College Trigonometry

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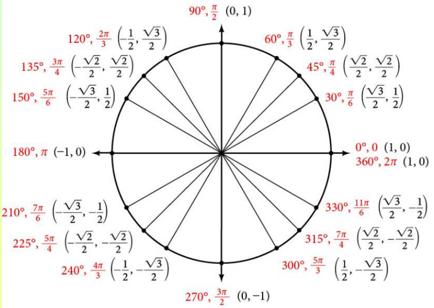
LSSU Math 131

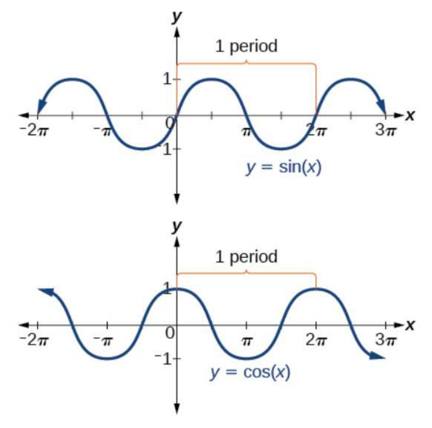
1. Periodic Functions
   - Graphs of the Sine and Cosine Functions
   - Graphs of the Other Trigonometric Functions
   - Inverse Trigonometric Functions

Subsection 1

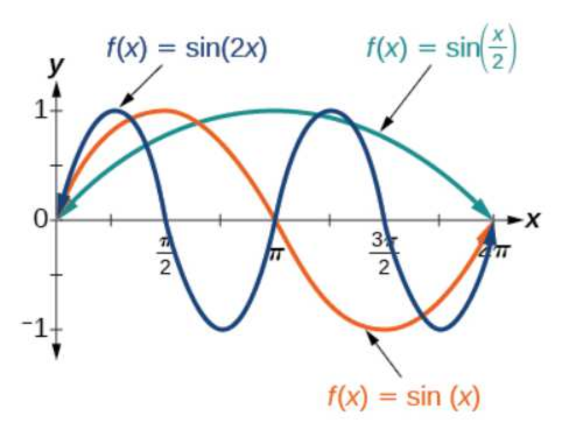
Graphs of the Sine and Cosine Functions

Period of the Sine or Cosine Function





How Horizontal Stretches and Compressions Affect Period



The period of $f(x) = A\sin(Bx)$ or $f(x) = A\cos(Bx)$ is $T = \frac{2\pi}{|B|}$.

Identifying the Period of a Sine or Cosine Function

- Determine the period of the function $f(x) = \sin\left(\frac{\pi}{6}x\right)$.

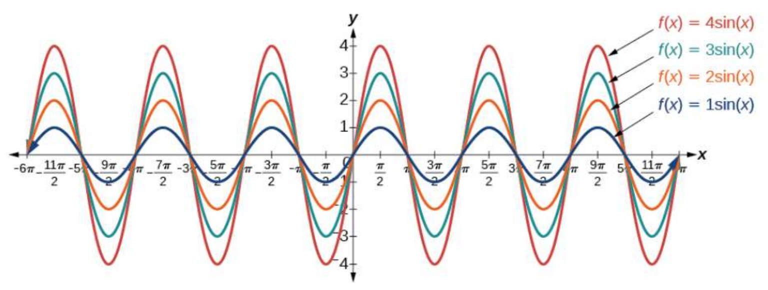
  $\sin x$ has period 2π.

  $f(x)$ is a horizontal stretch of $\sin x$ by a factor of $\frac{1}{\frac{\pi}{6}} = \frac{6}{\pi}$.

  So its period is

$$2\pi \cdot \frac{6}{\pi} = 12.$$

How Vertical Stretches and Compressions Affect Amplitude



The amplitude of $f(x) = A\sin(Bx)$ or $f(x) = A\cos(Bx)$ is $|A|$.

Identifying the Amplitude of a Sine or Cosine Function

(a) Which transformation is applied in passing from $\sin x$ to $f(x) = -4\sin x$?

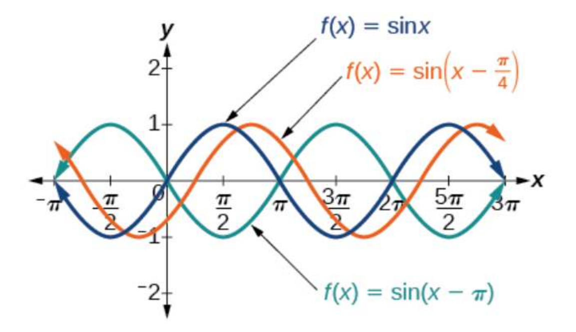
(b) What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$?

