Introduction to Topology

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LSSU Math 400

Subsection 1

Introduction

Types of Connectedness

- A subspace of a topological space is "connected" if it is all "of one piece", i.e., if it is impossible to decompose the subspace into two disjoint non-empty open sets.
- A second type of connectedness is called "path-connectedness" by which it is meant that each pair of points may be "connected" by a "path" or "arc".
- Path-connectedness is a stronger condition than connectedness, since each path-connected topological space is connected, whereas the converse is false.
- A topological space is "simply connected" if there are no holes in it to prevent the continuous shrinking of each closed arc to a point.

Subsection 2

Connectedness

Connected Topological Spaces

Definition (Connected Topological Space)

A topological space X is said to be **connected** if the only two subsets of X that are simultaneously open and closed are X itself and $\emptyset$.

A topological space which is not connected is said to be **disconnected**.

- A topological space X is disconnected if and only if there are two non-empty open subsets P and Q whose union is X and whose intersection is empty: In this event P is the complement of Q and therefore both open and closed, whereas P is neither X nor $\emptyset$.
- A topological space X is disconnected if and only if there are two non-empty closed subsets F and G whose union is X and whose intersection is empty.

Connected Subsets

- Every subset A of a topological space X is itself a topological space in the relative topology. We say that A is connected if the topological space A with the relative topology is connected:

Definition (Connected Subset)

A subset A of a topological space X is said to be **connected** if the only two subsets of A that are simultaneously relatively open and relatively closed in A are A and $\emptyset$.

- Thus, the statement, A is connected, has the same meaning whether the reference is to A as a topological space or as a subspace of some larger topological space.

Examples of Connected and Disconnected Subsets

- We shall shortly see that intervals such as $[a, b]$ and (a, b) are connected subsets of the real line $\mathbb{R}$.
- As an example of a subset of the real line that is disconnected, let $A = [0, 1] \cup (2, 3)$.
  - $[0, 1]$ is a relatively closed subset of A since $[0, 1]$ is closed in $\mathbb{R}$.
  - $[0, 1]$ is a relatively open subset of A, since $[0, 1] = (-\frac{1}{2}, \frac{3}{2}) \cap A$.

  Finally, $[0, 1] \neq \emptyset$ and $[0, 1] \neq A$. Hence A is disconnected.

  By the same token, the "open interval" $(2, 3)$ is also both relatively open and relatively closed in A.

Characterization of Disconnectedness

Lemma

Let A be a subspace of a topological space X. Then A is disconnected if and only if there exist two open subsets P and Q of X such that $A \subseteq P \cup Q$, $P \cap Q \subseteq C(A)$ and and $P \cap A \neq \emptyset$, $Q \cap A \neq \emptyset$.

- First, suppose that A is disconnected. Then, there is a subset P' of A that is different from $\emptyset$ and from A and is both relatively open and relatively closed. This implies that the complement of P' in A, $C_A(P')$, is also different from $\emptyset$ and from A and relatively open. Thus $P' = P \cap A$ and $C_A(P') = Q \cap A$, where P and Q are open subsets of X. We therefore have:
  - $A = P' \cup C_A(P') \subseteq P \cup Q$, for $P' \subseteq P$ and $C_A(P') \subseteq Q$;
  - $P \cap Q \cap A = (P \cap A) \cap (Q \cap A) = P' \cap C_A(P') = \emptyset$. So $P \cap Q \subseteq C(A)$.
  - $P' = P \cap A$ and $C_A(P') = Q \cap A$ are non-empty.

Characterization of Disconnectedness (Cont'd)

- Conversely, given open sets P and Q satisfying the stated conditions, set $P' = P \cap A$ and $Q' = Q \cap A$. Then:
  - $A = A \cap (P \cup Q) = (A \cap P) \cup (A \cap Q) = P' \cup Q'$;
  - $P' \cap Q' = (A \cap P) \cap (A \cap Q) = \emptyset$. Hence $P' = C_A(Q')$.

  Thus, P' is both relatively open and relatively closed in A. Since $P' \neq \emptyset$ and $P' \neq A$ (for Q' is non-empty), A is disconnected.
- Using closed sets:

Lemma

Let A be a subspace of a topological space X. Then A is disconnected if and only if there exist two closed subsets F and G of X such that $A \subseteq F \cup G$, $F \cap G \subseteq C(A)$ and $F \cap A \neq \emptyset$, $G \cap A \neq \emptyset$.

Continuous Maps Preserve Connectivity

- Connectedness is preserved under continuous mappings:

Theorem

Let X and Y be topological spaces and let $f : X \to Y$ be continuous. If A is a connected subset of X, then $f(A)$ is a connected subset of Y.

- Suppose $f(A)$ is not connected. Then, there are open subsets P' and Q' of Y, such that $f(A) \subseteq P' \cup Q'$, $P' \cap Q' \subseteq C(f(A))$ and $P' \cap f(A) \neq \emptyset$, $Q' \cap f(A) \neq \emptyset$. Since f is continuous, $P = f^{-1}(P')$ and $Q = f^{-1}(Q')$ are open subsets of X. We have:
  - $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(P' \cup Q') = P \cup Q$;
  - $P \cap Q = f^{-1}(P' \cap Q') \subseteq f^{-1}(Cf(A)) = C(f^{-1}(f(A))) \subseteq C(A)$;
  - $P \cap A \neq \emptyset$ and $Q \cap A \neq \emptyset$.

  Thus, A is not connected. It follows that, if A is connected, then $f(A)$ must also be connected.

Consequences of Preservation of Connectivity

Corollary

Let X and Y be topological spaces, let $f : X \to Y$ be a continuous mapping of X onto Y, and let X be connected. Then Y is connected.

Corollary

Let X and Y be homeomorphic topological spaces. Then X is connected if and only if Y is connected.

- A property of a topological space is said to be a **topological property** if each topological space homeomorphic to the given space must also possess this property.

  Thus, connectedness is a topological property.

Characterization of Connectedness

Lemma

Let $Y = \{0, 1\}$. A topological space X is **connected** if and only if the only continuous mappings $f : X \to Y$ are the constant mappings.

- Let $f : X \to Y$ be a continuous non-constant mapping. Then $P = f^{-1}(\{0\})$ and $Q = f^{-1}(\{1\})$ are both non-empty. Thus, $P \neq \emptyset$ and $P \neq X$. $\{0\}$ and $\{1\}$ are open subsets of Y and f is continuous, therefore P and Q are open subsets of X. But $P = C(Q)$, so P is both open and closed and consequently X is disconnected.

  If X is disconnected, there are non-empty open subsets P, Q of X such that $P \cap Q = \emptyset$ and $P \cup Q = X$. Define a mapping $f : X \to Y$ by $f(x) = 0$ if $x \in P$, $f(x) = 1$ if $x \in Q$. f is continuous, for there are four open subsets, $\emptyset, \{0\}, \{1\}$, and Y of Y and $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{0\}) = P$, $f^{-1}(\{1\}) = Q$, and $f^{-1}(Y) = X$ are open.

Connectivity of the Product of Connected Spaces

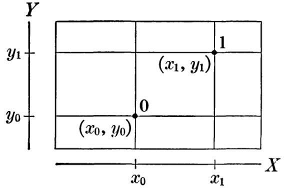
Theorem

Let X and Y be connected topological spaces. Then $X \times Y$ is connected.

- We shall show that the only continuous mappings $f : X \times Y \to \{0, 1\}$ are constant mappings.

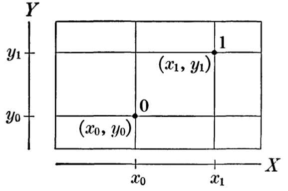
  Suppose, on the contrary, that there is a continuous mapping $f : X \times Y \to \{0, 1\}$ that is not constant. Then there are points (x_0, y_0), (x_1, y_1), such that $f(x_0, y_0) = 0$, $f(x_1, y_1) = 1$.

  If we picture $f(x, y)$ as a number attached to the point (x, y), then we have the situation on the right: Suppose $f(x_1, y_0) = 0$. We then define an "imbedding" $i_{x_1} : Y \to X \times Y$ by $i_{x_1}(y) = (x_1, y)$. i_{x_1} is continuous.



  Hence the composite mapping $fi_{x_1} : Y \to \{0, 1\}$ is continuous. (fi_{x_1} is essentially f restricted to the points of the form (x_1, y).)

Connectivity of the Product of Connected Spaces



But $fi_{x_1}(y_0) = f(x_1, y_0) = 0$ and $fi_{x_1}(y_1) = f(x_1, y_1) = 1$. Thus, in this case, there is a non-constant mapping of Y into $\{0, 1\}$, contradicting the connectedness of Y.

