College Trigonometry

George Voutsadakis[1]

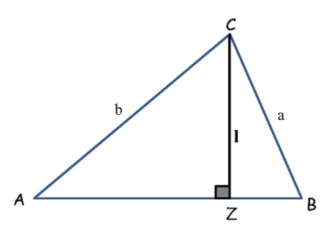
[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 131

Subsection 1

The Law of Sines

Law of Sines



- Consider an arbitrary triangle $\triangle ABC$;
- Let CZ be the perpendicular from C on AB;
- By right triangle trigonometry, we have $\sin A = \frac{\ell}{b}$ and $\sin B = \frac{\ell}{a}$;
- Therefore, $\ell = b \sin A = a \sin B$, showing that
$$\frac{a}{\sin A} = \frac{b}{\sin B};$$

The Law of Sines

If A, B, C are the measures of the angles of a triangle and a, b, c the lengths of the sides opposite to those angles, respectively, then
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C};$$

Solving a Triangle: The ASA Case

- Solve the triangle $\triangle ABC$, if $A = 42^\circ$, $B = 63^\circ$ and $c = 18$ cm. Since the sum of all three angles of a triangle is 180°, we get that

$$C = 180^\circ - (A + B) = 180^\circ - 42^\circ - 63^\circ = 75^\circ;$$

Now the Law of Sines yields:

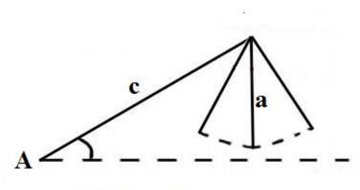
$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \Rightarrow \quad a = \frac{c \sin A}{\sin C} = \frac{18 \cdot \sin 42^\circ}{\sin 75^\circ} \text{ cm};$$

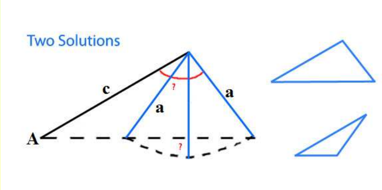
One more application of the Law of Sines yields

$$\frac{b}{\sin B} = \frac{c}{\sin C} \quad \Rightarrow \quad b = \frac{c \sin B}{\sin C} = \frac{18 \cdot \sin 63^\circ}{\sin 75^\circ} \text{ cm};$$

Solving a Triangle: The SSA Case

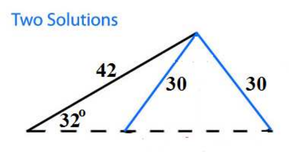
- Suppose we know sides a and c and angle A;
- Solving $\triangle ABC$ may result into none, one or two solutions, depending on the relationship between the height of the triangle from angle B and the lengths of a, c;
  - If $\angle A$ is acute, then
    - $a < h$: there is no solution;
    - $a = h$: there is a single solution; a right triangle;
    - $h < a < c$: there are two possible triangles;
    - $a \geq c$: there is a single solution, not a right triangle;
  - If $\angle A$ is obtuse, then
    - $a \leq c$: no solution;
    - $a > c$: one solution;





Example

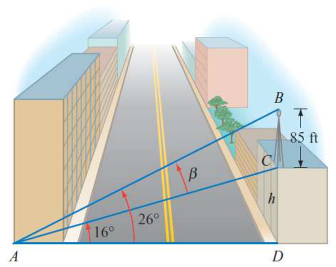
- Suppose that $A = 32^\circ$, $c = 42$ and $a = 30$; Find the measure of C; Note that for the height h corresponding to angle B we have
$$\sin A = \frac{h}{c} \quad \Rightarrow \quad h = c \sin A = 42 \sin 32^\circ \approx 22.26;$$
Therefore, $h < a < c$, which means that there are two solutions for $\triangle ABC$:



For $\angle C$, we have, using the Law of Sines:
$$\frac{c}{\sin C} = \frac{a}{\sin A} \quad \Rightarrow \quad \sin C = \frac{c \sin A}{a} = \frac{42 \sin 32^\circ}{30} \approx 0.742$$
$$\Rightarrow \quad C \approx \sin^{-1} 0.742 \text{ or } 180^\circ - \sin^{-1} 0.742$$
$$\Rightarrow \quad C \approx 47.89^\circ \text{ or } 132.11^\circ;$$

An Application: A Radio Antenna

An 85 feet high antenna is at the top of an office building; At distance AD from the base of the building, the angle of elevation to the top of the antenna is 26° and the angle of elevation to the bottom of the antenna is 16°; What is the height of the building?



