Introduction to Topology

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Subsection 1

Introduction

Compactness

- A closed and bounded subset A of the real line $\mathbb{R}$ is characterized by the fact that for each collection $\{O_\alpha\}_{\alpha\in I}$ of open subsets of $\mathbb{R}$, such that $A \subseteq \bigcup_{\alpha\in I} O_\alpha$, there is a finite sub-collection $O_{\alpha_1}, O_{\alpha_2}, \ldots, O_{\alpha_n}$, with $A \subseteq \bigcup_{i=1}^n O_{\alpha_i}$.
- If the latter property holds in a particular topological space, the space is said to be "compact".
- The closed and bounded subsets of $\mathbb{R}^n$ are precisely the compact subspaces of $\mathbb{R}^n$.

  This fact can be either proved directly or established by proving that the product of two compact spaces is itself compact.
- In metrizable spaces compactness is equivalent with each infinite subset having a "point of accumulation".
- Compactness, like connectedness and arcwise connectedness, is a "global" property, in that it depends on the nature of the entire space.

Subsection 2

Compact Topological Spaces

Coverings

Definition (Covering)

Let X be a set, B a subset of X, and $\{A_\alpha\}_{\alpha \in I}$ an indexed family of subsets of X. The collection $\{A_\alpha\}_{\alpha \in I}$ is called a **covering** of B or is said to **cover** B if $B \subseteq \bigcup_{\alpha \in I} A_\alpha$. If, in addition, the indexing set I is finite, $\{A_\alpha\}_{\alpha \in I}$ is called a **finite covering** of B.

- Example: Let X be a topological space and, for each $x \in X$, let N_x be a neighborhood of x. Then $\{N_x\}_{x \in X}$ is a covering of X.
- Example: For each integer n, let $A_n = [n, n+1]$. Then $\{A_n\}_{n \in \mathbb{Z}}$, ($\mathbb{Z}$ the set of integers) is a covering of the set $\mathbb{R}$ of real numbers.
- Example: For each ordered pair (m, n) of integers we let $A_{m,n}$ be the set of points $(x_1, x_2) \in \mathbb{R}^2$, such that $m \leq x_1 \leq m+1$, $n \leq x_2 \leq n+1$. Then $\{A_{m,n}\}_{(m,n) \in \mathbb{Z} \times \mathbb{Z}}$ is a covering of $\mathbb{R}^2$.
- Example: Let $X = \mathbb{R}$ and let $B = (0, 1]$. Set $A_1 = (\frac{1}{2}, 2)$, $A_2 = (\frac{1}{3}, 1)$, ..., $A_n = (\frac{1}{n+1}, \frac{1}{n-1})$, ... Then $\{A_n\}_{n \in \mathbb{N}}$ covers B.

Subcoverings

Definition (Subcovering)

Let X be a set and let $\{A_\alpha\}_{\alpha \in I}$, $\{B_\beta\}_{\beta \in J}$ be two coverings of a subset C of X. If for each $\alpha \in I$, $A_\alpha = B_\beta$, for some $\beta \in J$, then the covering $\{A_\alpha\}_{\alpha \in I}$ is called a **subcovering** of the covering $\{B_\beta\}_{\beta \in J}$.

- Thus $\{A_\alpha\}_{\alpha \in I}$ is a subcovering of $\{B_\beta\}_{\beta \in J}$ if "every A_α is a B_β".
- In particular, if $\{B_\beta\}_{\beta \in J}$ is a covering of a subset C, and I is a subset of J, such that $\{B_\beta\}_{\beta \in I}$ is also a covering of C, then $\{B_\beta\}_{\beta \in I}$ is a subcovering of $\{B_\beta\}_{\beta \in J}$.

  Example: Let $\mathbb{Q}$ be the set of rational numbers and for each $q \in \mathbb{Q}$, set $B_q = [q, q+1]$. Then $\{B_q\}_{q \in \mathbb{Q}}$ is a covering of the real numbers $\mathbb{R}$. Let $\mathbb{Z}$ be the set of integers and $A_n = [n, n+1]$. Then $\{A_n\}_{n \in \mathbb{Z}}$ is a subcovering of $\{B_q\}_{q \in \mathbb{Q}}$.

Open Coverings

- Suppose that $f : X \to Y$ is a continuous function from a topological space X to a metric space Y. Given $\epsilon > 0$, the continuity of f gives rise to a covering of X in the following manner:

  For each $x \in X$, given this $\epsilon > 0$, there is an open neighborhood U_x of x such that $f(U_x) \subseteq B(f(x); \epsilon)$. The family $\{U_x\}_{x \in X}$ of these subsets of X is a covering of X.
- This covering has the additional property that it is composed of open sets. Such a covering is referred to as an "open" covering:

Definition (Open Covering)

Let X be a topological space and B a subset of X. A covering $\{A_\alpha\}_{\alpha \in I}$ of B is said to be an **open covering** of B if for each $\alpha \in I$, A_α is an open subset of X.

Compacts Topological Spaces and Compact Subsets

Definition (Compact Topological Space)

A topological space X is said to be **compact** if, for each open covering $\{U_\alpha\}_{\alpha \in I}$ of X, there is a finite subcovering $\{U_\beta\}_{\beta \in J}$.

- X is compact if for each open covering $\{U_\alpha\}_{\alpha \in I}$ of X, there is a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of I, such that $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$ covers X.

Definition (Compact Subset)

A subset C of a topological space X is said to be compact, if C is a compact topological space in the relative topology.

- A topological space C may be a subspace of two distinct larger topological spaces X and Y.
  In this event the relative topology of C is the same whether we regard C as a subspace of X or of Y.
  So, the assertion "C is compact" depends only on C and its topology.

Compactness of Subspace In Terms of Topology of Space

- We may relate the compactness of a subspace C of a topological space X to the topology of X by means of the following theorem.

Theorem

A subset C of a topological space X is compact if and only if for each open covering $\{U_\alpha\}_{\alpha \in I}$ of C, U_α open in X, there is a finite subcovering of C.

- Let C be compact and let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of C. Then $\{U_\alpha \cap C\}_{\alpha \in I}$ is a covering of C by relatively open sets. Thus, there is a finite sub-covering $\{U_\alpha \cap C\}_{\alpha \in J}$ and $\{U_\alpha\}_{\alpha \in J}$ covers C.

  Conversely, suppose that for each open covering $\{U_\alpha\}_{\alpha \in I}$ of C there is a finite subcovering. Let $\{V_\beta\}_{\beta \in I}$ be a covering of C by relatively open subsets of C. For each $\beta \in I$, $V_\beta = U_\beta \cap C$, where U_β is open in X. Thus $\{U_\beta\}_{\beta \in I}$ is an open covering of C. By our hypothesis, there is a finite subcovering $U_{\beta_1}, \ldots, U_{\beta_m}$. Then $V_{\beta_1}, \ldots, V_{\beta_m}$ covers C and C is compact.

Compactness In Terms of Neighborhoods

- Compactness may be characterized in terms of neighborhoods:

Theorem

A topological space X is compact if and only if, whenever for each $x \in X$ a neighborhood N_x of x is given, there is a finite number of points $x_1, x_2, \ldots, x_n$ of X, such that $X = \bigcup_{i=1}^{n} N_{x_i}$.

- Suppose X is compact. Let N_x be a neighborhood of x, $x \in X$. For each x, there is an open set U_x, such that $x \in U_x \subseteq N_x$. Consequently, the family $\{U_x\}_{x \in X}$ is an open covering of X. Since X is compact, there is a finite subcovering $U_{x_1}, \ldots, U_{x_n}$. But $U_{x_i} \subseteq N_{x_i}$, for each i, whence $N_{x_1}, \ldots, N_{x_n}$ covers X.
  Conversely, assume the condition holds. Let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of X. Then, for each $x \in X$, there is an $\alpha = \alpha(x)$, such that $x \in U_\alpha$. Therefore, $N_x = U_\alpha$ is a neighborhood of x. By hypothesis, there are $x_1, \ldots, x_n$ of X, such that $N_{x_i} = U_{\alpha(x_i)}$, $i = 1, 2, \ldots, n$, covers X. Hence X is compact.

Compactness In Terms of Closed Sets

- In terms of closed sets, we have:

Theorem

A topological space is compact if and only if whenever a family $\{F_\alpha\}_{\alpha \in I}$ of closed sets is such that $\bigcap_{\alpha \in I} F_\alpha = \emptyset$, then there is a finite subset of indices $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, such that $\bigcap_{i=1}^n F_{\alpha_i} = \emptyset$.