Similarly, if $f(x_1, y_0) = 1$, we define an imbedding $i_{y_0} : X \to X \times Y$ by setting $i_{y_0}(x) = (x, y_0)$ and obtain a non-constant mapping $fi_{y_0} : X \to \{0, 1\}$, contradicting the connectedness of X.

It follows that there are no non-constant mappings of $X \times Y$ into $\{0, 1\}$. Therefore, $X \times Y$ is connected.

Corollary

If $X_1, X_2, \ldots, X_n$ are connected topological spaces, then $\prod_{i=1}^n X_i$ is a connected topological space.

Arbitrary Product Connectivity Lemma

- In an arbitrary product $\prod_{\alpha \in I} X_\alpha$ of connected spaces, altering a finite set of coordinates cannot change the value of a continuous function $f : X \to \{0, 1\}$.

Lemma

Let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of topological spaces each of which is connected. Let x and x' be two points of $X = \prod_{\alpha \in I} X_\alpha$, such that $x(\alpha) = x'(\alpha)$ except on a finite set of indices $I' \subseteq I$ and let $f : X \to \{0, 1\}$ be continuous. Then $f(x) = f(x')$.

- We define an "imbedding" of $\prod_{\alpha \in I'} X_\alpha$ into X. Let $J = I - I'$. So $x(\alpha) = x'(\alpha)$, for $\alpha \in J$. Given $z \in \prod_{\alpha \in I'} X_\alpha$, set $(j(z))(\alpha) = z(\alpha)$ for $\alpha \in I'$ and $(j(z))(\alpha) = x(\alpha)$ for $\alpha \in J$. Then $j : \prod_{\alpha \in I'} X_\alpha \to X$ and j is continuous, for each of the functions $p_\alpha j$ is continuous ($p_\alpha j = p_\alpha$, for $\alpha \in I'$, $p_\alpha j$ is constant, for $\alpha \in J$). Both x and x' are in the image of the connected set $\prod_{\alpha \in I'} X_\alpha$. Hence $f(x) = f(x')$.

Product of Connected Spaces

Theorem

$X = \prod_{\alpha \in I} X_\alpha$ is connected if each X_α is connected.

- Let $f : X \to \{0,1\}$ be continuous and let $w, x \in X$ be such that $f(w) = 0$. We will show that $f(x) = 0$. $\{0\}$ is a neighborhood of 0. Hence, there is a neighborhood N of w, such that $f(N) = 0$. It follows that there is a finite set of indices $I' = \{\alpha_1, \ldots, \alpha_k\}$ and neighborhoods N_{α_i} of $w(\alpha_i)$ in X_{α_i}, $i = 1, \ldots, k$, such that $p_{\alpha_i}^{-1}(N_{\alpha_i}) \cap \cdots \cap p_{\alpha_k}^{-1}(N_{\alpha_k}) \subseteq N$. Define a point $x' \in X$ by setting $x'(\alpha_i) = w(\alpha_i)$, $i = 1, \ldots, k$, $x'(\alpha) = x(\alpha)$, for all other $\alpha \in I$. Then $x' \in N$, whence $f(x') = 0$. Since $x(\alpha) = x'(\alpha)$ except for $\alpha \in I'$, by the preceding lemma, $f(x) = 0$.

Subsection 3

Connectedness on the Real Line

Intervals

Definition (Interval)

A subset A of the real line is called an **interval** if A contains at least two distinct points, and if given points $a, b \in A$ with $a < b$, then for each point x, such that $a < x < b$, it follows that $x \in A$.

- Thus, an interval contains all points between any two of its points.
- A closed interval $[a, b]$ or an open interval (a, b) is an interval in the sense of this definition.

Interval Notation

Definition

Let a be a real number. The subset of $\mathbb{R}$ consisting of all $x \in \mathbb{R}$, such that

- $a < x$ is denoted by $(a, +\infty)$;
- $x < a$ is denoted by $(-\infty, a)$;
- $a \leq x$ is denoted by $[a, +\infty)$;
- $x \leq a$, is denoted by $(-\infty, a]$.

Let $b \in \mathbb{R}$ with $a < b$. The subset of $\mathbb{R}$ consisting of all $x \in \mathbb{R}$, such that

- $a < x \leq b$ is denoted by $(a, b]$;
- $a \leq x < b$, is denoted by $[a, b)$.

We shall also denote $\mathbb{R}$ itself by $(-\infty, +\infty)$.

Complete List of Intervals

Theorem

A subset A of the real numbers is an interval if and only if it is of one of the following forms: (a, b), $[a, b)$, $(a, b]$, $[a, b]$, $(-\infty, a)$, $(-\infty, a]$, $(a, +\infty)$, $[a, +\infty)$, $(-\infty, +\infty)$.

- The "if" part is easy.

  Suppose A is an interval. If a point $x \notin A$, then either x is a lower bound of A or an upper bound of A: Otherwise there would be points with $a < x < b$ and we would obtain the contradiction $x \in A$. We distinguish four cases:

  - Case 1: A has neither an upper bound nor a lower bound. In this case $C(A)$ must be empty. So $A = (-\infty, +\infty)$.
  - Case 2: A has an upper bound but no lower bound. Since an interval is non-empty, A has a least upper bound a.

    Claim: If $x < a$, then $x \in A$.

Complete List of Intervals (Cont'd)

Suppose $x < a$. Then there is a point $a' \in A$ with $x < a' \leq a$. Since x cannot be a lower bound of A, there is a point $b \in A$ with $b < x$. But $b < x < a'$ and $a', b \in A$ imply that $x \in A$. Thus, $(-\infty, a) \subseteq A$. On the other hand, for $x > a$, $x \notin A$. It follows that A is either of the form $(-\infty, a]$ or $(-\infty, a)$, depending on whether $a \in A$ or $a \notin A$.

- Case 3: A has a lower bound but no upper bound. By reasoning similar to that of Case 2, one shows that A is either of the form $[a, +\infty)$ or $(a, +\infty)$, where a is the greatest lower bound of A.
- Case 4: A has a lower bound and an upper bound. Let a be the greatest lower bound of A and let b be the least upper bound of A. Since A contains at least two distinct points, $a < b$. A point x, if it is to lie in A, must therefore lie in $[a, b]$. So $A \subseteq [a, b]$.
  Claim: $a < x < b$ implies that $x \in A$.
  This implication follows from the fact that for any such point x, there must be points a' and b' with $a', b' \in A$ and $a \leq a' < x < b' \leq b$. Hence $(a, b) \subseteq A \subseteq [a, b]$. So, A must be of one of the forms (a, b), $[a, b)$, $(a, b]$, or $[a, b]$, depending on which, if any, of a, b belong to A.

Intervals and Connectivity

Theorem

A subset A of the real line that contains at least two distinct points is connected if and only if it is an interval.

- We shall first show that if A is not an interval then it is not connected: If A is not an interval, then there are points a, b, c, with $a < c < b$ and $a, b \in A$, whereas $c \notin A$. Let $P = (-\infty, c)$, $Q = (c, +\infty)$. By a preceding lemma, A is not connected.
- Conversely, if A is not connected, then A is not an interval:

  If A is not connected, there are closed subsets F and G of the real line such that $A \subseteq F \cup G$, $F \cap G \subseteq C(A)$ and both F and G contain a point of A. Assume that the notation is such that there is a point $a \in A \cap F$ and a point $b \in A \cap G$, with $a < b$. We find a point between a and b that is not in A.

Intervals and Connectivity (Cont'd)

- We have closed subsets F and G of the real line such that $A \subseteq F \cup G$, $F \cap G \subseteq C(A)$, $a \in A \cap F$, $b \in A \cap G$, with $a < b$.

  Let $G' = G \cap [a, b]$. Then G' is a closed non-empty subset of the real line. Consequently, it contains its greatest lower bound c. We cannot have $a = c$, for then $A \cap F \cap G \neq \emptyset$, contradicting $F \cap G \subseteq C(A)$. Thus, $a < c$.

  Next, let $F' = F \cap [a, c]$. F' is also a closed non-empty subset of the real line and therefore contains its least upper bound d.

  - In the event that $c = d$ we have $c \in F \cap G$, hence $c \notin A$ and A is not an interval.
  - Otherwise, $d < c$ and $(d, c) \cap (F \cup G) = \emptyset$ so that $(d, c) \cap A = \emptyset$, and again A does not contain a point between a and b and is therefore not an interval.