(a) We apply:

- a vertical reflection (around the x-axis);
- a vertical stretch by a factor of 4.

(b) The amplitude of $f(x)$ is $|-4| = 4$.

Phase Shift of a Function



The phase shift of $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$ can be found by rewriting as

$$f(x) = A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D \text{ or } f(x) = A\cos\left(B\left(x - \frac{C}{B}\right)\right) + D.$$

So it is equal to $\frac{C}{B}$.

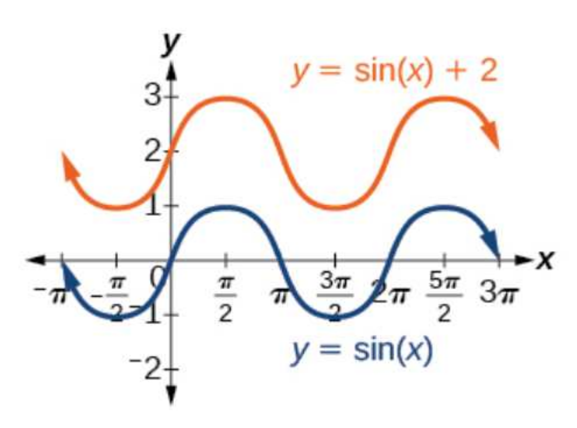
Identifying the Phase Shift of a Function

- Determine the direction and magnitude of the phase shift for $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$.

  $f(x)$ is obtained by $\sin x$ by applying
  - a horizontal shift left by $\frac{\pi}{6}$;
  - a vertical shift down by 2.

  Thus, the phase shift is $-\frac{\pi}{6}$.

Vertical Shift of a Function



The vertical shift of $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$ is D.

Identifying the Vertical Shift of a Function

- Determine the direction and magnitude of the vertical shift for $f(x) = \cos(x) - 3$.

  $f(x)$ is obtained by $\cos x$ by applying a vertical shift down by 3.

Multiple Transformations

- Given a sinusoidal function in the form

$$f(x) = A\sin(Bx - C) + D,$$

identify the amplitude, the period, the phase shift and the midline.

  - Determine the amplitude as $|A|$.
  - Determine the period as $P = \frac{2\pi}{|B|}$.
  - Determine the phase shift as $\frac{C}{B}$.
  - Determine the midline as $y = D$.

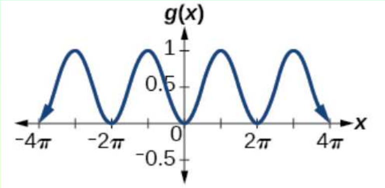
Identifying the Variations of a Sinusoidal Function

- Determine the midline, amplitude, period and phase shift of the function $y = 3\sin(2x) + 1$.

  We have:
  - Midline: $y = 1$.
  - Amplitude: $|A| = 3$.
  - Period: $P = \frac{2\pi}{|B|} = \frac{2\pi}{2} = \pi$.
  - Phase Shift: $\frac{C}{B} = 0$.

Identifying the Equation from a Graph

- Determine the formula for the cosine function



Follow one-by-one the features to determine A, B, C, D in the form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$.

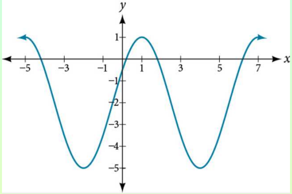
Here I follow sine:

- Amplitude: $|A| = \frac{1}{2}$.
- Period: $P = 2\pi$. So $B = 1$.
- Phase Shift: $\frac{C}{B} = \frac{\pi}{2}$. So $C = \frac{\pi}{2}$.
- Midline: $y = \frac{1}{2}$. So $D = \frac{1}{2}$.

We conclude that $f(x) = \frac{1}{2}\sin\left(x - \frac{\pi}{2}\right) + \frac{1}{2}$.