- Suppose X is compact. Let $\{F_\alpha\}_{\alpha \in I}$ be a family of closed sets such that $\bigcap_{\alpha \in I} F_\alpha = \emptyset$. Then $\bigcup_{\alpha \in I} C(F_\alpha) = C(\bigcap_{\alpha \in I} F_\alpha) = X$. Thus, $\{C(F_\alpha)\}_{\alpha \in I}$ is an open covering of X. So there is a finite subcovering $C(F_{\alpha_1}), \ldots, C(F_{\alpha_n})$. Then $\bigcap_{i=1}^n F_{\alpha_i} = C(\bigcup_{i=1}^n C(F_{\alpha_i})) = \emptyset$.
  Conversely, suppose that the condition holds. Let $\{O_\alpha\}_{\alpha \in I}$ be an open covering of X. Then $\{C(O_\alpha)\}_{\alpha \in I}$ is a family of closed sets such that $\bigcap_{\alpha \in I} C(O_\alpha) = \emptyset$. By hypothesis, there is finite $\{\alpha_1, \ldots, \alpha_n\}$, such that $\bigcap_{i=1}^n C(O_{\alpha_i}) = \emptyset$. It follows that $O_{\alpha_1}, \ldots, O_{\alpha_n}$ is a finite subcovering.

Images of Compact Sets Under Continuous Maps

Theorem

Let $f : X \to Y$ be continuous and let A be a compact subset of X. Then $f(A)$ is a compact subset of Y.

- Let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of $f(A)$. Thus $f(A) \subseteq \bigcup_{\alpha \in I} U_\alpha$. Consequently, $A \subseteq \bigcup_{\alpha \in I} f^{-1}(U_\alpha)$. Thus, $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ is a covering of A. Since f is continuous, $f^{-1}(U_\alpha)$ is an open subset of X, for each $\alpha \in I$. Therefore $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open covering of A. Since A is compact, there is a finite subcovering $f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_n})$ of A. But $A \subseteq f^{-1}(U_{\alpha_1}) \cup \cdots \cup f^{-1}(U_{\alpha_n})$ implies that $f(A) \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$. Since $\{U_\alpha\}_{\alpha \in I}$ was an arbitrary open covering of $f(A)$, by a preceding theorem $f(A)$ is compact.

Corollary

Let the topological spaces X and Y be homeomorphic. Then X is compact if and only if Y is compact.

NonCompact Subset of a Compact Space

- Not every subset of a compact space is itself compact.

  We shall see that the closed interval $[0,1]$ is compact, whereas the open interval $(0,1)$ is not compact.

  To show that $(0,1)$ is not compact, it suffices to find one open covering of $(0,1)$ that does not have a finite subcovering.

  To this end, for each integer $n = 3,4,5,\ldots$, let $U_n = (\frac{1}{n}, 1 - \frac{1}{n})$.
  - Then $\{U_n\}_{n=3,4,5,\ldots}$ is an open covering of $(0,1)$.
  - On the other hand, for each integer $k > 3$ we have $\frac{1}{k} \notin \bigcup_{n=3}^{k} U_n$. Thus the union of every finite subcollection of $\{U_n\}_{n=3,4,5,\ldots}$ must fail to contain some point of $(0,1)$. Hence there is no finite subcovering of $\{U_n\}_{n=3,4,5,\ldots}$.

Closed Subsets of Compact Spaces are Compact

Theorem

Let X be compact. Then each closed subset of X is compact.

- Let F be a closed subset of the compact space X. Let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of F. By adjoining the open set $O = C(F)$ to the family we obtain an open covering $\{V_\beta\}_{\beta \in J}$ of X. Since X is compact, there is a finite subcovering $V_{\beta_1}, V_{\beta_2}, \ldots, V_{\beta_m}$ of X. But each V_β is either equal to a U_α, for some $\alpha \in I$ or equal to O. If O occurs among $V_{\beta_1}, V_{\beta_2}, \ldots, V_{\beta_m}$, we may delete it to obtain a finite collection of the U_α's that covers $F = C(O)$.
- Thus, in a compact space, for each subset the property of being closed implies the property of being compact.

In a Hausdorff Space Compact Implies Closed

Theorem

Let X be a Hausdorff space. If $F \subseteq X$ is compact, then F is closed.

- We show $O = C(F)$ is open by showing that, for each $z \in O$, there is a neighborhood U of z contained in O, i.e., for which $U \cap F = \emptyset$. Fix $z \in O$. By the Hausdorff property, choose, for each $x \in F$, open neighborhoods U_x of z and V_x of x, such that $U_x \cap V_x = \emptyset$. The family $\{V_x\}_{x \in F}$ is an open covering of F. Since F is compact, there is a finite subcovering $V_{x_1}, \ldots, V_{x_n}$ of F. $U = X_{x_1} \cap \cdots \cap U_{x_n}$ is therefore a neighborhood of z. U cannot intersect F since it does not intersect each element $V_{x_1}, \ldots, V_{x_n}$ of a covering of F. Thus $U \subseteq O$. Hence O is a neighborhood of each of its points and $F = C(O)$ is closed.

Corollary

Let X be a compact Hausdorff space. Then a subset F of X is compact if and only if it is closed.

Compact to Hausdorff Space Homeomorphisms

Theorem

Let $f : X \to Y$ be a one-one continuous mapping of the compact space X onto a Hausdorff space Y. Then f is a homeomorphism.

- We define $g : Y \to X$ by setting $g(y) = x$ if $f(x) = y$, so that f and g are inverse functions. It remains to prove that g is continuous.

  We prove this by showing that, for each closed subset F of X, $g^{-1}(F)$ is a closed subset of Y.

  Given a closed subset F of X, F is compact. Hence, by a previous theorem, $f(F) = g^{-1}(F)$ is a compact subset of Y. By the preceding theorem, $g^{-1}(F)$ is a closed subset of Y.

  Thus, g is continuous and f is a homeomorphism.

Subsection 3

Compact Subsets of the Real Line

Compact Subsets of $\mathbb{R}$ are Closed and Bounded

Definition

A subset A of $\mathbb{R}^n$ is said to be bounded if there is a real number K, such that for each $x = (x_1, x_2, \ldots, x_n) \in A$, $|x_i| \leq K$, for $1 \leq i \leq n$.

- In particular a subset A of the real line $\mathbb{R}$ is bounded if A is contained in some closed interval $[-K, K]$, $K > 0$.
- Every closed interval $[a, b]$ is bounded for $[a, b] \subseteq [-K, K]$, where $K = \max\{|a|, |b|\}$.

Lemma

If A is a compact subset of $\mathbb{R}$ then A is closed and bounded.

- $\mathbb{R}$ satisfies the Hausdorff axiom, whence A is closed. For each positive integer n, let $O_n = (-n, n)$. $\mathbb{R} = \bigcup_{n \in \mathbb{N}} O_n$. Therefore $\{O_n\}_{n \in \mathbb{N}}$ is an open covering of A. Since A is compact, $A \subseteq O_{n_1} \cup \cdots \cup O_{n_q}$. If we set $k = \max\{n_1, n_2, \ldots, n_q\}$, then $O_{n_i} \subseteq O_k$, for $i = 1, 2, \ldots, q$, whence $A \subseteq O_k = (-k, k)$. Thus $A \subseteq [-k, k]$ and A is bounded.

Compactness of $[0, 1]$

Lemma

The closed interval $[0, 1]$ is compact.

- Let $\{O_\alpha\}_{\alpha \in I}$ be a covering of $[0, 1]$ by open sets. Assume that there is no finite subcovering of $\{O_\alpha\}_{\alpha \in I}$. In this event, at least one of the two closed intervals $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$ cannot be covered by a finite subcollection of the family $\{O_\alpha\}_{\alpha \in I}$. Let $[a_1, b_1]$ denote one of these two intervals of length $\frac{1}{2}$ that cannot be covered by a finite sub-collection of the family $\{O_\alpha\}_{\alpha \in I}$. We may now divide the interval $[a_1, b_1]$ into the two subintervals $[a_1, \frac{a_1+b_1}{2}]$ and $[\frac{a_1+b_1}{2}, b_1]$ of length $\frac{1}{4}$ and assert that at least one of these two intervals cannot be covered by a finite subcollection of $\{O_\alpha\}_{\alpha \in I}$. Let $[a_2, b_2]$ denote one of these two intervals of length $\frac{1}{4}$ that has the property that it cannot be covered by a finite subcollection of $\{O_\alpha\}_{\alpha \in I}$. We proceed to define a sequence $[a_0, b_0]$, $[a_1, b_1]$, $\ldots$, $[a_n, b_n]$, $\ldots$ of such intervals.