Subsection 4

Applications of Connectedness

Intermediate-Value Theorem

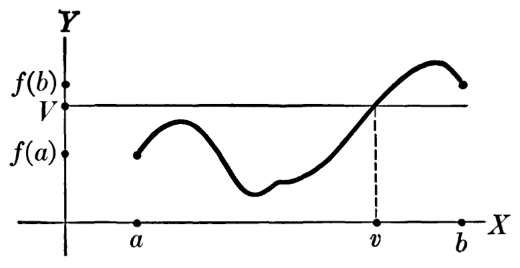
Theorem (Intermediate-Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be continuous and let $f(a) \neq f(b)$. Then, for each number V between $f(a)$ and $f(b)$, there is a point $v \in [a, b]$, such that $f(v) = V$.

- $[a, b]$ is connected, hence $f([a, b])$ is connected and is therefore an interval. Now, $f(a), f(b) \in f([a, b])$. Thus, if V is between $f(a)$ and $f(b)$, since $f([a, b])$ is an interval, $V \in f([a, b])$; that is, there is a $v \in [a, b]$, such that $f(v) = V$.

Illustrating the Intermediate-Value Theorem

- The Intermediate-Value Theorem states that for each V between $f(a)$ and $f(b)$, the horizontal line $y = V$ intersects the graph of $y = f(x)$ at some point (v, V), with $a < v < b$:



- If the domain of a continuous real-valued function contains an interval $[a, b]$, then its restriction to $[a, b]$ is continuous.
  So we can assert that f must assume at least once each value between $f(a)$ and $f(b)$ over the interval $[a, b]$.

Zeros and Fixed Points

- With $V = 0$ in the Intermediate-Value Theorem, we get

Corollary

Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a)f(b) < 0$, then there is an $x \in [a, b]$, such that $f(x) = 0$.

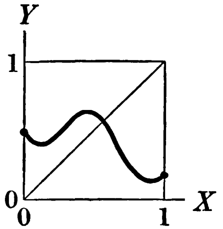
Corollary (Fixed-Point Theorem)

Let $f : [0, 1] \to [0, 1]$ be continuous. Then there is a $z \in [0, 1]$, such that $f(z) = z$.

- In the event that $f(0) = 0$ or $f(1) = 1$, the theorem is certainly true. Thus, it suffices to consider the case in which $f(0) > 0$ and $f(1) < 1$. Let $g : [0, 1] \to \mathbb{R}$ be defined by $g(x) = x - f(x)$. Note that, if $g(z) = 0$, $f(z) = z$. g is continuous. $g(0) = -f(0) < 0$, whereas $g(1) = 1 - f(1) > 0$. Consequently, by the corollary, there is a $z \in [0, 1]$, such that $g(z) = 0$. Hence $f(z) = z$.

Geometric Interpretation of the Fixed-Point Theorem

- Since $f : [0,1] \to [0,1]$, the graph of $y = f(x)$ is contained in the unit square defined by $0 \leq x \leq 1, 0 \leq y \leq 1$.
  The point $(z, f(z))$ given by the theorem lies on both the graph of $y = f(x)$ and the line $y = x$. Hence the theorem asserts that the graph of $y = f(x)$ intersects the line $y = x$ in this square:



Equivalently, in order for the curve which constitutes the graph to connect a point on the left-hand edge of the square with a point on the right-hand edge of the square, the curve must intersect the diagonal of the square.

The reason for the name "fixed-point theorem" is that, if we think of $f : [0,1] \to [0,1]$ as a transformation that carries each point x of $[0,1]$ into the point $f(x)$ of $[0,1]$, then to say that $f(z) = z$ is to say that the transformation f leaves z "fixed".

Homeomorphisms Preserve Fixed-Point Property

Theorem

Let X and Y be homeomorphic topological spaces. Then each continuous function $h : X \to X$ possesses a fixed point if and only if each continuous function $k : Y \to Y$ possesses a fixed point.

- Let $f : X \to Y$ and $g : Y \to X$ be a pair of continuous inverse functions. Let $k : Y \to Y$ be a continuous function so that we have the diagram. Suppose that each continuous function $h : X \to X$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "k"] \\ X & Y \arrow[l, "g"] \end{tikzcd}$$

possesses a fixed point. Then the function $h = gkf : X \to X$ is continuous and there is $z \in X$, such that $h(z) = z$. Let $w = f(z)$. We have $k(w) = k(f(z)) = f(g(k(f(z)))) = f(h(z)) = f(z) = w$. Thus, w is a fixed point of k.

Since the hypotheses are symmetric with regard to X and Y, it also follows that if each continuous function $k : Y \to Y$ has a fixed point then so does each continuous function $h : X \to X$.

Other Fixed-Point Theorems

- Any two closed intervals $[a, b]$ and $[c, d]$ are homeomorphic. Since a fixed-point theorem holds for $[0, 1]$, we obtain:

Corollary

Let $f : [a, b] \to [a, b]$ be continuous. Then there is a $z \in [a, b]$, such that $f(z) = z$.

- The Fixed-Point Theorem is a special case of the "Brouwer Fixed-Point Theorem":
- Recall that in $\mathbb{R}^n$, the unit n-cube $\mathbb{I}^n$ is defined as the set of points $(x_1, x_2, \ldots, x_n)$, whose coordinates satisfy the inequalities $0 \leq x_i \leq 1$, for $i = 1, 2, \ldots, n$.

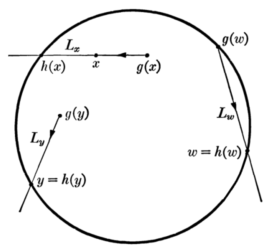
Theorem (Brouwer Fixed-Point Theorem)

Let $f : \mathbb{I}^n \to \mathbb{I}^n$ be continuous. Then there is a point $z \in \mathbb{I}^n$, such that $f(z) = z$.

The Brouwer Fixed-Point Theorem for $n = 2$

- We develop an argument for $n = 2$:

  On the basis of the preceding theorem, we work with a topological space homeomorphic to $\mathbb{I}^2$. Namely, with the unit disc, i.e., the set of points (x_1, x_2) in the plane whose coordinates satisfy the inequality $x_1^2 + x_2^2 \leq 1$. Let g be a continuous transformation of this disc into itself. Suppose that it were possible that for each point x of the disc, we had $g(x) \neq x$. Then, for each point x in the disc, there would be a unique half-line L_x emanating from $g(x)$ and passing through x:



  The half-line L_x will contain a point on the boundary of the disc other than $g(x)$. Let us call this point $h(x)$. In particular, if y is a boundary point of the disc, then $h(y) = y$. This is true even if $g(y)$ itself is a boundary point, as may be seen by examining the various cases.

The Brouwer Fixed-Point Theorem for $n = 2$ (Cont'd)

- Using the given transformation g, we have constructed a new transformation h, which has the property that it carries each point of the disc into a boundary point and leaves each boundary point fixed. (h is called a "retraction" since it retracts or pulls the interior of the disc onto its boundary while leaving the boundary fixed.)

  The transformation h is continuous, for the image $h(x)$ will vary by a small amount if we suitably restrict the variation of x.

  Our intuition tells us that no continuous transformation such as h can exist: If there were a function such as h, we should be able to retract the head of a drum onto the rim. Intuitively we know that we can do so only by ripping the drum head someplace, i.e., by introducing a discontinuity.

  Since there is no function such as the retraction h, we have obtained a contradiction. Therefore our supposition that g did not have a fixed point is untenable.

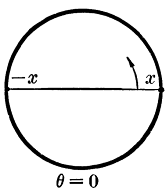
Antipodal Points on Spheres

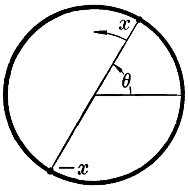
- The n-sphere, S^n, is the set of points $(x_1, x_2, \ldots, x_{n+1})$ in $\mathbb{R}^{n+1}$ satisfying the equation
$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1,$$
with the relative topology.
- If $(x_1, x_2, \ldots, x_{n+1})$ is in S_n, the pair of points $(x_1, x_2, \ldots, x_{n+1})$ and $(-x_1, -x_2, \ldots, -x_{n+1})$ are called a **pair of antipodal points**.
- Given $x = (x_1, x_2, \ldots, x_{n+1}) \in S^n$, it is convenient to denote $(-x_1, -x_2, \ldots, -x_{n+1})$ by $-x$ and call $-x$ the **antipodal point** of x.
- A pair $x, -x$ of antipodal points is the pair of end points of a diameter of the sphere.
- We are particularly interested in the 1-sphere, S^1, which is a circle.