Identifying the Equation from a Graph

- Determine the equation for the sinusoidal function



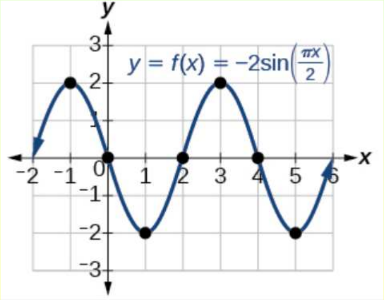
Follow one-by-one the features to determine A, B, C, D in the form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$.
I follow cosine:
  - Amplitude: $|A| = 3$.
  - Period: $P = \frac{2\pi}{|B|} = 6$. So $B = \frac{\pi}{3}$.
  - Phase Shift: $\frac{C}{B} = 1$. So $C = \frac{\pi}{3}$.
  - Midline: $y = -2$. So $D = -2$.

We conclude that $f(x) = 3\cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) - 2$.

Graphing and Identifying the Amplitude and Period

- Sketch a graph of $f(x) = -2\sin\left(\frac{\pi x}{2}\right)$.

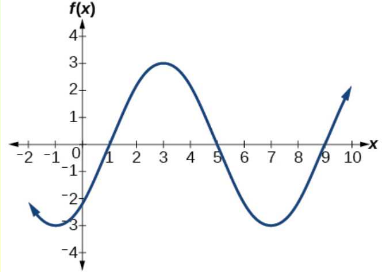
  Find the amplitude, period, phase shift and vertical shift:
  - Amplitude: $|A| = 2$.
  - Period: $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 4$.
  - Phase Shift: $\frac{C}{B} = 0$.
  - Vertical Shift: $D = 0$.



Graphing a Transformed Sinusoid

- Sketch a graph of $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.

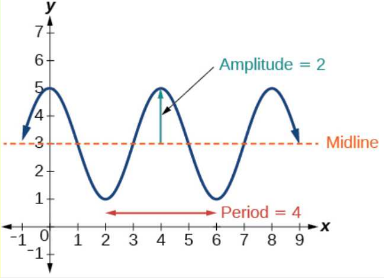
  Find the amplitude, period, phase shift and vertical shift:
  - Amplitude: $|A| = 3$.
  - Period: $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{4}} = 8$.
  - Phase Shift: $\frac{C}{B} = \frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1$.
  - Vertical Shift: $D = 0$.



Identifying the Properties of a Sinusoidal Function

- Given $y = -2\cos\left(\frac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift and vertical shift. Then graph the function.

  We find the features:
  - Amplitude: $|A| = 2$.
  - Period: $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 4$.
  - Phase Shift: $\frac{C}{B} = \frac{-\pi}{\frac{\pi}{2}} = -2$.
  - Vertical Shift: $D = 3$.



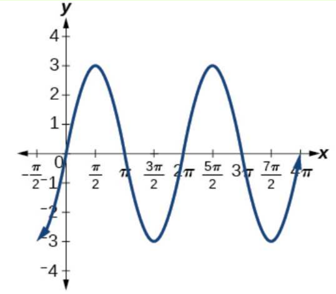
Finding the Vertical Component of Circular Motion

- A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y-coordinate of the point as a function of the angle of rotation.

  The y-coordinate for the unit circle is $y = \sin x$.

  Since the circle has radius 3, we get

$$f(x) = 3\sin x.$$



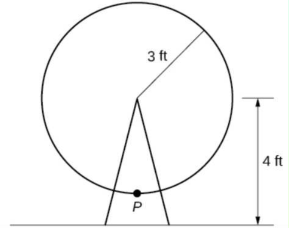
Finding the Vertical Component of Circular Motion

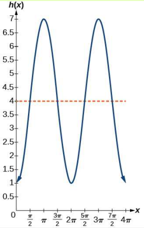
- A circle with radius 3 ft is mounted with its center 4 ft off the ground.

  The point closest to the ground is labeled P, as shown in the figure.

  Sketch a graph of the height above the ground of P as the circle is rotated.

  Then find a function that gives the height in terms of the angle of rotation.





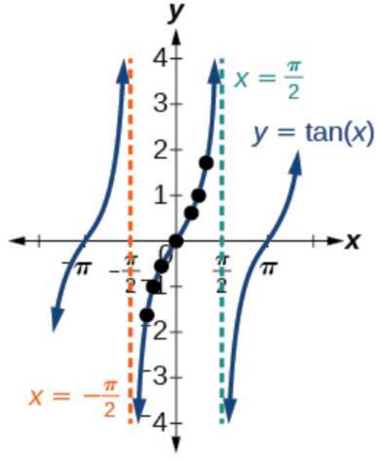
- Amplitude: $|A| = 3$.
- Period: $P = 2\pi$. So $B = 1$.
- Phase Shift: $\frac{C}{B} = \pi$. So $C = \pi$.
- Vertical Shift: $D = 1$.