Constructing the Sequence of Intervals

- Assume that for $r = 0, 1, 2, \ldots, n$, we have defined intervals $[a_r, b_r]$, such that:
  1. $[a_0, b_0] = [0, 1]$;
  2. $b_r - a_r = \frac{1}{2^r}$, for $r = 0, 1, \ldots, n$;
  3. for $r = 0, 1, \ldots, n-1$, either $[a_{r+1}, b_{r+1}] = [a_r, \frac{a_r+b_r}{2}]$ or $[a_{r+1}, b_{r+1}] = [\frac{a_r+b_r}{2}, b_r]$;
  4. for $r = 0, 1, \ldots, n$, no finite subcollection of $\{O_\alpha\}_{\alpha \in I}$ covers $[a_r, b_r]$.

  We then define $[a_{n+1}, b_{n+1}]$ in the following manner: In view of 4, at least one of the two intervals $[a_n, \frac{a_n+b_n}{2}]$, $[\frac{a_n+b_n}{2}, b_n]$ cannot be covered by a finite subcollection of $\{O_\alpha\}_{\alpha \in I}$. Denote by $[a_{n+1}, b_{n+1}]$ whichever of these two intervals cannot be covered by a finite subcollection of $\{O_\alpha\}_{\alpha \in I}$.

  Then Conditions 2,3 and 4 will also hold for $[a_{n+1}, b_{n+1}]$.

  By induction, we may define a sequence $[a_0, b_0]$, $[a_1, b_1]$, $[a_2, b_2]$, $\ldots$ of such intervals for which Conditions 1 through 4 are true.

Using the Sequence to Prove Compactness of $[0, 1]$

- By Conditions 3, $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$. It follows that for each pair of positive integers m and n, $a_m \leq b_n$. Thus each b_n is an upper bound of the set $\{a_0, a_1, \ldots\}$. Let a be the least upper bound of this set. Then $a \leq b_n$ for each n. Hence a is a lower bound of the set $\{b_0, b_1, \ldots\}$. Let b be the greatest lower bound of the latter set. We therefore have $a \leq b$. But, by the definition of a and b, we must have $a_n \leq a \leq b \leq b_n$, for each n. Hence, by Condition 2, $b - a \leq \frac{1}{2^n}$, for each n, and we conclude that $a = b$.

  We are now in a position to obtain a contradiction to Condition 4, from which it will follow that our assumption that there is no finite subcovering of $[0, 1]$ is untenable.

Obtaining the Contradiction

- $\{O_\alpha\}_{\alpha \in I}$ covers $[0,1]$ and $a = b \in [0,1]$. Therefore, $a \in O_\beta$, for some $\beta \in I$. Since O_β is open, there is an $\epsilon > 0$ such that $B(a;\epsilon) \subseteq O_\beta$. Let us choose the positive integer N large enough so that $\frac{1}{2^N} < \epsilon$. Then $b_N - a_N < \epsilon$. Now $a = b \in [a_N, b_N]$. Therefore, $a - a_N < \frac{1}{2^N} < \epsilon$ and $b - b_N < \frac{1}{2^N} < \epsilon$. Consequently, $[a_N, b_N] \subseteq B(a;\epsilon) \subseteq O_\beta$. Thus, $[a_N, b_N]$ may be covered by a finite subcollection (namely, one!) of the family $\{O_\alpha\}_{\alpha \in I}$.

  The assumption that no finite subcollection of $\{O_\alpha\}_{\alpha \in I}$ covers $[0,1]$ leads to a contradiction.

  We conclude that $[0,1]$ is compact.
- The gist of the argument is that if no finite subcollection of $\{O_\alpha\}_{\alpha \in I}$ covers $[0,1]$, then:
  - No finite subcollection of $\{O_\alpha\}_{\alpha \in I}$ covers a sequence of subintervals whose lengths approach zero;
  - On the other hand, if the length of one of these subintervals is small enough it is contained in some O_β.

The Heine-Borel Theorem

- Since each closed interval $[a, b]$ is homeomorphic to the closed interval $[0, 1]$ and compactness is a topological property, we obtain:

Corollary

Each closed interval $[a, b]$ is compact.

- The next theorem, which characterizes the compact subsets of the real line, is frequently referred to as the **Heine-Borel Theorem**.

Theorem (Heine-Borel)

A subset A of the real line is compact if and only if A is closed and bounded.

- The "only if" is given by a preceding lemma.
  Conversely, if A is closed and bounded, A is a closed subset of a closed interval $[-K, K]$ for some $K > 0$. But $[-K, K]$ is a compact space. Therefore, by a previous theorem, A is compact.

Subsection 4

Products of Compact Spaces

Compactness in terms of Coverings by Base for Open Sets

- A base for the open sets of a topological space Z is a collection $\mathcal{B}$ of open subsets s.t. each open subset of Z is a union of members of $\mathcal{B}$.

Lemma

Let $\mathcal{B}$ be a base for the open sets of a topological space Z. If, for each covering $\{B_\beta\}_{\beta \in J}$ of Z by members of $\mathcal{B}$, there is a finite subcovering, then Z is compact.

- Assume $\{O_\alpha\}_{\alpha \in I}$ is an open covering of Z. For each $\alpha \in I$, O_α is a union of members of $\mathcal{B}$. Let J be an indexing set for all the basic sets B_β that occur in the expression of some O_α as a union of members of $\mathcal{B}$. Thus $\bigcup_{\alpha \in I} O_\alpha \subseteq \bigcup_{\beta \in J} B_\beta$ and, hence, $\{B_\beta\}_{\beta \in J}$ is a covering of Z by members of $\mathcal{B}$. It follows from our hypothesis that there is a finite subcovering $B_{\gamma_1}, B_{\gamma_2}, \ldots, B_{\gamma_n}$ of Z. Since each B_{γ_i} occurs in the expression of some $O_{\alpha_i}, \alpha_i \in I$, as a union of members of $\mathcal{B}$, $B_{\gamma_i} \subseteq O_{\alpha_i}$. Consequently, $O_{\alpha_1}, O_{\alpha_2}, \ldots, O_{\alpha_n}$ must cover Z. Therefore, Z is compact.

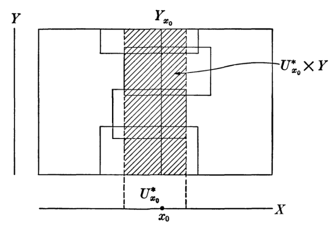
Product of Two Compact Spaces

- If X and Y are topological spaces, then a base for open sets of $X \times Y$ is the collection of sets of the form $U \times V$, where U is open in X and V is open in Y.

Theorem

Let X and Y be compact topological spaces. Then $X \times Y$ is compact.

- By the lemma, it suffices to prove that each covering of $X \times Y$ by sets of the form $U \times V$, U open in X, V open in Y, has a finite subcovering. Let $\{U_\alpha \times V_\alpha\}_{\alpha \in I}$ be such a covering.
For each $x_0 \in X$, consider the subset Y_{x_0} of $X \times Y$ consisting of all points (x_0, y), $y \in Y$. The open covering is necessarily an open covering of Y_{x_0}. But Y_{x_0} is homeomorphic to Y and hence compact. We may therefore find a finite subset I_{x_0} of I such that $\{U_\alpha \times V_\alpha\}_{\alpha \in I_{x_0}}$ covers Y_{x_0}.



Product of Two Compact Spaces (Cont'd)

- We may also assume that $x_0 \in U_\beta$, for each $\beta \in I_{x_0}$, for otherwise we may delete $U_\beta \times V_\beta$ and still cover Y_{x_0}. The set $U^*_{x_0} = \bigcap_{\alpha \in I_{x_0}} U_\alpha$ is a finite intersection of open sets containing x_0 and is therefore an open set containing x_0. We assert that $\{U_\alpha \times V_\alpha\}_{\alpha \in I_{x_0}}$ is an open covering of $U^*_{x_0} \times Y$. For, suppose $(x, y) \in U^*_{x_0} \times Y$. The point (x_0, y) is in $U_\beta \times V_\beta$ for some $\beta \in I_{x_0}$. Since $x \in U^*_{x_0}$, $x \in U_\alpha$, for all $\alpha \in I_{x_0}$. It follows that $(x, y) \in U_\beta \times V_\beta$, $\beta \in I_{x_0}$, proving our assertion.