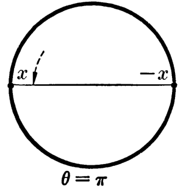
Matching Values on Antipodal Points

- Consider a continuous function $f : S^1 \to \mathbb{R}$. Define $F : S^1 \to \mathbb{R}$ by $F(x) = f(x) - f(-x)$, for $x \in S^1$. We can show that $F(z) = 0$, for some $x \in S^1$, i.e., $f(z) = f(-z)$, or f has the same value at one or more pairs of antipodal points.

  A value of F is determined by a diameter of the circle with antipodal points x and $-x$. If we rotate this through π radians, then the initial value of F is opposite in sign to the final value of F:

  Since F is continuous, its value must be zero for some position θ, with $0 \leq \theta \leq \pi$, where θ is the angle through which the diameter has been rotated, i.e., $f(z) = f(-z)$.

The Borsuk-Ulam Theorem

Theorem

Let $f : S^1 \to \mathbb{R}$ be continuous, then there exists a pair of antipodal points such that $f(z) = f(-z)$.

- This theorem is the case $n = 1$ of the Borsuk-Ulam Theorem.

Theorem

Let $f : S^n \to \mathbb{R}^n$ be continuous. Then there exists a pair of antipodal points $z, -z \in S^n$, such that $f(z) = f(-z)$.

- The case $n = 2$ answers a question about map making: The 2-sphere S^2 may be thought of as the surface of a globe. In this case the Borsuk-Ulam Theorem gives a negative answer to the question, "Is it possible to draw a map of the surface of the earth on a flat sheet of paper so that distinct points on the surface of the earth correspond to distinct points on the map, and nearby points on the surface of the earth correspond to nearby points on the map?"

Subsection 5

Components and Local Connectedness

Component of a Point in a Topological Space

- In any topological space X, each point a belongs to a maximal connected subset of X called the "component of a".

Theorem

Let X be a topological space. For each point $a \in X$, there is a non-empty subset $\mathrm{Cmp}(a)$, called the **component of** a, with the property that:

- $\mathrm{Cmp}(a)$ is connected;
- If D is any connected subset of X containing a, then $D \subseteq \mathrm{Cmp}(a)$.

- There are connected subsets of X containing a for $\{a\}$ is such a subset. Let I be an indexing set for the family of connected subsets $\{D_\alpha\}_{\alpha \in I}$ containing a. We set $\mathrm{Cmp}(a) = \bigcup_{\alpha \in I} D_\alpha$. Thus, if D is any connected subset of X, containing a, $D = D_\beta$, for some $\beta \in I$, whence $D \subseteq \mathrm{Cmp}(a)$.

  It remains to prove that $\mathrm{Cmp}(a)$ is connected.

Proving that Cmp(a) is Connected

- Assume that Cmp(a) is not connected. Then, there are nonempty relatively open subsets A and B of Cmp(a), such that:
  - $A \cap B = \emptyset$ and
  - $A \cup B = \text{Cmp}(a)$.

  Assume the notation is such that $a \in A$ and let b be a point of B. Since $b \in \text{Cmp}(a)$, $b \in D_\gamma$, for some connected subset D_γ of X containing a. Now $D_\gamma \subseteq \text{Cmp}(a)$. Hence $A' = A \cap D_\gamma$ and $B' = B \cap D_\gamma$ are non-empty relatively open subsets of D_γ. Furthermore, $A' \cap B' = \emptyset$ and $A' \cup B' = D_\gamma \cap (A \cup B) = D_\gamma$. Consequently, the supposition that Cmp(a) is not connected yields the contradiction that D_γ is not connected.

  Therefore, Cmp(a) is connected.

The Component Equivalence

Lemma

In a topological space X, let $b \in \mathrm{Cmp}(a)$. Then $\mathrm{Cmp}(b) = \mathrm{Cmp}(a)$.

- $b \in \mathrm{Cmp}(a)$ and $\mathrm{Cmp}(a)$ is a connected set containing b. By the theorem, $\mathrm{Cmp}(a) \subseteq \mathrm{Cmp}(b)$. But $a \in \mathrm{Cmp}(a)$. So $a \in \mathrm{Cmp}(b)$. By the same reasoning $\mathrm{Cmp}(b) \subseteq \mathrm{Cmp}(a)$. Hence $\mathrm{Cmp}(a) = \mathrm{Cmp}(b)$.

Corollary

In a topological space X, define $a \sim b$ if $b \in \mathrm{Cmp}(a)$. Then $\sim$ is an equivalence relation.

- Since $\{a\}$ is connected, $a \in \mathrm{Cmp}(a)$. So $a \sim a$.
  If $a \sim b$ or $b \in \mathrm{Cmp}(a)$, then, by the lemma, $\mathrm{Cmp}(a) = \mathrm{Cmp}(b)$. We have already seen that $a \in \mathrm{Cmp}(a)$. So $a \in \mathrm{Cmp}(b)$ and $b \sim a$.
  If $a \sim b$ and $b \sim c$, then as before, $\mathrm{Cmp}(a) = \mathrm{Cmp}(b) = \mathrm{Cmp}(c)$. Hence $c \in \mathrm{Cmp}(a)$ and $a \sim c$.

Partitions

- A subset of X that is a component of some point $a \in X$ is called a **component** of X.
  - The components are the equivalence classes under the relation $b \in \mathrm{Cmp}(a)$.
  - They constitute a partition of X into maximal connected subsets in the sense of the following definition.

Definition (Partition)

Let X be a set and $\{P_\alpha\}_{\alpha \in I}$ an indexed family of nonempty subsets of X. $\{P_\alpha\}_{\alpha \in I}$ is called a **partition** of X if:

(i) $X = \bigcup_{\alpha \in I} P_\alpha$;

(ii) If $\alpha, \beta \in I$, $\alpha \neq \beta$, then $P_\alpha \cap P_\beta = \emptyset$.

Closure and Connectivity

Theorem

Let A be a connected subset of a topological space X and let $A \subseteq B \subseteq \overline{A}$. Then B is also connected.

- We show that, if B is not connected, then A is not connected. Suppose there are open subsets P, Q of X such that $P \cap Q \subseteq C(B)$, $B \subseteq P \cup Q$, $P \cap B \neq \emptyset$ and $Q \cap B \neq \emptyset$. Then $A \subseteq P \cup Q$ and since $C(B) \subseteq C(A)$, $P \cap Q \subseteq C(A)$. To prove that A is not connected we must show that $P \cap A \neq \emptyset$ and $Q \cap A \neq \emptyset$. If $P \cap A = \emptyset$, then A would be contained in the closed set $C(P)$, hence $\overline{A} \subseteq C(P)$ or $P \cap \overline{A} = \emptyset$. But this last relation would imply that $P \cap B = \emptyset$. Thus, $P \cap A \neq \emptyset$. Similarly, $Q \cap A \neq \emptyset$.

Corollary

The closure of a connected set is connected.

Components are Closed

Corollary

In a topological space, each component is a closed set.

- Let A be a component, say $A = \mathrm{Cmp}(a)$. Then $\overline{A}$ is a connected set containing a. Therefore, $\overline{A} \subseteq \mathrm{Cmp}(a) = A$. But $A \subseteq \overline{A}$. Hence $A = \overline{A}$ and A is closed.
- It might be thought that a component must also be an open set, but it need not be as the following example shows:

  Example: Let X be the subspace of the real line consisting of the points 0 and all numbers of the form $\frac{1}{n}$, with n a positive integer.

  The only connected set containing 0 is $\{0\}$. Thus $\mathrm{Cmp}(0) = \{0\}$.

  On the other hand $\{0\}$ is not a neighborhood of 0 in X. Hence, $\{0\}$ is not an open subset of X.

Locally Connectedness

- A sufficient condition for the components in a space to be open is that the space be "locally connected".

Definition (Locally Connected Space)

A topological space X is said to be **locally connected at a point** $a \in X$ if each neighborhood N of a contains a connected neighborhood U of a.
A topological space X is said to be **locally connected** if it is locally connected at each of its points.

Lemma

Let X be a locally connected topological space and let Q be a component. Then Q is an open set.

- Let $a \in Q$. Since X is locally connected there is at least one connected neighborhood N of a. But $Q = \mathrm{Cmp}(a)$. Hence $N \subseteq Q$. So Q is a neighborhood of a. Thus, Q is a neighborhood of each of its points. Therefore, Q is open.

Local Connectedness and Bases of Neighborhoods

- If X is locally connected at a, then there are "arbitrarily small" connected neighborhoods of a: Given any neighborhood N of a, there is a connected neighborhood that is at least as "small" as N.
- An equivalent formulation of local connectedness is obtained by utilizing the concept of basis for the neighborhoods at a:

Lemma

A topological space is locally connected at a point $a \in X$ if and only if there is a basis for the neighborhoods at a composed of connected subsets of X.