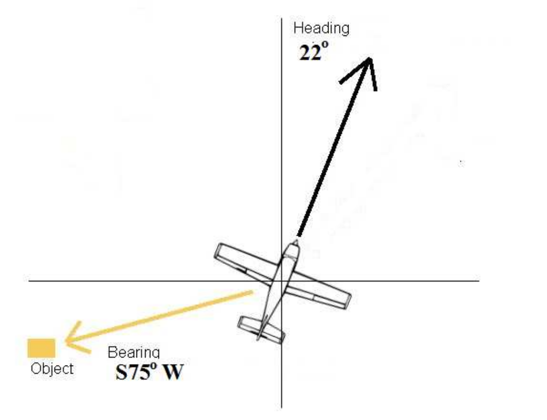
We first use Law of Sines to calculate the distance AC from the point A of the observer to the top of the building:

$$\frac{BC}{\sin\beta} = \frac{AC}{\sin B} \quad \Rightarrow \quad AC = \frac{BC\sin B}{\sin\beta} = \frac{85\cdot\sin 64^\circ}{\sin 10^\circ};$$

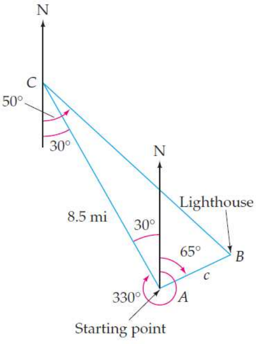
Finally, we apply right triangle trigonometry to compute h:

$$\sin 16^\circ = \frac{h}{AC} \quad \Rightarrow \quad h = AC\sin 16^\circ = \frac{85\sin 64^\circ \sin 16^\circ}{\sin 10^\circ} \approx 121 \text{ feet};$$

Definition of Heading and Bearing



An Application: Navigation



A ship with a heading of 330°first sighted a lighthouse at a bearing of N65°E; After traveling 8.5 miles, the ship observed the lighthouse at a bearing of S50°E; Find the distance from the ship to the lighthouse when the first sighting was made.

In $\triangle ABC$ we know the measures of all angles:

$$\begin{aligned} \angle CAB &= 65^\circ + 30^\circ = 95^\circ; \\ \angle BCA &= 50^\circ - 30^\circ = 20^\circ; \\ B &= 180^\circ - 95^\circ - 20^\circ = 65^\circ; \end{aligned}$$

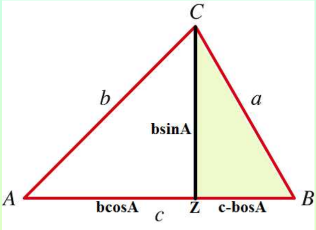
Thus, we may apply the Law of Sines to calculate the distance c from the ship to the lighthouse at the moment of the first sighting:

$$\frac{b}{\sin B} = \frac{c}{\sin C} \quad \Rightarrow \quad c = \frac{b \sin C}{\sin B} = \frac{8.5 \cdot \sin 20^\circ}{\sin 65^\circ} \approx 3.2 \text{ miles};$$

Subsection 2

The Law of Cosines and Area

The Law of Cosines



- Consider an arbitrary triangle $\triangle ABC$;
- By right triangle trigonometry, we have $\sin A = \frac{CZ}{b}$ and $\cos A = \frac{AZ}{b}$;
- Therefore, $CZ = b \sin A$ and $BZ = AB - AZ = c - b\cos A$;
- Therefore, we get
$$a^2 = (b\sin A)^2 + (c - b\cos A)^2 = b^2\sin^2 A + c^2 - 2bc\cos A + b^2\cos^2 A = b^2(\sin^2 A + \cos^2 A) + c^2 - 2bc\cos A = b^2 + c^2 - 2bc\cos A;$$

The Law of Cosines

If A, B, C are the measures of the angles of a triangle and a, b, c the lengths of the sides opposite to those angles, respectively, then $a^2 = b^2 + c^2 - 2bc\cos A$, $b^2 = a^2 + c^2 - 2ac\cos B$ and $c^2 = a^2 + b^2 - 2ab\cos C$.