  Now $\{U^*_x\}_{x \in X}$ is an open covering of the compact space X, hence, there is a finite subcovering $U^*_{x_1}, U^*_{x_2}, \ldots, U^*_{x_n}$ of X. Set $I^* = I_{x_1} \cup \cdots \cup I_{x_0}$. The finite family $\{U_\alpha \times V_\alpha\}_{\alpha \in I^*}$ is a covering of $X \times Y$: Given a point $(x, y) \in X \times Y$, $x \in U^*_{x_i}$, for some x_i, so that $(x, y) \in U^*_{x_i} \times Y$. By our previous assertion $(x, y) \in U_\beta \times V_\beta$ for some $\beta \in I_{x_i}$. Thus, $(x, y) \in U_\alpha \times V_\alpha$, for some $\alpha \in I^*$. We have thus established that $\{U_\alpha \times V_\alpha\}_{\alpha \in I^*}$ is a finite subcovering. Therefore, $X \times Y$ is compact.

Product of Compact Spaces

- If $X_1, X_2, \ldots, X_n$ are topological spaces, one may distinguish between $\prod_{i=1}^{n} X_i$ and $(\prod_{i=1}^{n-1} X_i) \times X_n$:
  - The points of the first space are n-tuples $(x_1, x_2, \ldots, x_n)$;
  - The points of the second space are ordered pairs $((x_1, x_2, \ldots, x_{n-1}), x_n)$ whose first elements are $(n-1)$-tuples.
- Nevertheless, these two spaces are certainly homeomorphic, using $(x_1, x_2, \ldots, x_n) \mapsto ((x_1, x_2, \ldots, x_{n-1}), x_n)$.
- Thus, applying induction, we get

Corollary

Let $X_1, X_2, \ldots, X_n$ be compact topological spaces. Then $\prod_{i=1}^{n} X_i$ is also compact.

- Tychonoff's Theorem (not proven here) asserts that the product of an arbitrary family of compact spaces is compact.

Compactness of the Unit n-Cube

- The unit n-cube I^n is the subset of $\mathbb{R}^n$ consisting of all points $x = (x_1, x_2, \ldots, x_n)$, such that $0 \leq x_i \leq 1$, for $i = 1, 2, \ldots, n$.
- As a subspace of $\mathbb{R}^n$, I^n has the same topology as the product space $I \times I \times \cdots \times I$ (n-factors).
- Since $I = [0, 1]$ is compact, as a special case of the preceding corollary we have:

Corollary

The unit n-cube I^n is compact.

Characterization of Compactness in $\mathbb{R}^n$

Theorem

A subset A of $\mathbb{R}^n$ is compact if and only if A is closed and bounded.

- The proof that, if A is compact, then A is closed and bounded is similar to the proof of this fact for a subset of the real line.

  Conversely, we first show that each closed "cube" is compact. The collection of points $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$, such that $|x_i| \leq K$, for $i = 1, 2, \ldots, n$, denoted by M_K, is a "cube" of width $2K$ with center at the origin. M_K is homeomorphic to the unit n-cube I^n, for the function $F : I^n \to M_K$ defined by

  $$F(x_1, \ldots, x_n) = (2Kx_1 - K, \ldots, 2Kx_n - K)$$

  is a homeomorphism. Since I^n is compact, M_K is compact.

  Now suppose A is closed and bounded. Then A is a closed subset of the compact cube M_K, for some K, whence A is compact.

Subsection 5

Compact Metric Spaces

Accumulation Points

- A metric space (X, d) is said to be **compact** or is called a **compactum** if its associated topological space is compact.
- A basic result is that a metric space is compact if and only if every infinite subset has at least one "point of accumulation".

Definition (Accumulation Point)

Let X be a topological space and A a subset of X. A point $a \in X$ is called an **accumulation point** of A if each neighborhood of a contains infinitely many distinct points of A.

- In referring to the accumulation points of a set A, care must be taken to specify of which topological space A is to be considered a subset.

  Example: In the real line $\mathbb{R}$, the subset $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$ has the accumulation point 0. In the topological space $(0, +\infty)$, the same set A has no accumulation points.

Accumulation and Limit Points in Hausdorff Spaces

- In a metric space we defined a as a limit point of a subset A if every neighborhood of a contains a point of A different from a.
- If we use the same definition in a topological space every accumulation point of A is also a limit point of A.
- In Hausdorff spaces, and hence in metric spaces, the two coincide.

Lemma

Let X be a Hausdorff space and A a subset of X. A point $a \in X$ is an accumulation point of A if and only if a is a limit point of A.

- Suppose a is not an accumulation point of A. Then there is a neighborhood N of a that contains at most a finite collection $\{a_1, a_2, \ldots, a_p\}$ of points of A distinct from a. For each a_i, $i = 1, 2, \ldots, p$, find a neighborhood U_i of a and a neighborhood V_i of a_i such that $U_i \cap V_i = \emptyset$. Then $N \cap U_1 \cap U_2 \cap \cdots \cap U_p$ is a neighborhood of a that contains no points of A other than possibly a.

Existence of Limit Points in Compact Spaces

Theorem

Let X be a compact space. Then every infinite subset K of X has at least one limit point in X.

- Suppose K is a subset of X that has no limit points. For each $x \in K$, there is a neighborhood N_x of x, such that $N_x \cap K = \{x\}$. K is closed and hence compact. Therefore, there are points $x_1, x_2, \ldots, x_m$, such that $N_{x_1}, N_{x_2}, \ldots, N_{x_m}$ cover K. It follows that $K = \{x_1, x_2, \ldots, x_m\}$ and K is finite.
- For compact Hausdorff spaces, and in particular for compact metric spaces, the theorem gives that every infinite subset A of X has at least one point of accumulation in X.
- We show that the converse holds for metric spaces.

First Property of Metric Spaces

Lemma

Let (X, d) be a metric space such that every infinite subset of X has at least one accumulation point in X. Then, for each positive integer n, there is a finite set of points $x_1^n, \ldots, x_p^n$, of X, such that the collection of open balls $B(x_1^n; \frac{1}{n}), \ldots, B(x_p^n; \frac{1}{n})$ covers X.

- Suppose there was an n, such that no finite collection of balls of radius $\frac{1}{n}$ covered X. Choose $x_1 \in X$. $B(x_1; \frac{1}{n})$ certainly does not cover X. Hence, there is $x_2 \in X$, such that $x_2 \notin B(x_1; \frac{1}{n})$. $B(x_1; \frac{1}{n}) \cup B(x_2; \frac{1}{n})$ does not cover X. Hence, there is $x_3 \in X$, such that $x_3 \notin B(x_1; \frac{1}{n}) \cup B(x_2; \frac{1}{n})$. Continuing, construct $x_1, x_2, \ldots, x_k, \ldots$ in X, such that for $k > 1$, $x_k \notin \bigcup_{i=1}^{k-1}(x_i; \frac{1}{n})$. Thus, $d(x_k, x_{k'}) \geq \frac{1}{n}$, if $k \neq k'$. The set $\{x_1, x_2, \ldots, x_k, \ldots\}$ is infinite and therefore has a point of accumulation $x \in X$. The neighborhood $B(x; \frac{1}{2n})$ contains infinitely many points of $\{x_1, x_2, \ldots, x_k, \ldots\}$, in particular, two $x_k, x_{k'}$, with $k \neq k'$. Since $x_k, x_{k'} \in B(x; \frac{1}{2n})$, we obtain $d(x_k, x_{k'}) < \frac{1}{n}$.

Second Property of Metric Spaces

Lemma

Let (X, d) be a metric space such that each infinite subset of X has at least one point of accumulation. Then for each open covering $\{O_\alpha\}_{\alpha \in I}$ of X, there is a positive number ϵ, such that each open ball $B(x; \epsilon)$ is contained in an element O_β of this covering.

- Suppose not. For each $n = 1, 2, \ldots$, there is an open ball $B(x_n; \frac{1}{n})$, such that $B(x_n; \frac{1}{n}) \not\subseteq O_\alpha$, for each $\alpha \in I$. Let $A = \{x_1, x_2, \ldots\}$.
  - If A is finite, some point $x \in X$ occurs infinitely often in the sequence $x_1, x_2, \ldots$. Since $\{O_\alpha\}_{\alpha \in I}$ covers X, $x \in O_\beta$, for some $\beta \in I$. Since O_β is open, there is a $\delta > 0$, such that $B(x; \delta) \subseteq O_\beta$. Choose n, so that $\frac{1}{n} < \delta$ and $x_n = x$, Then $B(x_n; \frac{1}{n}) = B(x; \frac{1}{n}) \subseteq O_\beta$, a contradiction.
  - If $A = \{x_1, x_2, \ldots\}$ is infinite it has at least one point of accumulation x. Again $x \in O_\beta$, for some $\beta \in I$. Thus, $B(x; \delta) \subseteq O_\beta$, for some $\delta > 0$. There are infinitely many points of A in the neighborhood $B(x; \frac{\delta}{2})$ of x. Hence, we may choose an n, such that $\frac{1}{n} < \frac{\delta}{2}$ and $x_n \in B(x; \frac{\delta}{2})$. We then have $B(x_n; \frac{1}{n}) \subseteq B(x; \delta) \subseteq O_\beta$, a contradiction.