  $f(x) = 3\cos(x - \pi) + 1$.

Subsection 2

Graphs of the Other Trigonometric Functions

The Graph of Tangent

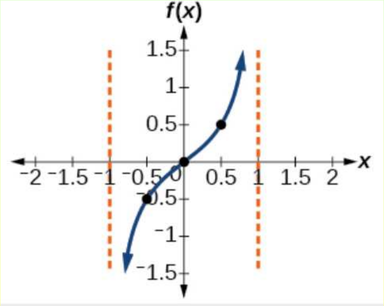


Sketching a Compressed Tangent

- Sketch a graph of one period of the function $y = 0.5\tan\left(\frac{\pi}{2}x\right)$.

  The given function is obtained by $y = \tan x$ by:
  - A horizontal compression by a factor of $\frac{2}{\pi}$;
  - A vertical compression by a factor of 0.5.

  So its graph is the one shown below.

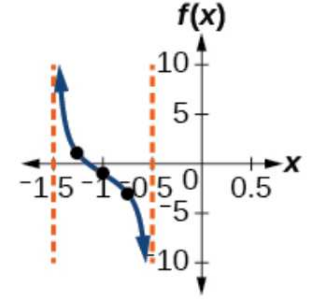


Graphing One Period of a Shifted Tangent Function

- Given the function $y = A\tan(Bx - C) + D$:
  - Identify the vertical stretching/compressing factor, $|A|$.
  - Identify B and determine the period, $P = \frac{\pi}{|B|}$.
  - Identify C and determine the phase shift, $\frac{C}{B}$.
  - Draw the graph of
    $$y = A\tan(Bx)$$
    shifted to the right by $\frac{C}{B}$ and up by D.

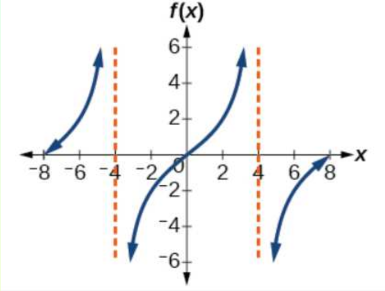
Graphing One Period of a Shifted Tangent Function

- Graph one period of the function $y = -2\tan(\pi x + \pi) - 1$.
  We have:
  - Vertical stretch by a factor of $|A| = 2$.
  - Period $P = \frac{\pi}{|B|} = \frac{\pi}{\pi} = 1$.
  - Phase Shift $\frac{C}{B} = \frac{-\pi}{\pi} = -1$.
  - Vertical shift $D = -1$.



Identifying the Graph of a Stretched Tangent

- Find a formula for the function graphed in

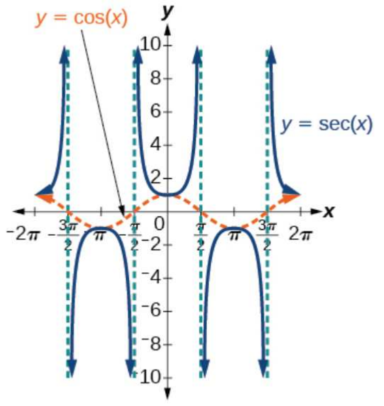


We want to identify A, B, C and D in $y = A\tan(Bx - C) + D$:

- Vertical stretch: $A = 2$.
- Period: $\frac{\pi}{|B|} = 8$. So $B = \frac{\pi}{8}$.
- Phase shift: $\frac{C}{B} = 0$. So $C = 0$.
- Vertical shift: $D = 0$.

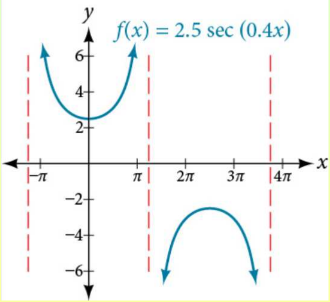
Thus, $y = 2\tan\left(\frac{\pi}{8}x\right)$.