Solving Triangles

- The SAS Case: In $\triangle ABC$, $B = 110^\circ$, $a = 10$ cm and $c = 15$ cm; Find the size b;

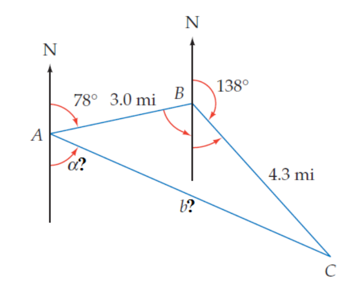
$$b^2 = a^2 + c^2 - 2ac\cos B = 10^2 + 15^2 - 2 \cdot 10 \cdot 15 \cdot \cos 110^\circ = 100 + 225 - 300 \cdot \cos 110^\circ \approx 427.606$$
$$\Rightarrow \quad b \approx \sqrt{427.606} \approx 20.68;$$

- The SSS Case: In $\triangle ABC$, $a = 32$ feet, $b = 20$ feet and $c = 40$ feet; Find B;

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$\Rightarrow \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{32^2 + 40^2 - 20^2}{2 \cdot 32 \cdot 40} = 0.86875$$
$$\Rightarrow \quad B = \cos^{-1} 0.86875 \approx 29.69^\circ;$$

Application: Navigation

A boat sailed 3 miles at a heading of 78° and then turned to a heading of 138° and sailed another 4.3 miles; Find the distance and the bearing of the boat from the starting point;



In $\triangle ABC$ we have: $B = 78° + (180° - 138°) = 120°$;

Thus, we may apply the Law of Cosines to calculate b and, then A:

$$b = \sqrt{a^2 + c^2 - 2ac\cos B} = \sqrt{4.3^2 + 3^2 - 2 \cdot 4.3 \cdot 3 \cdot \cos 120°} \approx 6.4 \text{ miles}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \approx \frac{6.4^2 + 3^2 - 4.3^2}{2 \cdot 6.4 \cdot 3} \approx 0.82$$

$$\Rightarrow \quad A \approx \cos^{-1} 0.82 \approx 34.96°;$$

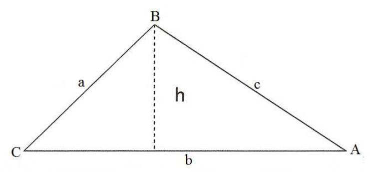
Therefore the bearing from the starting point is $\alpha \approx 180° - (78° + 34.96°) \approx 67.04°$, i.e., S67.04°E;

Law of Sines or Law of Cosines?

- The Law of Sines is used for the following cases:
  - ASA: Measures of two angles and length of included side known;
  - AAS: Measures of two angles and length of a side opposite one of these angles known;
  - SSA: The lengths of two sides and the measure of an angle opposite one of these sides known (**ambiguous case**);
- The Law of Cosines is used for the following cases:
  - SSS: The lengths of all three sides known;
  - SAS: Lengths of two sides and measure of the included angle known;

Area of a Triangle

- Consider an arbitrary triangle $\triangle ABC$;
- Let h be the length of the height to side AC;



- Notice $\sin A = \frac{h}{c}$, whence $h = c \sin A$;
- Thus, the area K of the triangle is given by $K = \frac{1}{2} bh = \frac{1}{2} bc \sin A$;

Area of a Triangle

The area K of $\triangle ABC$ is one half the product of the lengths of any two sides and the sine of the included angle:

$$K = \frac{1}{2} bc \sin A = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B;$$