Lebesgue Number of an Open Covering

- The number ϵ of the preceding lemma depends on the particular open covering considered.
- Given the open covering $\{O_\alpha\}_{\alpha \in I}$, if the number ϵ has the property that, for each $x \in X$, $B(x; \epsilon) \subseteq O_\beta$, for some $\beta \in I$, then each number ϵ', with $0 < \epsilon' < \epsilon$ also has this property.
- The least upper bound of the set of numbers having this property is called the **Lebesgue number**, ϵ_L, of the open covering $\{O_\alpha\}_{\alpha \in I}$.

Corollary

Let (X, d) be a metric space such that each infinite subset of X has an accumulation point. Then each open covering $\{O_\alpha\}_{\alpha \in I}$ of X has a Lebesgue number ϵ_L.

Metric Bolzano-Weierstraß Implies Compactness

- A topological space X is said to have the **Bolzano-Weierstraß property** if each infinite subset of X has at least one point of accumulation.
- A metric space that has the Bolzano-Weierstraß property is compact.

Theorem

Let (X, d) be a metric space that has the property that every infinite subset of X has at least one accumulation point. Then X is compact.

- Let $\{O_\alpha\}_{\alpha \in I}$ be an open covering and let ϵ_L be its Lebesgue number. We choose n so that $\frac{1}{n} < \epsilon_L$.
  - By the First Lemma, there is a finite set $\{x_1, x_2, \ldots, x_p\}$ of points of X, such that the open balls $B(x_1; \frac{1}{n}), \ldots, B(x_p; \frac{1}{n})$ cover X.
  - By the Second Lemma, for each $i = 1, 2, \ldots, p$, there is a $\beta_i \in I$, such that $B(x_i; \frac{1}{n}) \subseteq O_{\beta_i}$.

  Thus, the collection $O_{\beta_1}, \ldots, O_{\beta_p}$ is a finite subcovering of $\{O_\alpha\}_{\alpha \in I}$.

Compactness and Accumulation Points in Metric Spaces

- We have now proved the main result of this section.

Theorem

Let (X, d) be a metric space. Each infinite subset of X has at least one accumulation point if and only if X is compact.

- Recall the fact that a subspace X of Euclidean n-space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Corollary

Let X be a subspace of $\mathbb{R}^n$. Then the following three properties are equivalent:

1. X is compact.
2. X is closed and bounded.
3. Each infinite subset of X has at least one point of accumulation in X.

Uniform Continuity

Definition (Uniform Continuity)

Let $f : (X, d) \to (Y, d')$ be a function from a metric space (X, d) to a metric space (Y, d'). f is said to be **uniformly continuous** if, for each positive number ϵ, there is a $\delta > 0$, such that, whenever $d(x, y) < \delta$, then $d'(f(x), f(y)) < \epsilon$.

- If the function $g : X \to Y$ is continuous, then for each $\epsilon > 0$, there is $\delta > 0$, where δ may depend on both the choice of x and ϵ, such that $d(x, a) < \delta$ implies $d'(g(x), g(a)) < \epsilon$.
- If, however, g is uniformly continuous, then given $\epsilon > 0$, the number δ may be used at each point $x \in X$, that is, uniformly throughout X, to yield $d'(g(x), g(a)) < \epsilon$ if $d(x, a) < \delta$:

Corollary

If $f : X \to Y$ is uniformly continuous, then f is continuous.

Continuity Does Not Imply Uniform Continuity

- A continuous function need not be uniformly continuous.
  Example: Consider $f : (0,1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$.
  Given $\epsilon = 1$, we shall show that there does not exist a $\delta > 0$, such that
  $$|x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| < 1.$$
  Given any $\delta > 0$ we can choose n large enough so that, if $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$, we have
  $$x - y = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \delta.$$
  On the other hand,
  $$\left| f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right) \right| = |n - (n+1)| = 1.$$
- Note that, in this example, the interval $(0,1]$ is not compact.

Continuity from a Compact Metric Space

Theorem

Let $f : (X, d) \to (Y, d')$ be a continuous function from a compact metric space X to a metric space Y. Then f is uniformly continuous.

- Given $\epsilon > 0$, for each $x \in X$, there is a $\delta_x > 0$, such that if $y \in B(x; \delta_x)$, then $f(y) \in B(f(x); \frac{\epsilon}{2})$. The collection $\{B(x; \delta_x)\}_{x \in X}$ is an open covering of X. Since X is compact, this open covering has a Lebesgue number. Choose δ to be a positive number less than this Lebesgue number. Suppose $z, z' \in X$ and $d(z, z') < \delta$. Then z and z' are in a ball of radius less than δ. It follows that $z, z' \in B(x; \delta_x)$, for some $x \in X$. Consequently, $f(z), f(z') \in B(f(x); \frac{\epsilon}{2})$. Hence

$$d'(f(z), f(z')) \leq d'(f(z), f(x)) + d'(f(x), f(z')) < \epsilon.$$

Subsection 6

Compactness and the Bolzano-Weierstraß Theorem

Compactness and the Bolzano-Weerstraß Property

- We proved that a metric space is compact if and only if each infinite subset has at least one accumulation point.
- We also showed that the first property implies the second for Hausdorff spaces.
- Compactness is the stronger of the two properties since there are examples of topological spaces that are not compact, but in which each infinite subset has a point of accumulation.
- Thus, the second property, the Bolzano-Weierstraß property, may be viewed as a weaker type of compactness.

Countable and Denumerable Sets

Definition (Countable Set)

A non-empty set X is said to be **countable** if there is an onto function $f : \mathbb{N} \to X$, where $\mathbb{N}$ is the set of positive integers.

- A finite set $X = \{x_1, x_2, \ldots, x_n\}$ is countable, for we may construct an onto function $f : \mathbb{N} \to X$ by setting $f(i) = x_i$, $1 \leq i \leq n$, and defining $f(i)$ for $i > n$ arbitrarily, say $f(i) = x_1$, $i > n$.
- A countable set that is not finite is called **denumerable**.
- In this case an onto function $f : \mathbb{N} \to X$ gives rise to an "enumeration" $x_1 = f(1), x_2 = f(2), \ldots, x_n = f(n), \ldots$ of the elements of X, so that we may write $X = \{x_1, x_2, \ldots, x_n, \ldots\}$.
- Since f may not be one-one, a given element $x \in X$ may occur more than once in this enumeration. By deleting all but the first occurrence of any element $x \in X$ and reducing the succeeding subscripts accordingly, it is possible to obtain an enumeration of X in which each element occurs one and only one time.

Consequences of Countability

Corollary

Let X and Y be non-empty sets. If X is countable and there is an onto function $g : X \to Y$, then Y is countable.

- Since X is countable, there is an onto function $f : \mathbb{N} \to X$, $\mathbb{N}$ the set of positive integers. The composite function $gf : \mathbb{N} \to Y$ is onto. Hence Y is countable.

Corollary

A non-empty subset of a countable set is countable.

- Let $A \subseteq X$, X countable, A non-empty. We may define an onto function $g : X \to A$ by setting $g(a) = a$, for $a \in A$, and defining g arbitrarily for points $x \notin A$.

Some Examples

- The set $\mathbb{N}$ of positive integers is countable, since the identity function $i : \mathbb{N} \to \mathbb{N}$ is onto.
- On the other hand, the collection $2^{\mathbb{N}}$ of subsets of $\mathbb{N}$ is not countable, since for an arbitrary set X there is no onto function $f : X \to 2^X$.
- A set that is not countable is called **uncountable**.
- Another example of an uncountable set is the set $\mathbb{R}$ of real numbers.
- We show next that the set $\mathbb{N} \times \mathbb{N}$ is countable!

Countability of $\mathbb{N} \times \mathbb{N}$

Theorem

Let $\mathbb{N}$ be the set of positive integers. Then $\mathbb{N} \times \mathbb{N}$ is countable.