The Graph of Secant



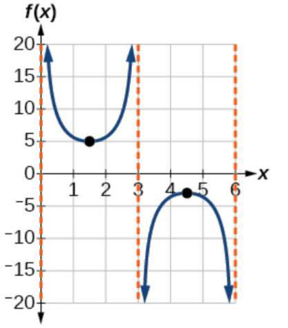
Graphing a Variation of the Secant Function

- Graph one period of $f(x) = 2.5\sec(0.4x)$.
  We have:
  - Vertical stretch by a factor of $|A| = 2.5$.
  - Period $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{2}{5}} = 5\pi$.
  - Phase Shift $\frac{C}{B} = 0$;
  - Vertical shift $D = 0$.



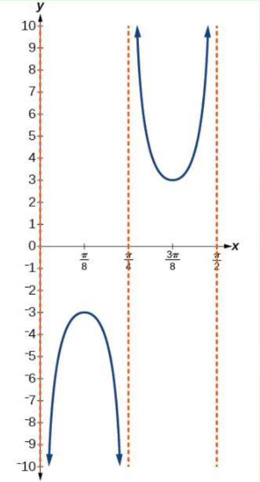
Graphing a Variation of the Secant Function

- Graph one period of $y = 4\sec\left(\frac{\pi}{3}x - \frac{\pi}{2}\right) + 1$.
  We have:
  - Vertical stretch by a factor of $|A| = 4$.
  - Period $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{3}} = 6$.
  - Phase Shift $\frac{C}{B} = \frac{\frac{\pi}{2}}{\frac{\pi}{3}} = \frac{3}{2}$.
  - Vertical shift $D = 1$.



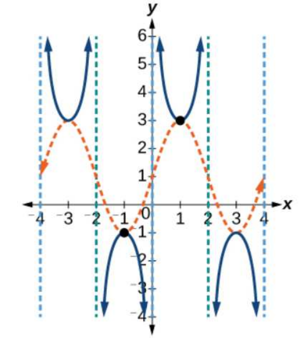
Graphing a Variation of the Cosecant Function

- Graph one period of $f(x) = -3\csc(4x)$.
  We have:
  - Vertical stretch by a factor of $|A| = 3$.
  - Period $P = \frac{2\pi}{|B|} = \frac{2\pi}{4} = \frac{\pi}{2}$.
  - Phase Shift $\frac{C}{B} = 0$.
  - Vertical shift $D = 0$.

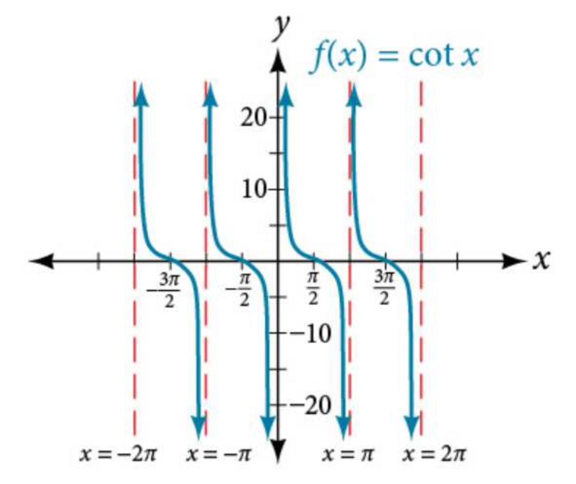


Graphing a Transformed Cosecant

- Sketch a graph of $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$.
  We have:
  - Vertical stretch by a factor of $|A| = 2$.
  - Period $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 4$.
  - Phase Shift $\frac{C}{B} = 0$.
  - Vertical shift $D = 1$.



The Graph of Cotangent

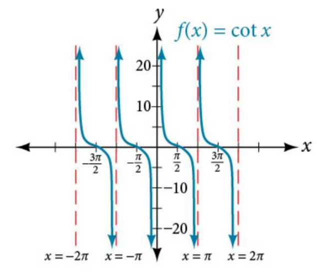


Graphing Variations of the Cotangent Function

- Determine the stretching factor, period, and phase shift of $y = 3\cot(4x)$, and then sketch a graph.

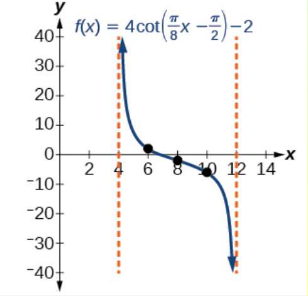
  We have:

  - Vertical stretch by a factor of $|A| = 3$.
  - Period $P = \frac{\pi}{|B|} = \frac{\pi}{4}$.
  - Phase Shift $\frac{C}{B} = 0$.
  - Vertical shift $D = 0$.