Examples

- Given $A = 62^\circ$, $b = 12$ meters and $c = 5$ meters, find the area of $\triangle ABC$;

$$K = \frac{1}{2} bc \sin A = \frac{1}{2} \cdot 12 \cdot 5 \cdot \sin 62^\circ \approx 26.49;$$

- Given $A = 32^\circ$, $C = 77^\circ$ and $a = 14$ inches, find the area of $\triangle ABC$;

  Using the Law of Sines, we get

$$\frac{b}{\sin B} = \frac{a}{\sin A} \quad \Rightarrow \quad b = \frac{a \sin B}{\sin A};$$

  Therefore, for the area K we obtain

$$K = \frac{1}{2} ab \sin C = \frac{1}{2} a \frac{a \sin B}{\sin A} \sin C = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin A}$$

$$\Rightarrow \quad K = \frac{1}{2} \cdot 14^2 \cdot \frac{\sin 71^\circ \sin 77^\circ}{\sin 32^\circ} \approx 170.38;$$

Heron's Formula for the Area

Heron's Formula for the Area of a Triangle

If a, b and c are the lengths of the sides of a triangle, then the area K of the triangle is given by

$$K = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{1}{2}(a+b+c).$$

- Example: Find the area of the triangle with $a = 7$ meters, $b = 15$ meters and $c = 12$ meters;

$$s = \tfrac{1}{2}(a+b+c) = \tfrac{1}{2}(7+15+12) = 17;$$
$$K = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{17 \cdot 10 \cdot 2 \cdot 5} = \sqrt{1700} = 10\sqrt{17} \text{ m}^2;$$

Application: Las Vegas Luxury

Each face of Luxor Hotel in Las Vegas is an isosceles triangle with a base of 646 feet and sides of length 576 feet; If the glass on the exterior costs $35 per square foot, what is the cost of the glass for one of the triangular faces of the hotel?

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(646+576+576) = 899;$$

$$K = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{899 \cdot 253 \cdot 323 \cdot 323} = \sqrt{23,729,318,063} \approx 154,043 \text{ ft}^2;$$

Therefore, the cost of the glass for one face is

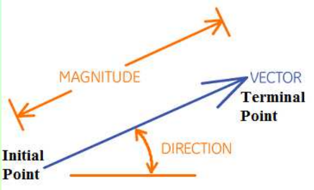
$$C = \$35/\text{ft}^2 \cdot 154,043 \text{ ft}^2 = \$5,391,505;$$

Subsection 3

Vectors

Vectors

- **Vector quantities** have a **magnitude** and a **direction**;



- Examples are force, velocity, displacement etc.;

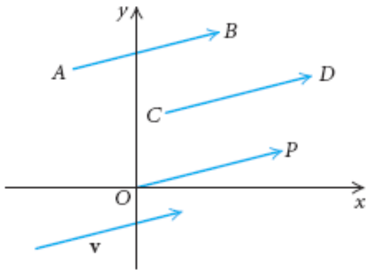
Definition of a Vector

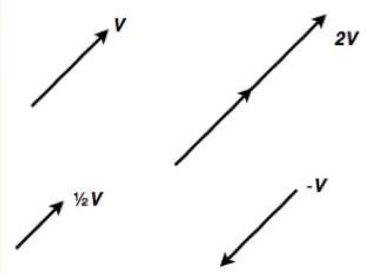
A **vector** is a directed line segment; The length of the line segment is the **magnitude** of the vector and the **direction** is measured by an angle.