- The elements of $\mathbb{N} \times \mathbb{N}$ may be arrayed in the form of an infinite matrix We may arrange these elements in the form of a sequence, $x_1 = f(1), x_2 = f(2), \ldots, x_k = f(k), \ldots$, by setting $x_1 = (1,1)$, $x_2 = (2,1)$, $x_3 = (1,2)$, $x_4 = (3,1)$, $\ldots$,

$$\begin{matrix} (1,1) & (1,2) & (1,3) & \ldots & (1,n) & \ldots \\ (2,1) & (2,2) & (2,3) & \ldots & (2,n) & \ldots \\ (3,1) & (3,2) & (3,3) & \ldots & (3,n) & \ldots \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots\vdots\vdots \\ (m,1) & (m,2) & (m,3) & \ldots & (m,n) & \ldots \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots\vdots\vdots \end{matrix}$$

  i.e., having exhausted the entries on the diagonal of this matrix from $(p,1)$ to $(1,p)$ we proceed to enumerate the entries on the diagonal from $(p+1,1)$ to $(1,p+1)$.

  To define the onto function $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ we note that there are $\frac{p(p+1)}{2}$ entries on or above the diagonal from $(p,1)$ to $(1,p)$. Hence, if $1 \leq j \leq p+1$, we are setting $x_{\frac{p^2+p}{2}+j} = f(\frac{p^2+p}{2}+j) = (p-j+2, j)$.

Countable Union of Countable Sets

- Note that the set $\mathbb{Q}^+$ of positive rational numbers is countable, since the function $h : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ defined by $h(r,s) = \frac{r}{s}$, $(r,s) \in \mathbb{N} \times \mathbb{N}$, is onto.

Corollary

Let $X_1, X_2, \ldots, X_n, \ldots$, be a sequence of sets, each of which is countable. Then $\bigcup_{i=1}^{\infty} X_i$ is a countable set.

- Since each X_i is countable, there is an onto function $f_i : \mathbb{N} \to X_i$, $i = 1, 2, \ldots, n, \ldots$. We define a function $F : \mathbb{N} \times \mathbb{N} \to \bigcup_{i=1}^{\infty} X_i$, by setting

$$F(i,j) = f_i(j), \quad (i,j) \in \mathbb{N} \times \mathbb{N}.$$

 F is onto: If $x \in \bigcup_{i=1}^{\infty} X_i$, $x \in X_i$, for some i, whence $x = f_i(j) = F(i,j)$, for some $(i,j) \in \mathbb{N} \times \mathbb{N}$.

 But $\mathbb{N} \times \mathbb{N}$ is countable. Therefore, $\bigcup_{i=1}^{\infty} X_i$ is countable.

Other Countable Sets

- A more direct proof of the preceding corollary can be given by utilizing the same array as before to display the elements of $\bigcup_{i=1}^{\infty} X_i$, entering the element $x_j^i = f_i(j) = F(i,j)$ in the ith row and jth column.

  One then enumerates the elements of $\bigcup_{i=1}^{\infty} X_i$ in accordance with the scheme adopted in the proof of the preceding theorem.
- In view of the fact that the set $\mathbb{Q}^+$ of positive rational numbers is countable, the set $\mathbb{Q}^-$ of negative rational numbers is also countable.

  Consequently, the set $\mathbb{Q}$ of all rational numbers is countable.
- Using the corollary, we may then assert that the collection B of all open intervals on the real line of the form $B(p;q)$, $q > 0$, with p and q rational, is also a countable set, for it is a countable union of sets each of which is countable.
- Thus, there is a countable basis for the open sets on the real line.

Bolzano-Weierstraß and Countable Open Coverings

- The Bolzano-Weierstraß property implies that each countable open covering has a finite subcovering:

Theorem

Let E be a subspace of a topological space X with the property that each infinite subset of E has a point of accumulation in E. Then every countable open covering of E has a finite subcovering.

- Let $O_1, O_2, \ldots, O_n, \ldots$ be a sequence of open subsets of X such that $E \subseteq \bigcup_{n=1}^{\infty} O_n$. Suppose that no finite subcollection covers E. Then, for each integer k, the open set $O_k^* = \bigcup_{n=1}^{k} O_n$ does not cover E. Hence, for each k, there is $x_k \in E$, such that $x_k \notin O_k^*$. $A = \{x_1, x_2, \ldots, x_k, \ldots\} \subseteq E$ must be infinite. Let $x \in E$ be a point of accumulation of A. Since $x \in E$, $x \in O_p$, for some index p. O_p is a neighborhood of x. Therefore, infinitely many of the points of A belong to O_p. In particular, for some $k > p$, we would have $x_k \in O_p \subseteq O_p^* \subseteq O_k^*$. This contradicts the choice of x_k.

Spaces with Countable Basis for Open Sets

- If a topological space X is such that every open covering has a countable subcovering, by virtue of the preceding theorem, the Bolzano-Weierstraß property implies compactness.
- A sufficient condition for every open covering to have a countable subcovering is given by the next theorem:

Theorem (Lindelöf's Theorem)

Let X be a topological space that has a countable basis for the open sets. Then each open covering $\{O_\alpha\}_{\alpha \in I}$ has a countable subcovering.

- Let $\mathcal{B} = \{B_\beta\}_{\beta \in J}$ be a countable basis for the open sets of X.
  Claim: For each point $x \in X$ and each open set O containing x, there is a basis element B_β, such that $x \in B_\beta \subseteq O$.
  Since $\mathcal{B}$ is a basis for the open sets, O is a union of elements of $\mathcal{B}$. Thus $O = \bigcup_{\beta \in J'} B_\beta$, for some subset J' of J. But $x \in O$. Hence $x \in B_\beta$, for some $\beta \in J'$. Moreover, $B_\beta \subseteq O$.

Spaces with Countable Basis for Open Sets (Cont'd)

- Now suppose that $\{O_\alpha\}_{\alpha \in I}$ is an open covering of X. We must find a countable subset $I' \subseteq I$, such that $\{O_\alpha\}_{\alpha \in I'}$ is a covering.

  For each $x \in X$ and each O_α containing x, we choose a B_β, such that $x \in B_\beta \subseteq O_\alpha$. The totality of sets B_β so chosen constitute a countable subfamily $\{B_\beta\}_{\beta \in J'}$ of the basis $\mathcal{B}$ and this subfamily covers X. Now, for each such B_β with $\beta \in J'$, let us choose a single index $\alpha = f(\beta) \in I$, such that $B_\beta \subseteq O_\alpha = O_{f(\beta)}$. The totality of sets O_α so chosen constitute a subfamily $\{O_\alpha\}_{\alpha \in I'} = \{O_{f(\beta)}\}_{\beta \in J'}$, which is also countable and must cover X, for $\bigcup_{\beta \in J'} B_\beta \subseteq \bigcup_{\alpha \in I'} O_\alpha$.

Corollary

Let X be a topological space that has a countable basis for the open sets. Then X is compact if and only if X has the Bolzano-Weierstraß property.

NonCompact Spaces with the Bolzano-Weierstraß Property

- We will not give an example of a topological space X that has the Bolzano-Weierstraß property, but is not compact.
- The preceding discussion shows that such a space must be found among those topological spaces which are not metrizable and do not possess a countable basis for the open sets.
- Those spaces which possess a countable base for the open sets are called **completely separable** or are said to satisfy the **second axiom of countability**.

Subsection 7

Surfaces by Identification

Continuous Function from Compact to Hausdorff Space

Lemma

Let $f : X \to Y$ be a continuous mapping of a compact space X onto a Hausdorff space Y. Then a subset B of Y is closed if and only if $f^{-1}(B)$ is a closed subset of X.

- If B is closed, $f^{-1}(B)$ is closed by continuity of f. If $f^{-1}(B)$ is closed, then $f^{-1}(B)$ is compact. But $B = f(f^{-1}(B))$. Hence B is compact. A compact subset of a Hausdorff space is closed.

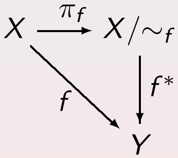
Corollary

Let $f : X \to Y$ be a continuous mapping of a compact space X onto a Hausdorff space Y. Then Y has the identification topology determined by f.

"Pasting" Points Together

- Let $\pi_f : X \to X/\sim_f$ be the identification map which carries each element $x \in X$ into its equivalence set determined by the relation $x \sim_f x'$ if $f(x) = f(x')$. $\sim_f$ is continuous. So $X/\sim_f$ is compact.

  By a previous theorem, there is a continuous map $f^* : X/\sim_f \to Y$ such that $f^* \pi_f = f$. f^* is one-one. Hence, again by a previous result, f^* is a homeomorphism.



