College Trigonometry

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 131

Subsection 1

Verifying and Using Trigonometric Identities

Trigonometric Identities

- Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1, \quad 1 + \cot^2\theta = \csc^2\theta, \quad 1 + \tan^2\theta = \sec^2\theta.$$

- Even-Odd Identities

$$\begin{array}{ccc} \sin(-\theta) = -\sin\theta & \cos(-\theta) = \cos\theta & \tan(-\theta) = -\tan\theta \\ \csc(-\theta) = -\csc\theta & \sec(-\theta) = \sec\theta & \cot(-\theta) = -\cot\theta \end{array}$$

- Reciprocal Identities

$$\begin{array}{ccc} \sin\theta = \frac{1}{\csc\theta} & \cos\theta = \frac{1}{\sec\theta} & \tan\theta = \frac{1}{\cot\theta} \\ \csc\theta = \frac{1}{\sin\theta} & \sec\theta = \frac{1}{\cos\theta} & \cot\theta = \frac{1}{\tan\theta} \end{array}$$

- Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

Verifying a Trigonometric Identity

- Verify $\tan\theta\cos\theta = \sin\theta$.

  We have

$$\begin{aligned} \tan\theta\cos\theta &= \frac{\sin\theta}{\cos\theta}\cos\theta \\ &= \sin\theta. \end{aligned}$$

Verifying the Equivalency Using the Even-Odd Identities

- Verify the following equivalency using the even-odd identities: $(1+\sin x)[1+\sin(-x)] = \cos^2 x$.

  We have

$$\begin{aligned}(1+\sin x)[1+\sin(-x)] &= (1+\sin x)(1-\sin x)\\ &= 1-\sin^2 x\\ &= \cos^2 x.\end{aligned}$$

Verifying a Trigonometric Identity Involving $\sec^2\theta$

- Verify the identity $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$.

  We have

$$\begin{aligned}\frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{\tan^2\theta}{\sec^2\theta} \\ &= \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}} \\ &= \sin^2\theta.\end{aligned}$$

Creating and Verifying an Identity

- Create an identity for the expression $2\tan\theta\sec\theta$ by rewriting strictly in terms of sine.

  We have

$$\begin{aligned} 2\tan\theta\sec\theta &= 2\frac{\sin\theta}{\cos\theta}\frac{1}{\cos\theta} \\ &= \frac{2\sin\theta}{\cos^2\theta} \\ &= \frac{2\sin\theta}{1-\sin^2\theta}. \end{aligned}$$

Verifying an Identity Using Even/Odd Identities

- Verify the identity: $\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} = \cos\theta - \sin\theta.$

  We get

$$\begin{aligned}
\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} &= \frac{\sin^2\theta-\cos^2\theta}{-\sin\theta-\cos\theta} \\
&= \frac{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}{-(\sin\theta+\cos\theta)} \\
&= \frac{\sin\theta-\cos\theta}{-1} \\
&= \cos\theta-\sin\theta.
\end{aligned}$$

Verifying an Identity Involving Cosines and Cotangents

- Verify the identity: $(1-\cos^2 x)(1+\cot^2 x)=1$.

  We have

$$\begin{aligned}(1-\cos^2 x)(1+\cot^2 x) &= \sin^2 x \csc^2 x \\ &= \sin^2 x \cdot \frac{1}{\sin^2 x} \\ &= 1.\end{aligned}$$

Rewriting Using Difference of Squares

- Rewrite the trigonometric expression using the difference of squares: $4\cos^2\theta - 1$.

  We have

$$\begin{aligned} 4\cos^2\theta - 1 &= (2\cos\theta)^2 - 1^2 \\ &= (2\cos\theta + 1)(2\cos\theta - 1). \end{aligned}$$

Simplify by Rewriting and Using Substitution

- Simplify the expression by rewriting and using identities: $\csc^2\theta - \cot^2\theta$.