- The starting point A is called the **initial point** or **tail** and the ending point B is called the **terminal point** or the **head** of the vector;
- A vector with tail A and head B is denoted $\overrightarrow{AB}$ or **AB**;
- The magnitude of this vector is denoted $\|\overrightarrow{AB}\| = \|\mathbf{AB}\|$;

Equivalence and Scalar Multiplication

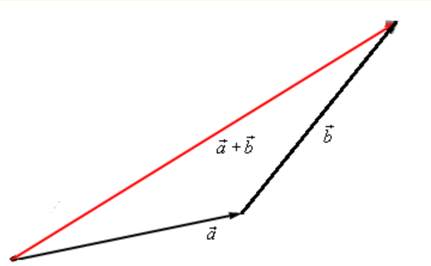
- Two vectors are **equivalent** if they have the same magnitude and the same direction;
- **Scalar multiplication** is the multiplication of a vector by a real number; If the real number is positive, then the magnitude changes but the direction does not; If the number is negative then the magnitude changes and the direction is reversed;

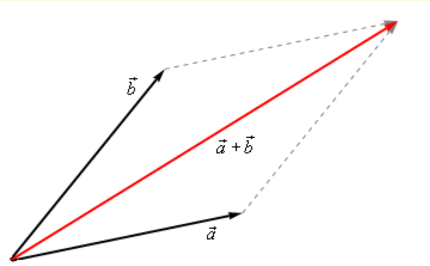




Sum or Resultant Vector

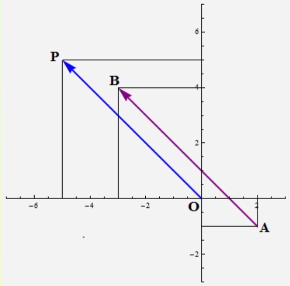
- The **resultant** or **sum** of two vectors is the vector that has the same effect as the combined application of the two vectors;
- The resultant can be computed using
  - the triangle method; or
  - the parallelogram method;





Standard Position

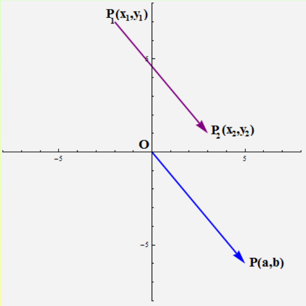
- A vector can be moved in the plane as long as its magnitude and direction are not changed;
- For instance, the vector **AB** with $A(2,-1)$ and $B(-3,4)$ may be moved so that its initial point is at the origin O; Then its terminal point becomes $P(-5,5)$;



- Because **OP** and **AB** have same magnitude and direction, they are equivalent: $\mathbf{OP} = \mathbf{AB}$;

Standard Position and Components

- When a vector is placed with initial point at the origin O it is said to be in **standard position**;
- Then, the coordinates of its terminal point P are its **components**;
- Consider the vector $\mathbf{P_1P_2}$, with $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$;



Its components a, b are given by

$$a = x_2 - x_1 \quad \text{and} \quad b = y_2 - y_1;$$

Then one uses the notation

$$\mathbf{P_1P_2} = \langle a, b \rangle;$$

Vector Operations Using Components

Definitions of Fundamental Vector Operations

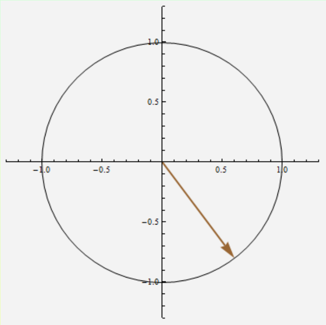
If $\mathbf{v} = \langle a, b\rangle$ and $\mathbf{w} = \langle c, d\rangle$ are two vectors and k a real number, then

1. $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$;
2. $\mathbf{v} + \mathbf{w} = \langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle$;
3. $k\mathbf{v} = k\langle a, b\rangle = \langle ka, kb\rangle$;

- Example: Let $\mathbf{v} = \langle -2, 3\rangle$ and $\mathbf{w} = \langle 4, -1\rangle$; Find
  - $\|\mathbf{w}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$;
  - $\mathbf{v} + \mathbf{w} = \langle -2, 3\rangle + \langle 4, -1\rangle = \langle -2 + 4, 3 - 1\rangle = \langle 2, 2\rangle$;
  - $-3\mathbf{v} = -3\langle -2, 3\rangle = \langle 6, -9\rangle$;
  - $2\mathbf{v} - 3\mathbf{w} = 2\langle -2, 3\rangle - 3\langle 4, -1\rangle = \langle -4, 6\rangle - \langle 12, -3\rangle = \langle -16, 9\rangle$;