Graphing a Modified Cotangent

- Sketch a graph of one period of the function $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.
  We have:
  - Vertical stretch by a factor of $|A| = 4$.
  - Period $P = \frac{\pi}{|B|} = \frac{\pi}{\frac{\pi}{8}} = 8$.
  - Phase Shift $\frac{C}{B} = \frac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4$.
  - Vertical shift $D = -2$.



Subsection 3

Inverse Trigonometric Functions

Things to Recall about Inverse Functions

- Only one-to-one functions (that is, functions passing the horizontal line test) have inverses.
- If f is a 1-1 function, f^{-1} denotes its inverse function.
- The key relation between inverse functions is that

$$y = f(x) \quad \text{if and only if} \quad x = f^{-1}(y).$$

  That is the roles of input and output values are interchanged.
- Consequently domains and ranges are also interchanged.
- For x in the domain of f and y in the domain of f^{-1} we have:

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y.$$

- The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric with respect to $y = x$.

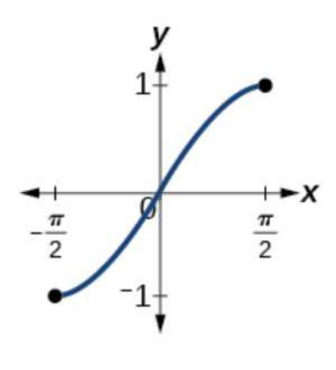
Writing a Relation for an Inverse Function

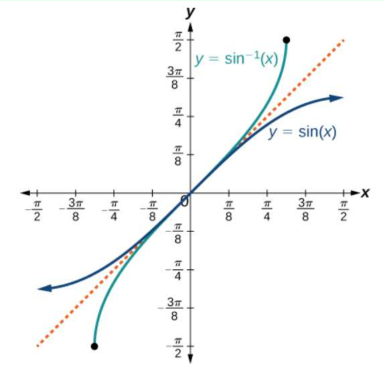
- Given $\sin\left(\frac{5\pi}{12}\right) \approx 0.96593$, write a relation involving the inverse sine.

  Since roles of input and output are switched:

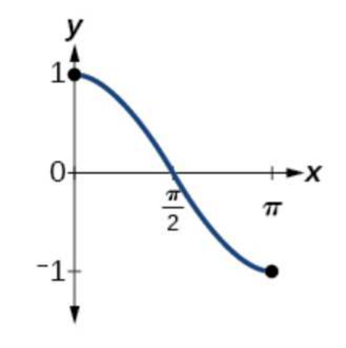
$$\sin^{-1}(0.96593) \approx \frac{5\pi}{12}.$$

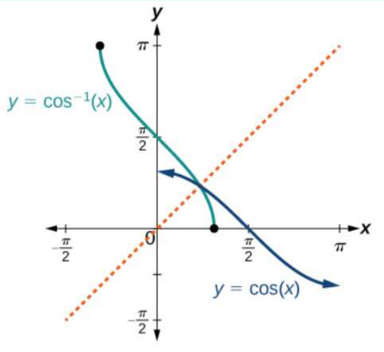
Inverse Sine



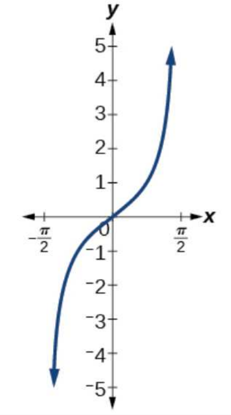


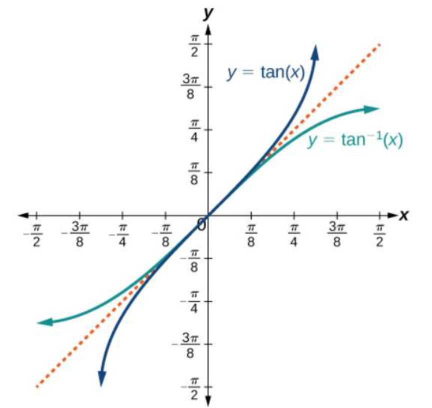
Inverse Cosine





Inverse Tangent





Evaluating Inverse Trigonometric Functions

- Evaluate each of the following.
  - $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ (since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$).
  - $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ (since $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$).
  - $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$ (since $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$).
  - $\tan^{-1}(1) = \frac{\pi}{4}$ (since $\tan\left(\frac{\pi}{4}\right) = 1$).

