified that we have constructed a category $C_{\mathcal{G}}$ of groups in $\mathcal{G}$ and homomorphisms.

Functors

- A transformation from one category to another which preserves the structure of a category is called a "functor".

Definition

Let C and C' be categories with objects A and A' respectively. A **functor** $F : C \to C'$ is a pair of functions F_1 and F_2 such that:

- $F_1 : A \to A'$ and
- for each ordered pair X, Y of objects of A, $F_2 : H(X, Y) \to H'(F_1(X), F_1(Y))$, so that:
  - $F_2(1_X) = 1_{F_1(X)}$ and
  - $F_2(gf) = F_2(g)F_2(f)$, for $f \in H(X, Y)$, $g \in H(Y, Z)$.

Functors Diagrammatically

- Denote an element $f \in H(X, Y)$ by $X \xrightarrow{f} Y$.
- If $F : C \to C'$ is a functor, we have:
  - $$F_1(X) \xrightarrow{F_2(f)} F_1(Y)$$
  - F_2 preserves identities
    $$F_1(X) \xrightarrow{F_2(1_X) = 1_{F_1(X)}} F_1(X)$$
  - If the diagram on the left
    $$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g} \\ & & Z \end{array} \qquad\qquad \begin{array}{ccc} F_1(X) & \xrightarrow{F_2(f)} & F_1(Y) \\ & {\scriptstyle F_2(h)}\searrow & \downarrow{\scriptstyle F_2(g)} \\ & & F_1(Z) \end{array}$$
    is commutative, then so is the one on the right, i.e., F carries commutative diagrams into commutative diagrams.

Examples of Functors

- The passage from a metric space (X, d) to its associated topological space $(X, \mathcal{T})$ is an example of a functor from C_M to C_T.
- A functor from C_T to itself: Let Z be a fixed topological space.
  - To each topological space $X \in C_T$ associate the topological space $F_1(X) = X \times Z$.
  - To each continuous function $f \in H(X, Y)$ associate the function $F_2(f)$ defined by

$$(F_2(f))(x, z) = (f(x), z), \text{ for } (x, z) \in F_1(X).$$

$$\begin{array}{ccc} X \times Z & \xrightarrow{F_2(f)} & Y \times Z \\ (x, z) & \longmapsto & (f(x), z) \end{array}$$

Then $F_2(f) : F_1(X) \to F_1(Y)$ is continuous.

It can be verified that $F = (F_1, F_2)$ is a functor.