Unit Vectors

- A **unit vector** is one whose magnitude is 1;
- Example: Verify that $\mathbf{v} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$ is a unit vector;

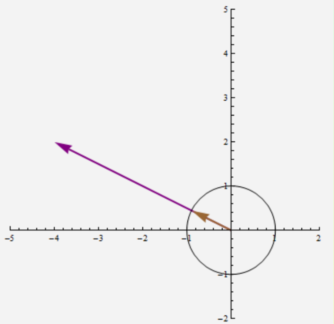


We have

$$\|\mathbf{v}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{1} = 1;$$

Unit Vector in a Given Direction

- Example: Find a unit vector **u** in the direction of the vector $\mathbf{v} = \langle -4, 2 \rangle$;



$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{(-4)^2 + 2^2}}\langle -4, 2 \rangle = \langle -\frac{4}{\sqrt{20}}, \frac{2}{\sqrt{20}} \rangle = \langle -\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \rangle;$$

Unit Vectors **i** and **j**

Definitions of Vectors **i** and **j**

$$\mathbf{i} = \langle 1, 0 \rangle, \qquad \mathbf{j} = \langle 0, 1 \rangle$$

- Example: Write the vector $\langle 3, 7 \rangle$ in terms of the unit vectors **i** and **j**;

$$\langle 3, 7 \rangle = \langle 3, 0 \rangle + \langle 0, 7 \rangle = 3\langle 1, 0 \rangle + 7\langle 0, 1 \rangle = 3\mathbf{i} + 7\mathbf{j};$$

Representation of a Vector in Terms of **i** and **j**

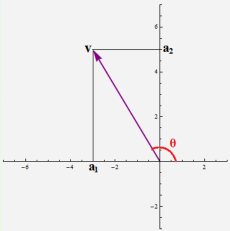
If $\mathbf{v}$ is a vector and $\mathbf{v} = \langle a_1, a_2 \rangle$, then $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j}$.

- Example: Given $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = 5\mathbf{i} + 3\mathbf{j}$, find $3\mathbf{v} - 2\mathbf{w}$;

$$3\mathbf{v} - 2\mathbf{w} = 3(3\mathbf{i} - 4\mathbf{j}) - 2(5\mathbf{i} + 3\mathbf{j}) = (9\mathbf{i} - 12\mathbf{j}) - (10\mathbf{i} + 6\mathbf{j}) =$$
$$(9 - 10)\mathbf{i} + (-12 - 6)\mathbf{j} = -\mathbf{i} - 18\mathbf{j};$$

Horizontal and Vertical Components

- Consider the vector $\mathbf{v} = \langle a_1, a_2 \rangle$;



- Its magnitude is $\|\mathbf{v}\| = \sqrt{a_1^2 + a_2^2}$;
- Recall the definitions of sine and cosine of the angle θ with initial side the positive x-axis and terminal side the vector $\mathbf{v}$:

$$\cos\theta = \frac{a_1}{\|\mathbf{v}\|} \quad \text{and} \quad \sin\theta = \frac{a_2}{\|\mathbf{v}\|};$$

- Thus, we obtain $a_1 = \|\mathbf{v}\| \cos\theta$ and $a_2 = \|\mathbf{v}\| \sin\theta$;

Horizontal and Vertical Components of a Vector

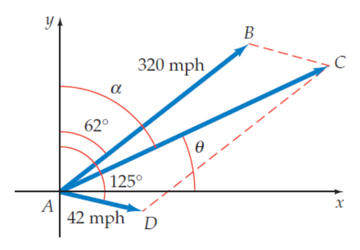
If $\mathbf{v} = \langle a_1, a_2 \rangle$, with $\mathbf{v} \neq \mathbf{0}$, then

$$a_1 = \|\mathbf{v}\| \cos\theta \quad \text{and} \quad a_2 = \|\mathbf{v}\| \sin\theta;$$

The **horizontal component** of $\mathbf{v}$ is $\|\mathbf{v}\| \cos\theta$ and the **vertical component** is $\|\mathbf{v}\| \sin\theta$.