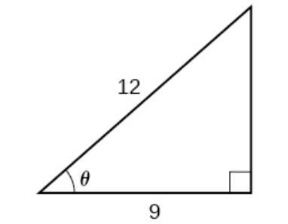
Evaluating the Inverse Sine on a Calculator

- Evaluate $\sin^{-1}(0.97)$ using a calculator.

  I included this to emphasize that we should check the **MODE** so that we calculate in degrees or radians as intended.

Applying the Inverse Cosine to a Right Triangle

- Solve the triangle in the figure for the angle θ.



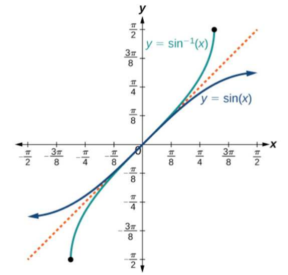
We have

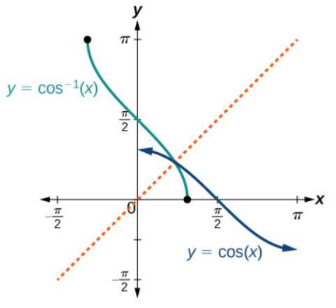
$$\cos\theta = \frac{9}{12} = \frac{3}{4}.$$

Therefore,

$$\theta = \cos^{-1}\left(\frac{3}{4}\right) \approx 41.41^\circ.$$

Using Inverse Trigonometric Functions





- Evaluate the following:
  - (a) $\sin^{-1}(\sin(\frac{\pi}{3})) = \sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$.
  - (b) $\sin^{-1}(\sin(\frac{2\pi}{3})) = \sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$.
  - (c) $\cos^{-1}(\cos(\frac{2\pi}{3})) = \cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$.
  - (d) $\cos^{-1}(\cos(-\frac{\pi}{3})) = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$.

Composition of an Inverse Sine with a Cosine

- Evaluate $\sin^{-1}(\cos(\frac{13\pi}{6}))$.

  We have
  $$\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}.$$

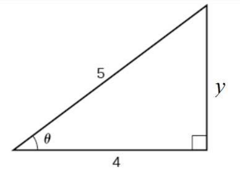
Composition of a Sine with an Inverse Cosine

- Find an exact value for $\sin(\cos^{-1}(\frac{4}{5}))$.

  Set $\theta = \cos^{-1}(\frac{4}{5})$.

  This yields $\cos\theta = \frac{4}{5}$.

  So we get:

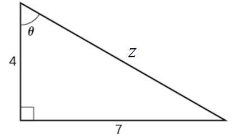


  Use the Pythagorean Theorem to find $y = \sqrt{5^2 - 4^2} = \sqrt{9} = 3$.

  Now we get $\sin(\cos^{-1}(\frac{4}{5})) = \sin\theta = \frac{3}{5}$.

Composition of a Sine with an Inverse Tangent

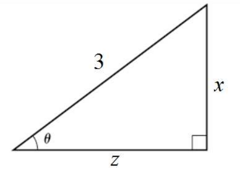
- Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.
  Set $\theta = \tan^{-1}\left(\frac{7}{4}\right)$.
  This yields $\tan\theta = \frac{7}{4}$.
  So we get:



  Use the Pythagorean Theorem to find $z = \sqrt{7^2 + 4^2} = \sqrt{65}$.
  Now we get $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right) = \sin\theta = \frac{7}{\sqrt{65}}$.

Cosine of the Inverse Sine of an Expression

- Find a simplified expression for $\cos(\sin^{-1}(\frac{x}{3}))$ for $-3 \leq x \leq 3$.
  Set $\theta = \sin^{-1}(\frac{x}{3})$.
  This yields $\sin\theta = \frac{x}{3}$.
  So we get:



  Use the Pythagorean Theorem to find $z = \sqrt{3^2 - x^2} = \sqrt{9 - x^2}$.
  Now we get $\cos(\sin^{-1}(\frac{x}{3})) = \cos\theta = \frac{\sqrt{9-x^2}}{3}$.