- First, suppose that X is locally connected at a and let U_a be the collection of connected neighborhoods of a. Since every neighborhood N of a contains an element of U_a, U_a is a basis for the nbhds at a. Conversely, suppose there is a basis U_a for the neighborhoods at a consisting of connected sets. Then each neighborhood N of a must contain an element of U_a. Therefore, X is locally connected at a.

Subsection 6

Path Connected Topological Spaces

Paths Connecting Points in $\mathbb{R}^3$

- In the three-dimensional geometry of the calculus, we discuss parametric curves $x = f(t), y = g(t), z = h(t)$.
- It is generally understood that f, g, h are at least continuous, if not differentiable, over an interval $[a, b]$, whence $F(t) = (f(t), g(t), h(t))$ defines a continuous function $F : [a, b] \to \mathbb{R}^3$.
- The curve is the image of $[a, b]$ under F, i.e., $F([a, b])$.
- We may think of this curve as "connecting" the two points $F(a) = (f(a), g(a), h(a))$ and $F(b) = (f(b), g(b), h(b))$.
- Given two points $A, B \in \mathbb{R}^3$, the question of whether there is a curve "connecting" A and B is identical to the question of whether there is a continuous $F : [a, b] \to \mathbb{R}^3$, such that $F(a) = A$ and $F(b) = B$.
- The interval $[a, b]$ may be restricted to $[0, 1]$:
  Using any homeomorphism $\varphi : [0, 1] \to [a, b]$, one may show that the required $F : [a, b] \to \mathbb{R}^3$ exists if and only if a corresponding $G = F\varphi : [0, 1] \to \mathbb{R}^3$ exists.

Path Connectedness

Definition (Path)

Let X be a topological space. A continuous function $f : [0,1] \to X$ is called a **path** in X. The path f is said to **connect** or **join** the point $f(0)$ to the point $f(1)$. $f(0)$ is called the **initial point** and $f(1)$ is called the **terminal point** of the path f.
If f is a path in X, $f([0,1])$ is called a **curve** in X.

Definition (Path Connectedness)

A topological space X is said to be **path-connected** if, for each pair of points $u, v \in X$, there is a path f connecting u to v.
A non-empty subset A of a topological space X is said to be **path-connected** if the topological space A in the relative topology is path-connected.

Examples of Path Connected Spaces

- The real line $\mathbb{R}$ is a path-connected space: If a, b are two real numbers, the path $f : [0,1] \to \mathbb{R}$ defined by

$$f(t) = a + (b - a)t = (1 - t)a + tb,$$

  for $t \in [0,1]$ connects $f(0) = a$ and $f(1) = b$.
- $\mathbb{R}^n$ is also path-connected.
  - We can either directly join a given pair x, y of points of $\mathbb{R}^n$ by a path, or
  - use the general result that if X and Y are path-connected spaces, then so is $X \times Y$.
- Another significant class of path-connected spaces is the spheres, $S^n, n > 0$.
- A path f in a topological space X whose initial and terminal points coincide is called a **closed path** or a **loop** in X.

Images of Path-Connected Spaces

- If f is a path in a topological space X and g is a continuous mapping of X into a second topological space Y, then the composite function $gf : [0,1] \to Y$ is a path in Y.

Theorem

Let Y be a topological space. If there exists a path-connected topological space X and a continuous mapping $g : X \to Y$, which is onto, then Y is path-connected.

- Let $a, b \in Y$. Since $g : X \to Y$ is onto, there are points $a', b' \in X$, such that $g(a') = a$, $g(b') = b$. Since X is path-connected, there is a path f in X joining a' to b'. Hence, the path gf joins a to b.
- Note the necessity of the requirement that $g : X \to Y$ be onto.
- It follows that given homeomorphic topological spaces X and Y, X is path-connected if and only if Y is path-connected.

  Thus, path-connectedness is a topological property.

Path-Connectedness and Connectedness

- Path-connectedness is a stronger property than connectedness, i.e., if a topological space X is path-connected, then X is connected.

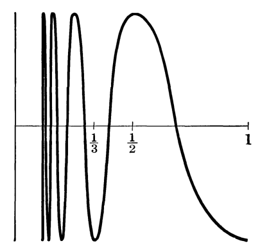
Theorem

If X is a path-connected topological space, then X is connected.

- Suppose X were not connected. Then there is a proper subset P of X which is both open and closed. Since P is proper, there is a point $a \in P$ and a point $b \in C(P)$. Let $f : [0,1] \to X$ be a path from a to b. $f^{-1}(P)$ is a proper subset of $[0,1]$ for $0 \in f^{-1}(P)$, $1 \notin f^{-1}(P)$. Since f is continuous, $f^{-1}(P)$ is both open and closed. But this contradicts the fact that $[0,1]$ is connected. Therefore, X is connected.

A Connected but Non Path Connected Space

- A topological space that is connected but not path-connected is the subspace of the plane consisting of the set of points (x, y) such that either
$$x = 0, -1 \leq y \leq 1 \quad \text{or} \quad 0 < x \leq 1, y = \cos\frac{\pi}{x}.$$



It is not difficult to prove that this space is connected. First of all let us decompose this space into two subsets Z_1 and Z_2, where:

  - Z_1 is the set of points $(0, y), -1 \leq y \leq 1$, on the Y-axis;
  - Z_2 is the complementary set consisting of those points $(x, y), 0 < x \leq 1$ and $y = \cos\frac{\pi}{x}$.

The function $F(t) = (t, \cos\frac{\pi}{t})$ defines a continuous mapping of the connected interval $(0, 1]$ onto Z_2. Hence Z_2 is connected.

Z is Connected but not Path-Connected

- To prove that the entire space $Z = Z_1 \cup Z_2$ is connected, we shall prove that $\overline{Z_2} = Z$, i.e. $Z_1 \subseteq \overline{Z_2}$.
  This is so because there are points of Z_2 arbitrarily close to each point of Z_1. Let $(0, y) \in Z_1$ and let $\epsilon > 0$ be given. We may find an even integer N sufficiently large so that $\frac{1}{N} < \epsilon$. Now $\cos \frac{\pi}{1/N} = 1$ and $\cos \frac{\pi}{1/(N+1)} = -1$. By the intermediate value theorem, there is a number $t \in [\frac{1}{N+1}, \frac{1}{N}]$, such that $\cos \frac{\pi}{t} = y$. The point $(t, \cos \frac{\pi}{t})$ is in Z_2 and its distance from $(0, y)$ is less than ϵ. Thus $Z_1 \subseteq \overline{Z_2}$ and $\overline{Z_2}$ is the entire space Z. By a preceding corollary, Z is connected.
- Suppose there was a path $F : [0, 1] \to Z$ with initial point $F(0) = (0, 1) \in Z_1$ and terminal point $F(1) = (1, -1) \in Z_2$. Write $F(t) = (F_1(t), F_2(t))$. Then F_1 and F_2 are continuous functions and $F_1(0) = 0$, $F_1(1) = 1$. The set $U = F_1^{-1}(\{0\})$ is a closed bounded subset of the real numbers. Hence, it contains its least upper bound t^*. Since $F_1(1) \neq 0$, $t^* < 1$. We show F_2 cannot be continuous at t^*.

Z is not Path Connected

Claim: F_2 cannot be continuous at t^*.

For each value of t, such that $t^* < t \leq 1$, we have $F_1(t) > 0$, hence $F(t) \in Z_2$ and $F_2(t) = \cos \frac{\pi}{F_1(t)}$. We show that for each $\delta > 0$, with $t^* + \delta \leq 1$, there is a value of t, such that $|t^* - t| < \delta$, whereas $|F_2(t^*) - F_2(t)| \geq 1$.

First, $F_1(t^* + \delta) > 0$, hence, there is an even integer N sufficiently large so that $F_1(t^*) = 0 < \frac{1}{N+1} < \frac{1}{N} < F_1(t^* + \delta)$. By the intermediate-value theorem, we may find $u, v \in [t^*, t^* + \delta]$, such that $F_1(u) = \frac{1}{N+1}$, $F_1(v) = \frac{1}{N}$. Since $u, v > t^*$, we have

$F_2(u) = \cos \frac{\pi}{F_1(u)} = \cos (N+1)\pi = -1,$

$F_2(v) = \cos \frac{\pi}{F_1(v)} = \cos N\pi = 1.$

- If $F_2(t^*) \geq 0$, $|F_2(t^*) - F_2(u)| \geq 1 \geq \epsilon$;
- If $F_2(t^*) \leq 0$, $|F_2(t^*) - F_2(v)| \geq 1 \geq \epsilon$.