Application: Air Speed

An airplane is traveling with an airspeed of 320 mph and a heading of 62°; A wind of 42 mph is blowing at a heading of 125°; Find the ground speed and the course of the airplane;



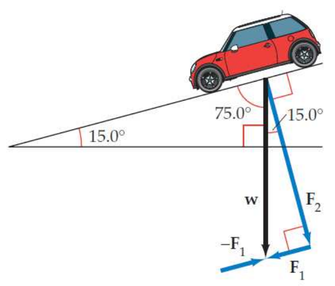
$\mathbf{AB} = 320\cos 28^\circ\mathbf{i} + 320\sin 28^\circ\mathbf{j}$;
$\mathbf{AD} = 42\cos(-35^\circ)\mathbf{i} + 42\sin(-35^\circ)\mathbf{j}$;
$\mathbf{AC} = [320\cos 28^\circ + 42\cos(-35^\circ)]\mathbf{i} + [320\sin 28^\circ + 42\sin(-35^\circ)]\mathbf{j} \approx (282.5 + 34.4)\mathbf{i} + (150.2 - 24.1)\mathbf{j} = 316.9\mathbf{i} + 126.1\mathbf{j}$;

Therefore,

$$\|\mathbf{AC}\| = \sqrt{(316.9)^2 + (126.1)^2} \approx 340;$$
$$\alpha = 90^\circ - \theta \approx 90^\circ - \tan^{-1}\frac{126.1}{316.9} \approx 68^\circ;$$

Application: Force

The car of the picture weighs 2855 pounds; What magnitude force is needed to keep the car from rolling down the ramp? What magnitude force does the car exert against the ramp?



$$\sin 15^\circ = \frac{\|\mathbf{F_1}\|}{\|\mathbf{w}\|} \quad \Rightarrow \quad \|\mathbf{F_1}\| = \|\mathbf{w}\| \sin 15^\circ = 2855 \sin 15^\circ \approx 739 \text{ pounds;}$$

$$\cos 15^\circ = \frac{\|\mathbf{F_2}\|}{\|\mathbf{w}\|} \quad \Rightarrow \quad \|\mathbf{F_2}\| = \|\mathbf{w}\| \cos 15^\circ = 2855 \cos 15^\circ \approx 2758 \text{ pounds;}$$

Dot Product

Definition of Dot Product

Given $\mathbf{v} = \langle a, b \rangle$ and $\mathbf{w} = \langle c, d \rangle$, the **dot product** of $\mathbf{v}$ and $\mathbf{w}$ is given by

$$\mathbf{v} \cdot \mathbf{w} = ac + bd.$$

- Example: Find the dot product of $\mathbf{v} = \langle 6, -2 \rangle$ and $\mathbf{w} = \langle -3, 4 \rangle$;

$$\mathbf{v} \cdot \mathbf{w} = 6 \cdot (-3) + (-2) \cdot 4 = -18 - 8 = -26;$$

Properties of the Dot Product

In the following $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors and a is a scalar:

1. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$;
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$;
3. $a(\mathbf{u} \cdot \mathbf{v}) = (a\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (a\mathbf{v})$;
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$;
5. $\mathbf{0} \cdot \mathbf{v} = 0$;
6. $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$;
7. $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$;

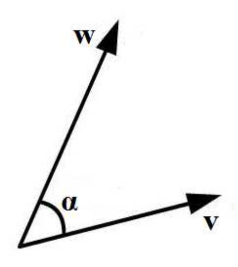
Magnitude, Angle and the Dot Product

Magnitude of a Vector in Terms of the Dot Product

If $\mathbf{v} = \langle a, b \rangle$, then $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.

Alternative Formula for the Dot Product

If $\mathbf{v}$ and $\mathbf{w}$ are two nonzero vectors and α is the smallest nonnegative angle between $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \alpha$.