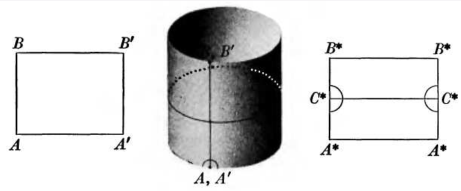
Corollary

The mapping $f^* : X/\sim_f \to Y$ induced by a continuous function $f : X \to Y$ of a compact space onto a Hausdorff space is a homeomorphism.

- One may think of a point $\overline{x} \in X/\sim_f$ as being represented by "pasting" together the various points in X.

A Cylinder

- We start with a rectangle with four corner vertices A, B, B', A'.

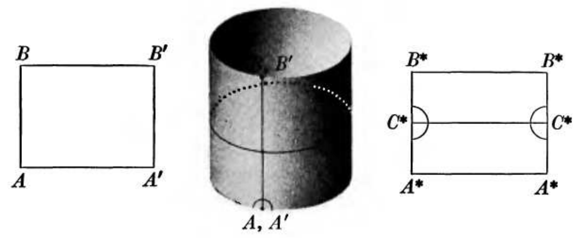


We identify the edge AB with the edge $A'B'$ in such a way that A is identified with A' and B with B'.

We obtain a surface that is homeomorphic to the cylinder.

- We may picture the cylinder as being the topological space obtained by:
  - Replacing both A and A' by a new point A^*;
  - Replacing both B and B' by a new point B^*;
  - Similarly any pair of corresponding points C and C' on the respective edges AB and $A'B'$ is replaced by a new point C^*.
    A neighborhood of this new point C^* would contain the interior of the small semi-circles shown.

An Additional Remark on the Cylinder



- Note that if in this figure we join C^* to itself by the path represented by the horizontal line, the space consisting of the points of this line would be homeomorphic to a circle.

  In fact, it consists of an interval whose end points have been identified.

Restricting the Domain of an Identification

Lemma

Let X and Y be topological spaces, let $f : X \to Y$ be a continuous function that is onto, and let Y have the identification topology induced by f. If $B \subseteq Y$ is such that $A = f^{-1}(B)$ is closed, then the subspace B of Y has the identification topology induced by the restriction $f\mid_A : A \to B$.

- We must show that a subset F of B is closed in B if and only if $f\mid_A^{-1}(F)$ is closed in A.
  The restriction $f\mid_A$ of the continuous function f to $A = f^{-1}(B)$ is continuous. So, if F is closed in B, then $f\mid_A^{-1}(F)$ is closed in A.
  Conversely, suppose that $f\mid_A^{-1}(F)$ is closed in A. Then, since A is closed in X, $f\mid_A^{-1}(F)$ is closed in X.
  If we prove that $f\mid_A^{-1}(F) = f^{-1}(F)$, it will follow that F is closed in Y and consequently in B, for Y has the identification topology and therefore $f^{-1}(F)$ closed in X implies F closed in Y.

Restricting the Domain of an Identification (Cont'd)

- It remains to prove $f\mid_A^{-1}(F) = f^{-1}(F)$.

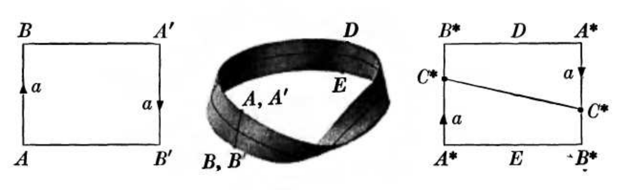
  Suppose that $x \in f^{-1}(F)$. To show that $x \in f\mid_A^{-1}(F)$, we must show that $x \in A$ and $f\mid_A(x) \in F$. But, if $x \in f^{-1}(F)$, then $f(x) \in F \subseteq B$, whence $x \in f^{-1}(B) = A$. Thus, x is in the domain of $f\mid_A$ and $f\mid_A(x) = f(x) \in F$. Hence $x \in f^{-1}(F)$ implies that $x \in f\mid_A^{-1}(F)$.

  Conversely, if $x \in f\mid_A^{-1}(F)$, then $f\mid_A(x) \in F$. Now $f\mid_A(x) = f(x)$. Thus, $f(x) \in F$ and $x \in f^{-1}(F)$.

  It follows that $f\mid_A^{-1}(F) = f^{-1}(F)$.

The Möbius Strip or Band

- Start with the rectangle with vertices labeled in the order A, B, A', B':

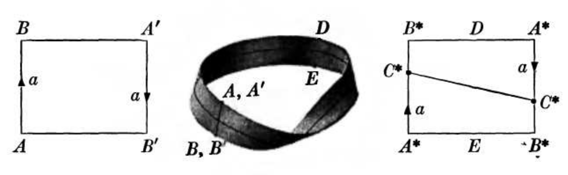


  We identify the edge AB with the edge $B'A'$ by first giving the rectangular strip a 180 degree twist, so that the vertices A and A' coincide and the vertices B and B' coincide.

- A topologically equivalent space is obtained by identifying pairs of points such as A, A', replacing them by a single new point A^*.

Some Properties of the Möbius Strip

- The Möbius strip has many curious properties.



  - The oblique line in joining C^* to itself is homeomorphic to a circle.
  - The upper horizontal line running from B^* through D to A^* is homeomorphic to an interval.
  - If on the Möbius strip we trace out the curve from B^* through D to A^* and continue on through E back to B^* we trace out an interval with its end-points identified, that is, a circle.

    Thus the Möbius strip is a surface whose bounding curve is a circle.
  - If the Möbius strip is cut down its center, the resulting surface will not be disconnected: We may still connect a point of the upper half rectangle to a point of the lower half rectangle by joining both of them to the bounding curve $B^*DA^*EB^*$.

Orientable and Non-Orientable Surfaces

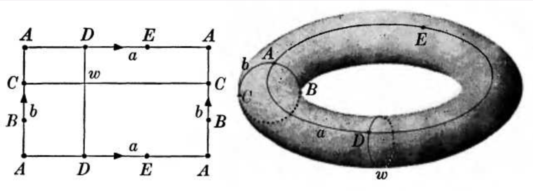
- If an arrowhead is placed on a circle we say that the circle is **oriented**.
  - The sense of rotation indicated by the arrowhead is then called the **positive orientation**;
  - The opposite sense of rotation is called the **negative orientation**.
- An oriented circle in the plane can be moved about in the plane in an arbitrary manner but will always be oriented in the same sense when it returns to its original position.

  For this reason the plane is said to be **orientable**.
- On the Möbius strip an oriented circle can be moved around the strip, say along the oblique line with its center initially at C^*, and when it returns to its original position the orientation will have been reversed.

  Surfaces with this property are called **non-orientable**.

The Torus

- We identify the edges of a rectangle according to the following scheme:



The resulting topological space is called a **torus**.

- A torus is topologically the surface of a donut or a rubber tire.
- We may view the torus as being obtained in two steps:
  - First, we identify the two opposite edges labeled a of the rectangle to obtain a cylinder;
  - Second, we identify the two resulting circular edges (labeled b) of the cylinder to obtain the torus.

Composing Identifications

Proposition

Let X, Y, Z be topological spaces, let $f : X \to Y$ and $g : Y \to Z$ be continuous and onto. If Y has the identification topology induced by $f : X \to Y$ and Z has the identification topology induced by $g : Y \to Z$, then Z has the identification topology induced by $gf : X \to Z$.

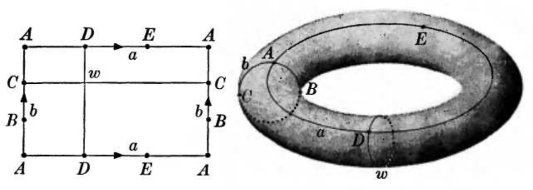
- Clearly, if F is a closed subset of Z, then $(gf)^{-1}(F)$ is a closed subset of X, for gf is continuous.

  Conversely, suppose $(gf)^{-1}(F) = f^{-1}(g^{-1}(F))$ is a closed subset of X. Since Y has the identification topology induced by $f : X \to Y$, $g^{-1}(F)$ is a closed subset of Y. Similarly, since Z has the identification topology induced by $g : Y \to Z$, $g^{-1}(F)$ closed in Y implies that F is closed in Z.

  Thus F is closed if and only if $(gf)^{-1}(F)$ is closed, i.e., Z has the identification topology induced by $gf : X \to Z$.

The Topology of the Torus

- Topologically, the torus is the product of two circles.