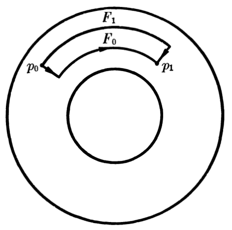
This contradicts the continuity of F_2 at t^*. Thus, no path such as F exists. Therefore, the space Z is not path-connected.

Subsection 7

Homotopic Paths and the Fundamental Group

Paths in an Annulus

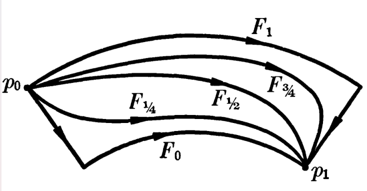
- The collection of points on and between the two concentric circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$ is called an **annulus**.
- It is easy to see that this annulus is path-connected.
- Given $p_0 = (x_0, y_0)$ and $p_1 = (x_1, y_1)$, one may construct a path from p_0 to p_1 by: First traversing the radius of p_0 until we reach a point whose distance from the origin is the same as that of p_1; Then traversing in a clockwise direction the circular arc from this point to p_1.



  Let us call this path F_0.

  A second path F_1 from p_0 to p_1: First traverses in a clockwise direction a circular arc from p_0 to the radius of p_1; Then traverses this radius until p_1 is reached.
- In a given unit of time it is possible to smoothly deform the path F_0 into the path F_1 (with p_0, p_1 fixed throughout the deformation).

Deforming F_0 into F_1

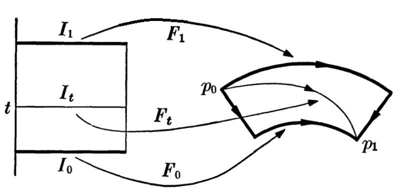


- This deformation might be carried out so that:
  - At time $t = \frac{1}{4}$ the string lies over the curve $F_{1/4}$;
  - At time $t = \frac{1}{2}$ the string lies over $F_{1/2}$;
  - At time $t = \frac{3}{4}$ the string lies over $F_{3/4}$.
- We may thus conceive of the deformation of the path F_0 into the path F_1 as being accomplished by constructing an entire family of paths F_t, for $0 \leq t \leq 1$, such that, if t and t' are close, then the paths F_t and $F_{t'}$ are "close".

Basing the Paths on Different Intervals

- The concept of regarding two paths as being "close" implies the introduction of a topology in the set of paths.
- We regard the unit of time as a unit interval on a line.
- Instead of viewing the two original paths F_0 and F_1 as being defined on the same unit interval,

  let us view F_0 as temporarily being defined on the homeomorphic image I_0 of the unit interval, where I_0 is the set of points $(x, 0)$ in the plane with $0 \leq x \leq 1$.



- Similarly, let us view F_1 as being defined on I_1, where I_1 is the set of points $(x, 1)$, $0 \leq x \leq 1$.
- For each value of t, $0 \leq t \leq 1$, we may view the path F_t as being defined on the homeomorphic image of the unit interval I_t, where I_t is the set of points (x, t), $0 \leq x \leq 1$.

Idea Behind Closeness of Paths

- If we have such a situation, we may define a function $H : I^2 \to X$, where
  - I^2 is the unit square;
  - X is our annulus,

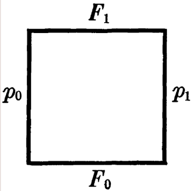
  by setting $H(x,t) = F_t(x,t)$.
- If we insist on viewing each path F_t as being defined on the same unit interval I, we may still obtain the same function H by setting $H(x,t) = F_t(x)$.
- We introduce the concept of closeness amongst paths by requiring that the function $H : I^2 \to X$ be continuous.

Homotopic Paths

Definition (Homotopic Paths)

Let F_0, F_1 be two paths in a topological space X with the same initial point $p_0 = F_0(0) = F_1(0)$ and the same terminal point $p_1 = F_0(1) = F_1(1)$. F_0 is said to be **homotopic** to F_1 if there is a continuous function $H : I^2 \to X$, such that

$$\begin{array}{rcll} H(0,t) & = & p_0, & 0 \leq t \leq 1, \\ H(1,t) & = & p_1, & 0 \leq t \leq 1, \\ H(x,0) & = & F_0(x), & 0 \leq x \leq 1, \\ H(x,1) & = & F_1(x), & 0 \leq x \leq 1. \end{array}$$



The function H is called a **homotopy connecting F_0 to F_1**.

- In this event we say that the path F_0 is **deformable** into the path F_1 with fixed end points.

Homotopy is an Equivalence Relation

Theorem

Let F_0, F_1, F_2 be three paths in a topological space X with the same initial point p_0 and the same terminal point p_1.

(i) F_0 is homotopic to itself.

(ii) If F_0 is homotopic to F_1, then F_1 is homotopic to F_0.

(iii) If F_0 is homotopic to F_1 and F_1 is homotopic to F_2, then F_0 is homotopic to F_2.

(i) To show that F_0 is homotopic to itself we need only define $H : I^2 \to X$ by $H(x,t) = F_0(x)$.

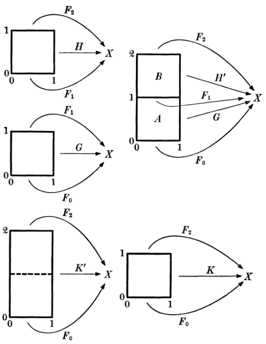
(ii) If F_0 is homotopic to F_1, there is a homotopy $H : I^2 \to X$ from F_0 to F_1. For each $(x,t) \in I^2$, set $H'(x,t) = H(x, 1-t)$. Then H' is easily seen to be a homotopy from F_1 to F_0.

Homotopy is Transitive

(iii) Let G be a homotopy from F_0 to F_1 and let H be a homotopy from F_1 to F_2. We construct a homotopy from F_0 to F_2 in stages:

- Alter H to an H' defined for (x, t') with $1 \leq t' \leq 2$, so that G and H' together constitute a function K' defined for (x, t) with $0 \leq t \leq 2$.
- Compress K' to a function K again defined on I^2.

Let $H'(x, t') = H(x, t' - 1)$, $0 \leq x \leq 1, 1 \leq t' \leq 2$. We then have two functions G and H', G defined on the subset $A = I^2$ of the plane and H' defined on the subset B consisting of the points (x, t), such that $0 \leq x \leq 1$ and $1 \leq t \leq 2$. The set $A \cap B$ consists of the points $(x, 1)$, and, therefore, we have $G(x, 1) = F_1(x)$, $H'(x, 1) = H(x, 0) = F_1(x)$, i.e., G and H' agree in their common domain of definition.



Homotopy is Transitive (Cont'd)

Lemma

Let A, B be closed subsets of a topological space Z. Let $g : A \to X$ and $h : B \to X$ be continuous functions with the property that for $z \in A \cap B$, $g(z) = h(z)$. Then the function $k : A \cup B \to X$ defined by

$$k(z) = \begin{cases} g(z), & \text{if } z \in A \\ h(z), & \text{if } z \in B \end{cases}$$

is a continuous extension of g and h.

- Let U be a closed subset of X. Then $g^{-1}(U)$ is a relatively closed subset of A and, since A is closed, $g^{-1}(U)$ is a closed subset of Z. Similarly, $h^{-1}(U)$ is a closed subset of Z. But $k^{-1}(U) = g^{-1}(U) \cup h^{-1}(U)$. Hence $k^{-1}(U)$ is closed. So k is continuous.

Finishing the Proof of Transitivity

- The function $K' : A \cup B \to X$ defined by
$K'(x,t) = \begin{cases} G(x,t), & \text{if } (x,t) \in A \\ H'(x,t), & \text{if } (x,t) \in B \end{cases}$ is continuous.
We finally "compress" K' to the function $K : I^2 \to X$ defined by
$$K(x,t) = K'(x,2t), \quad (x,t) \in I^2.$$
Summarizing, for $(x,t) \in I^2$,
  - if $0 \leq t \leq \frac{1}{2}$, $K(x,t) = K'(x,2t) = G(x,2t)$.
  - if $\frac{1}{2} \leq t \leq 1$, $K(x,t) = K'(x,2t) = H'(x,2t) = H(x,2t-1)$.

  From these two equations it follows that:
  - $K(0,t)$ is the initial point of F_0 and F_2;
  - $K(1,t)$ is the terminal point of F_0 and F_2;
  - $K(x,0) = G(x,0) = F_0(x)$, whereas $K(x,1) = H(x,1) = F_2(x)$.