Angle Between Two Vectors

Angle Between Two Vectors

If $\mathbf{v}$ and $\mathbf{w}$ are two nonzero vectors and α is the smallest nonnegative angle between $\mathbf{v}$ and $\mathbf{w}$, then $\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$ and $\alpha = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$.

- Example: Find the measure of the smallest nonnegative angle between the vectors $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{w} = -\mathbf{i} + 5\mathbf{j}$;

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{2\cdot(-1)+(-3)\cdot 5}{\sqrt{2^2+(-3)^2}\sqrt{(-1)^2+5^2}} = \frac{-17}{\sqrt{13}\sqrt{26}} = \frac{-17}{13\sqrt{2}} = -\frac{17\sqrt{2}}{26};$$

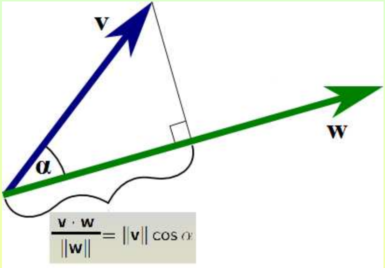
$$\alpha = \cos^{-1}\left(-\tfrac{17\sqrt{2}}{26}\right) \approx 157.6^\circ;$$

Scalar Projection

Scalar Projection of **v** Onto **w**

If **v** and **w** are two nonzero vectors and α is the angle between **v** and **w**, then the scalar projection of **v** onto **w**, $\text{proj}_{\mathbf{w}}\mathbf{v}$, is given by

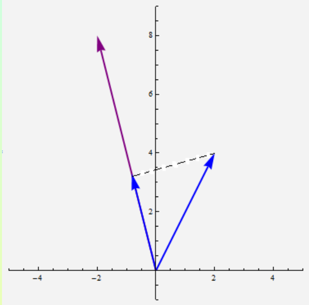
$$\text{proj}_{\mathbf{w}}\mathbf{v} = \|\mathbf{v}\| \cos \alpha;$$



- Since $\cos \alpha = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$, we also get $\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|}$;

Example

Given $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{w} = -2\mathbf{i} + 8\mathbf{j}$, find $\text{proj}_{\mathbf{w}}\mathbf{v}$;



$$\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|} = \frac{2\cdot(-2)+4\cdot 8}{\sqrt{(-2)^2+8^2}} = \frac{28}{\sqrt{68}} = \frac{28}{2\sqrt{17}} = \frac{14\sqrt{17}}{17};$$

Parallel and Perpendicular Vectors

- Two vectors are **parallel** when the angle α between them is 0° or 180°;
- Two vectors are **perpendicular** or **orthogonal** when the angle between them is 90°;
- Two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel if and only if there exists a real number c, such that $\mathbf{w} = c\mathbf{v}$;
- Two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.

Application: Work

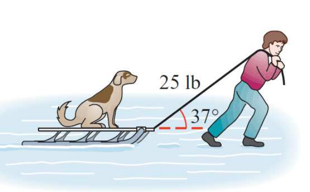
Definition of Work

The work W done by a force $\mathbf{F}$ applied along a displacement $\mathbf{s}$ is

$$W = \mathbf{F} \cdot \mathbf{s} = \|\mathbf{F}\| \|\mathbf{s}\| \cos \alpha, \ \alpha \text{ angle between } \mathbf{F} \text{ and } \mathbf{s};$$

- If the child pulls the sled a horizontal distance of 7 feet, what is the work done?

  $$W = \mathbf{F} \cdot \mathbf{s} = \|\mathbf{F}\| \|\mathbf{s}\| \cos \alpha = 25 \cdot 7 \cdot \cos 37^\circ \approx 140 \text{ ft-lb};$$



- What is the work done in moving the box 15 feet along the ramp?

  $$W = \mathbf{F} \cdot \mathbf{s} = \|\mathbf{F}\| \|\mathbf{s}\| \cos \alpha = 50 \cdot 15 \cdot \cos 27^\circ \approx 670 \text{ ft-lb};$$