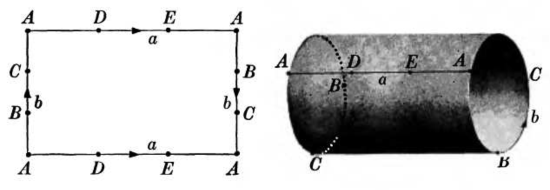
- An arbitrary point w of the torus may be written as $w = (C, D)$, where C is a point of the circle b and D a point of the circle a.
- It is clear that the product of a neighborhood of C and a neighborhood of D is a neighborhood of w.

  Conversely, a neighborhood of w contains the product of a neighborhood of C and a neighborhood of D.

  Thus, the topology of the torus is the topology of the product of two circles.

The Klein Bottle

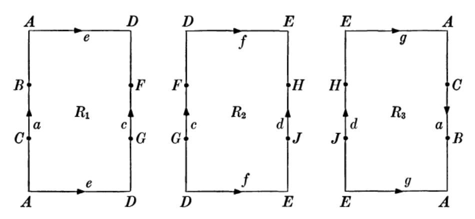
- Another surface resulting from the identification of opposite pairs of edges of a rectangle is called the **Klein bottle**.

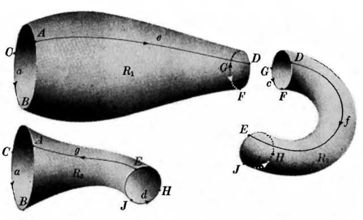


  The Klein bottle may be obtained by:
  - First identifying the edges labeled *a* to obtain a cylinder.
  - Then identifying the two circles labeled *b*, not like in the torus, but, rather, introducing a “twist”.
- In visualizing the Klein bottle, there is no way to identify these two circular edges of the cylinder without forcing the surface of the Klein bottle to intersect or pass through itself.
- So we construct the Klein bottle in several pieces.

Starting with Three Cylinders

- We start with three rectangles.



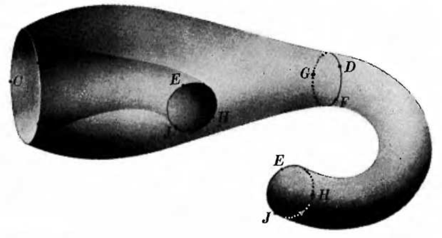


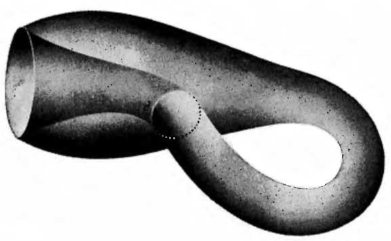
If the rectangles R_1 and R_2 are joined along the edge labeled c and the rectangles R_2 and R_3 are joined along the edge labeled d, we obtain the rectangle and identifications of the preceding figure.

- If, in these three rectangles, we first identify the pairs of edges labeled e, f, and g respectively, we obtain three cylinders that are homeomorphic to three corresponding surfaces labeled R_1, R_2, R_3.

Pasting the Three Cylinders

- To construct the Klein bottle we identify the three cylinders along the pairs of circular edges a, c and d, respectively:
  - We join R_1 and R_3 along the circles a, so that R_3 lies inside R_1.
  - We join R_1 and R_2 along the circles labeled c for the left figure:
  - We finally identify the two circles labeled d.

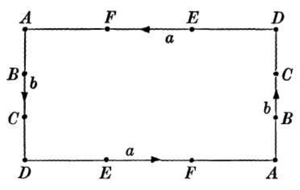


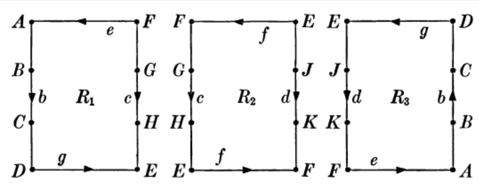


- In the right figure, the Klein bottle does not intersect itself along the circle d, but each point along d represents at the same time two points of the Klein bottle.

The Projective Plane

- The last surface we consider is obtained by identifying both of the pairs of opposite edges of a rectangle with a "twist".



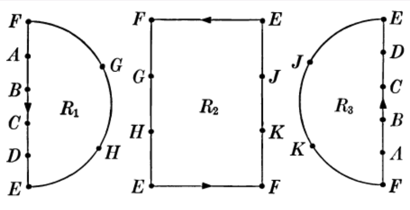


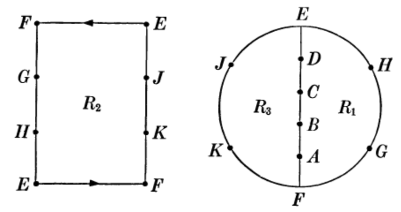
- In this figure not all vertices are identified with one another, but only diagonally opposite vertices are joined together.
- To relate this surface to some of the preceding surfaces, first separate the large rectangle into three smaller rectangles R_1, R_2, R_3.

  When re-identified along the pairs of edges labeled c, d, R_1, R_2, R_3 will again give us the rectangle and the identifications of the original.

Pasting the Rectangles Together

- Join the two edges labeled f in R_2 we obtain a Möbius strip.
- Distort (by homeomorphisms) R_1 and R_3 into semicircular regions.





- Join R_1 and R_3 along common edge $FABCDE$ to obtain the disc and the Möbius strip, with the indicated identifications.
- The surface we have been considering is therefore a Möbius strip whose boundary circle $FGHEJKF$ is to be attached to the boundary circle $FGHEJKF$ of a disc.
- This is homeomorphic to one of the models of the "real projective plane", namely, a disc with antipodal points identified.

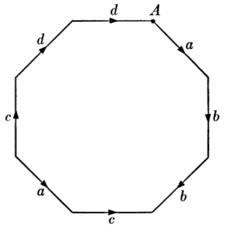
Analytic Model of the Real Projective Plane

- Let $A = \mathbb{R}^3 - \{(0,0,0)\}$ be the set of all ordered triples (x_1, x_2, x_3) of real numbers such that not all of x_1, x_2, x_3 are zero.
- Define an equivalence relation on A by $(x_1, x_2, x_3) \sim (y_1, y_2, y_3)$, if there is a real number $r \neq 0$, such that $rx_1 = y_1, rx_2 = y_2, rx_3 = y_3$.
- The collection of equivalence sets P is the **real projective plane**.
- A point $p \in P$ is the collection of all points on a given straight line through the origin of $\mathbb{R}^3$ other than the origin itself.
- The intersection of this equivalence set p with the unit sphere S^2 in $\mathbb{R}^3$ is a pair of antipodal points.
- If we confine ourselves to the hemisphere of S^2 lying above the plane $x_3 = 0$, each equivalence set p meets the hemisphere in either a single point in the interior of the hemisphere or in a pair of antipodal points on the equator or boundary of the hemisphere.
  - This upper hemisphere is topologically a disc.
  - Identifying antipodal points on the boundary yields an identification space which is equivalent to the real projective plane.

Closed 2-Manifolds

- The sphere, torus, Klein bottle, and projective plane are examples of a larger class of surfaces, called **closed 2-manifolds**, obtained by identifying pairs of edges of a polygon with $2n$ sides.

Example:



The figure indicates the surface that can be obtained by identifying pairs of sides of an octagon. With each such figure we may associate a "surface symbol".

We start at any vertex, such as A, and in clockwise order write down:

- the label of an edge, if its arrow is pointing clockwise;
- the label with an inverse sign if the arrow points counterclockwise.

Examples:

- A surface symbol for the surface of the figure is $abbc^{-1}a^{-1}cdd$.
- A surface symbol for the torus is $ab^{-1}a^{-1}b$.

Classification of Closed 2-Manifolds

- By the "cut-and-paste" method one can show that each 2-manifold is homeomorphic to a 2-manifold whose surface symbol is of one of the following four forms:
  - $abb^{-1}a^{-1}$: The first form indicates that the surface is homeomorphic to a sphere.
  - $a_1b_1a_1^{-1}b_1^{-1}\cdots a_pb_pa_p^{-1}b_p^{-1}$, $p \geq 1$: The second form includes the surface symbol of a torus and in general indicates that the surface is homeomorphic to a sphere with p handles.

  These two classes of surface are orientable and can all be constructed in three-dimensional Euclidean space.
  - $abab$: The third form indicates that the surface is homeomorphic to the projective plane. The projective plane is a disc to whose circular boundary has been attached a Möbius strip.
  - $a_1a_1\cdots a_qa_q$, $q > 1$: The fourth form consists of all surfaces obtained by attaching q Möbius strips to a sphere with q circular regions removed.