  Therefore, K is a homotopy from F_0 to F_2.

Homotopy Classes of Paths

- If F is a path that is homotopic to a path G we shall write $F \cong G$. Thus, by the theorem:
  - (i) $F_0 \cong F_0$;
  - (ii) If $F_0 \cong F_1$, then $F_1 \cong F_0$;
  - (iii) If $F_0 \cong F_1$ and $F_1 \cong F_2$, then $F_0 \cong F_2$.

  Thus $\cong$ is an equivalence relation. We shall denote the equivalence class of a path F by $[\![F]\!]$.

Definition (Homotopy Class of Paths)

An equivalence set of homotopic paths is called a **homotopy class of paths**.

Homotopy Classes of Closed Paths at a Point

Definition (Set of Homotopy Classes of Loops at a Point)

At a point z in a topological space Z the collection of homotopy classes of closed paths at z is denoted by $\Pi(Z, z)$.
Among these homotopy classes there is the homotopy class $[\![e_z]\!]$, where e_z is the **constant path** defined by $e_z(t) = z$, $0 \leq t \leq 1$.

- We will show that there is a natural procedure whereby $\Pi(Z, z)$ may be converted into a group with $[\![e_z]\!]$ as its identity.

Product or Concatenation of Paths

Definition

Let $F, G : I \to Z$ be closed paths at $z \in Z$. Define $F \cdot G : I \to Z$ by

$$F \cdot G = \begin{cases} F(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ G(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

- Since $F(1) = G(0) = z$, by the lemma, $F \cdot G$ is a closed path at z.
- $F \cdot G$ is called the **product** or **concatenation** of F and G or F **followed by** G.

Product or Concatenation of Classes of Paths

- The product $\cdot$ induces a product in $\Pi(Z, z)$:

Lemma

In $\Pi(Z, z)$, if $[\![F]\!] = [\![F']\!]$ and $[\![G]\!] = [\![G']\!]$, then $[\![F \cdot G]\!] = [\![F' \cdot G']\!]$.

- We are given homotopies $K, L : I^2 \to Z$ connecting F to F' and G to G', respectively. A concatenation of K and L yields a homotopy connecting $F \cdot G$ to $F' \cdot G'$. Let

$$\begin{array}{rcll} H(t,s) & = & K(2t, s), & 0 \leq t \leq \frac{1}{2}; \\ H(t,s) & = & L(2t-1, s), & \frac{1}{2} \leq t \leq 1. \end{array}$$

Note $K(1, s) = L(0, s) = z$. By the lemma, H is continuous. Moreover, $H(t, 0) = (F \cdot G)(t)$, $H(t, 1) = (F' \cdot G')(t)$, and $H(0, s) = H(1, s) = z$.

The $\Pi(Z,z)$ Operation and its Identity

Definition

In $\Pi(Z,z)$, let $[\![F]\!] \cdot [\![G]\!] = [\![F \cdot G]\!]$.

Lemma

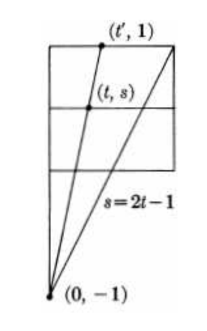
$[\![F]\!] \cdot [\![e_z]\!] = [\![e_z]\!] \cdot [\![F]\!] = [\![F]\!]$, for all $[\![F]\!] \in \Pi(Z,z)$.

- We shall first show that $[\![F]\!] \cdot [\![e_z]\!] = [\![F]\!]$. Define $H : I^2 \to Z$ by $H(t,s) = \begin{cases} F(\frac{2t}{s+1}), & \text{if } s \geq 2t-1 \\ z, & \text{if } s \leq 2t-1 \end{cases}$ If $(t,s) \in I^2$ and $s = 2t-1$, then $\frac{2t}{s+1} = 1$. Thus, by the continuity lemma, H is continuous. If $s = 1$, then $s \geq 2t-1$ and $H(t,1) = F(t)$. If $s = 0$, for $0 \leq t \leq \frac{1}{2}$, we have $s \geq 2t-1$ so that $H(t,0) = F(2t)$, while for $\frac{1}{2} \leq t \leq 1$, we have $2t-1 \geq s$ so that $H(t,0) = e_z(t) = z$. Therefore $H(t,0) = (F \cdot e_z)(t)$ and H connects $F \cdot e_z$ to F.

  For $[\![e_z]\!] \cdot [\![F]\!] = [\![F]\!]$, we define $H(t,s) = \begin{cases} z, & \text{if } s \geq 2t \\ F(\frac{2t-s}{2-s}), & \text{if } s \leq 2t \end{cases}$ and show that H connects F to $e_z \cdot F$.

Illustration of the Homotopy of the Proof

- By projecting the point $(t,s) \in I^2$, with $s \geq 2t-1$ onto the point $(t',1) \in I^2$ from the point $(0,-1)$, as s goes from 0 to 1 the interval $(t,0), 0 \leq t \leq \frac{1}{2}$ is gradually enlarged until it becomes the interval $(t',1)$, $0 \leq t' \leq 1$. By analytic geometry $t' = \frac{2t}{s+1}$. By setting $H(t,s) = F(\frac{2t}{s+1})$, we have arranged matters so that for a fixed s the interval $(t,s), 0 \leq t \leq \frac{s+1}{2}$, is mapped in such a way as to trace out the same path as F. Finally, the interval $(t,s), \frac{s+1}{2} \leq t \leq 1$, is mapped into z.



  Thus, we have started out along the interval $(t,0), 0 \leq t \leq 1$, mapping this horizontal interval by $F \cdot e_z$ and, gradually, as s increases, increased the length of the horizontal interval mapped using F and decreased to zero the length of the horizontal interval mapped by e_z.

Inverse Paths

Definition (Inverse Path)

Let $F : I \to Z$ be a path. Define $F^{-1} : I \to Z$ by $F^{-1}(t) = F(1-t)$.

- If F is a path from z to y, then F^{-1} is a path from y to z.
- If F is a closed path at z then F^{-1} is also a closed path at z which may be thought of as F traversed in the opposite sense.

Lemma

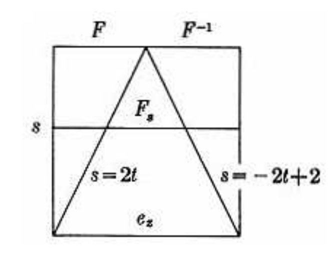
For each $[\![F]\!] \in \Pi(Z,z)$, $[\![F]\!] \cdot [\![F^{-1}]\!] = [\![F^{-1}]\!] \cdot [\![F]\!] = [\![e_z]\!]$.

- We must show that $F \cdot F^{-1} \cong e_z \cong F^{-1} \cdot F$. To show that $F \cdot F^{-1} \cong e_z$, define $H : I^2 \to Z$ as follows:
$$H(t,s) = \begin{cases} F(2t), & \text{if } s \geq 2t \\ F(s), & \text{if } s \leq 2t \text{ and } s \leq -2t+2 \\ F(2-2t), & \text{if } s \geq -2t+2 \end{cases}$$

$F \cdot F^{-1} \cong e_z \cong F^{-1} \cdot F$

- $H(t,s) = \begin{cases} F(2t), & \text{if } s \geq 2t \\ F(s), & \text{if } s \leq 2t \\ & \text{and } s \leq -2t+2 \\ F(2-2t), & \text{if } s \geq -2t+2 \end{cases}$

  Let $F_s(t) = H(t,s)$. The path F_s starts out by tracing the path of F at twice its normal rate until $s = 2t$. Then it remains stationary at $F(s)$ until $s = -2t+2$. The path F_s then returns to z backwards along this portion of the path of F, again at twice the normal rate.



  Since the various definitions of H agree when $s = 2t$ and $s = -2t+2$, H is continuous. By setting $s = 0$ and $s = 1$, H is easily seen to be a homotopy connecting e_z to $F \cdot F^{-1}$.

  Interchanging the roles of F and F^{-1} yields a homotopy connecting e_z to $F^{-1} \cdot F$.

The Product of Paths is Associative

- To complete the proof that $\Pi(Z,z)$ is a group we must show that the product is associative.

Lemma

$(\llbracket F\rrbracket \cdot \llbracket G\rrbracket) \cdot \llbracket K\rrbracket = \llbracket F\rrbracket \cdot (\llbracket G\rrbracket \cdot \llbracket K\rrbracket)$ for all $\llbracket F\rrbracket, \llbracket G\rrbracket, \llbracket K\rrbracket \in \Pi(Z,z)$.

- We must show that $(F \cdot G) \cdot K \cong F \cdot (G \cdot K)$. We define $H : I^2 \to Z$ as follows: $H(t,s) = \begin{cases} F(\frac{4t}{s+1}), & \text{if } 4t-1 \leq s \\ G(4t-s-1), & \text{if } 4t-2 \leq s \leq 4t-1 \\ K(\frac{4t-2s}{2-s}-1), & \text{if } s \leq 4t-2 \end{cases}$

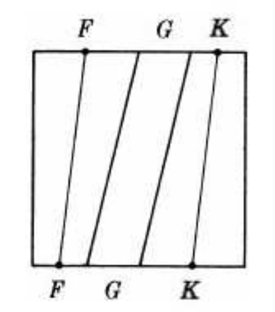
  The various definitions of H agree when $s = 4t-1$ and $s = 4t-2$.

  So H is continuous.

  Again by setting $s = 0$ and $s = 1$ it is easily seen that H is a homotopy connecting $(F \cdot G) \cdot K$ with $F \cdot (G \cdot K)$.

Illustration of the Homotopy

- A point (t,s), with $4t-1 \leq s$, is projected from $(0,-1)$ onto the point $(\frac{t}{s+1},0)$. This is mapped by $(F\cdot G)\cdot K$ into $((F\cdot G)\cdot K)(\frac{t}{s+1}) = F(\frac{4t}{s+1})$.
- Points (t,s), with $4t-2 \leq s \leq 4t-1$, are parallel projected onto $(\frac{4t-s}{4},0)$. These are mapped by $(F\cdot G)\cdot K$ into $((F\cdot G)\cdot K)(\frac{4t-s}{4}) = G(4t-s-1)$.
- A point (t,s), with $s \leq 4t-2$ is projected from $(1,2)$ onto the point $(\frac{2t-s}{2-s},0)$. This is mapped by $(F\cdot G)\cdot K$ into $((F\cdot G)\cdot K)(\frac{2t-s}{2-s}) = K(\frac{4t-2s}{2-s}-1)$.



Subsection 8

Simple Connectedness

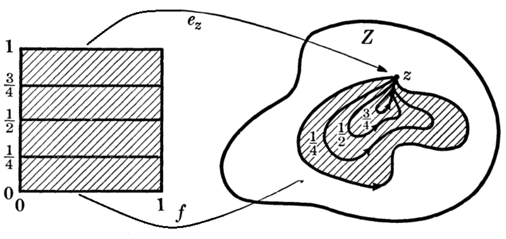
Simply Connected Topological Spaces

Definition (Simply Connected Topological Space)

A topological space Z is said to be **simply connected** if at each point $z \in Z$, there is only one homotopy class of closed paths.

- Thus if Z is simply connected, at each point the fundamental group $\Pi(Z, z)$ consists of precisely the identity element $[\![e_z]\!]$.

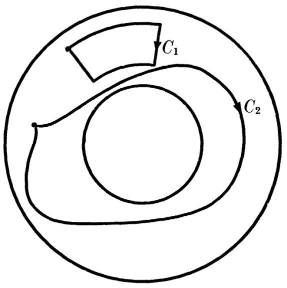
In this case there is for each closed path f at z a homotopy $H : I^2 \to Z$ which deforms f into the constant path e_z:



The possibility of carrying out the deformation corresponds to the fact that the curve traced out by f does not enclose any holes in the space Z.

Example: Annulus

- One can prove that an annulus is not simply connected:
  - Although a closed path such as C_1 is homotopic to a constant path,
  - A closed path such as C_2 is not homotopic to a constant path.



Comparing Closed Paths at Different Points

Theorem

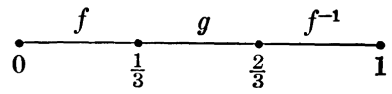
Let Z be a path-connected topological space and let $z \in Z$. Z is simply connected if and only if there is exactly one homotopy class of closed paths at z.

- In order to prove this theorem we must develop a procedure for comparing the homotopy classes of closed paths at different points.

Definition

Let f be a path in a topological space Z with $z = f(0)$ and $y = f(1)$. Let g be a closed path at y. Define $g_f : I \to Z$ by

$$g_f(t) = \begin{cases} f(3t), & \text{if } 0 \leq t \leq \frac{1}{3} \\ g(3t-1), & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3} \\ f(3-3t), & \text{if } \frac{2}{3} \leq t \leq 1 \end{cases}.$$



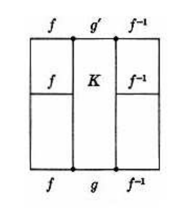
- If g is the constant path e_y the same homotopy used in the proof of the theorem shows that $(e_y)_f \cong e_z$.

Transferring Equivalence of Paths

Lemma

If $[\![g]\!] = [\![g']\!] \in \Pi(Z, y)$, then $[\![g_f]\!] = [\![g'_f]\!] \in \Pi(Z, z)$.

- Let $K : I^2 \to Z$ be the homotopy connecting g to g'. Define $H : I^2 \to Z$ by:
$$H(s,t) = \begin{cases} f(3t), & \text{if } 0 \leq t \leq \frac{1}{3} \\ K(3t-1, s), & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3} \\ f(3-3t), & \text{if } \frac{2}{3} \leq t \leq 1 \end{cases}.$$



  One verifies, as usual, that H is a homotopy connecting g_f to g'_f.
- If f' is homotopic to f, it is easily seen that we can use: a contraction of the homotopy to map the first strip and the reverse to map the third strip to obtain a homotopy connecting g_f and $g'_{f'}$:

Lemma

Let f and f' be homotopic paths with $f(0) = f'(0) = z$ and $f(1) = f'(1) = y$. Then for $[\![g]\!] \in \Pi(Z, y)$, $[\![g_f]\!] = [\![g_{f'}]\!]$.

Homomorphisms of Path Groups

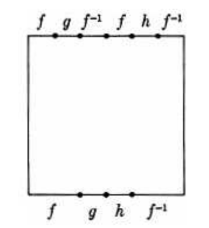
Definition (Translation of a Class of Paths)

Let $f : I \to Z$ be a path with $z = f(0)$ and $y = f(1)$. For $[\![g]\!] \in \Pi(Z, y)$, set $a_f([\![g]\!]) = [\![g_f]\!]$.

Proposition

$a_f : \Pi(Z, y) \to \Pi(Z, z)$ is a homomorphism and if $f \cong f'$, then $a_f = a_{f'}$.

- Since $(e_y)_f \cong e_z$, a_f carries the identity of $\Pi(Z, y)$ into that of $\Pi(Z, z)$. We must show that $a_f([\![g]\!] \cdot [\![h]\!]) = (a_f([\![g]\!])) \cdot (a_f([\![h]\!]))$, for $[\![g]\!], [\![h]\!] \in \Pi(Z, y)$. We have $a_f([\![g]\!] \cdot [\![h]\!]) = a_f([\![g \cdot h]\!]) = [\![(g \cdot h)_f]\!]$ and $(a_f([\![g]\!])) \cdot (a_f([\![h]\!])) = [\![g_f]\!] \cdot [\![h_f]\!] = [\![g_f \cdot h_f]\!]$.
  Thus, we must show that $(g \cdot h)_f \cong g_f \cdot h_f$.
  I^2 can be mapped along the lower edge by $(g \cdot h)_f$ and along the upper edge by $g_f \cdot h_f$.
  The last part follows by the preceding lemma.

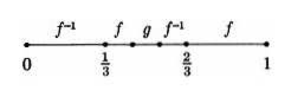


Isomorphism of the Fundamental Groups

Theorem

$a_f : \Pi(Z, y) \to \Pi(Z, z)$ and $a_{f^{-1}} : \Pi(Z, z) \to \Pi(Z, y)$ are inverse functions.

- Suppose $[\![g]\!] \in \Pi(Z, y)$. The figure shows how $(g_f)_{f^{-1}}$ is defined. A variant of the well-known construction used previously provides a homotopy connecting $(g_f)_{f^{-1}}$ to g.



  Thus $a_{f^{-1}}(a_f([\![g]\!])) = [\![(g_f)_{f^{-1}}]\!] = [\![g]\!]$. Similarly, $a_f a_{f^{-1}}$ is the identity.
- To prove the preceding theorem, suppose Z is path-connected and $\Pi(Z, y)$ consists of a single element $[\![e_y]\!]$. Note that, for any point $z \in Z$, there is a path f from z to y. Thus, $\Pi(Z, z) = a_f(\Pi(Z, y))$ is also a single element.