  We get

$$\begin{aligned} \csc^2\theta - \cot^2\theta &= 1 + \cot^2\theta - \cot^2\theta \\ &= 1. \end{aligned}$$

Subsection 2

Sum and Difference Identities

Sum and Difference Identities

- Sine of Sum/Difference:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha.$$

- Cosine of Sum/Difference:

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

- Tangent of Sum/Difference:

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}.$$

Finding Exact Values Using Cosine of Difference

- Using the formula for the cosine of the difference of two angles, find the exact value of $\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right)$.

  We have

$$\begin{aligned}\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right) &= \cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\ &= -\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \left(-\frac{\sqrt{2}}{2}\right)\cdot\frac{1}{2} \\ &= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\ &= \frac{-\sqrt{6}-\sqrt{2}}{4}.\end{aligned}$$

Finding Exact Values Using the Cosine of Sum

- Find the exact value of $\cos(75^\circ)$.

  We split 75° as a sum:

$$\begin{aligned}
\cos(75^\circ) &= cos(45^\circ + 30^\circ) \\
&= \cos(45^\circ)\cos(30^\circ) - \sin(45^\circ)\sin(30^\circ) \\
&= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\
&= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}.
\end{aligned}$$

Using Sum/Difference to Evaluate a Difference

- Use the sum and difference identities to evaluate the difference of the angles and show that $\sin(45^\circ - 30^\circ)$ equals $\sin(135^\circ - 120^\circ)$.
  This is asking us to compute both sines using the difference formulas and show that they are equal.

$$\begin{aligned}
\sin(45^\circ - 30^\circ) &= \sin(45^\circ)\cos(30^\circ) - \sin(30^\circ)\cos(45^\circ) \\
&= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}. \\
\sin(135^\circ - 120^\circ) &= \sin(135^\circ)\cos(120^\circ) - \sin(120^\circ)\cos(135^\circ) \\
&= \sin(45^\circ)[-\cos(60^\circ)] - \sin(60^\circ)[-\cos(45^\circ)] \\
&= \frac{\sqrt{2}}{2} \cdot \left(-\frac{1}{2}\right) - \frac{\sqrt{3}}{2} \cdot \left(-\frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}.
\end{aligned}$$

Exact Value of an Expression Involving Inverse Function

- Find the exact value of $\sin\left(\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{3}{5}\right)$.

  We get

$$\begin{aligned}\sin\left(\cos^{-1}\tfrac{1}{2}+\sin^{-1}\tfrac{3}{5}\right) &= \sin\left(\cos^{-1}\tfrac{1}{2}\right)\cos\left(\sin^{-1}\tfrac{3}{5}\right) \\ &\quad +\sin\left(\sin^{-1}\tfrac{3}{5}\right)\cos\left(\cos^{-1}\tfrac{1}{2}\right) \\ &= \tfrac{\sqrt{3}}{2}\cdot\tfrac{4}{5}+\tfrac{3}{5}\cdot\tfrac{1}{2} \\ &= \tfrac{4\sqrt{3}+3}{10}.\end{aligned}$$

Finding the Exact Value of an Expression Involving Tangent

- Find the exact value of $\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$.

  We get

$$\begin{aligned}\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right) &= \frac{\tan\left(\frac{\pi}{6}\right)+\tan\left(\frac{\pi}{4}\right)}{1-\tan\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{4}\right)} \\ &= \frac{\frac{\sqrt{3}}{3}+1}{1-\frac{\sqrt{3}}{3}\cdot 1} \\ &= \frac{\sqrt{3}+3}{3-\sqrt{3}}.\end{aligned}$$

Finding Multiple Sums and Differences of Angles

- Given $\sin\alpha = \frac{3}{5}$, $0 < \alpha < \frac{\pi}{2}$, $\cos\beta = -\frac{5}{13}$, $\pi < \beta < \frac{3\pi}{2}$, find $\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$, $\tan(\alpha+\beta)$, $\tan(\alpha-\beta)$.

  First compute $\cos\alpha$ and $\tan\alpha$ and $\sin\beta$ and $\tan\beta$.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5};$$

$$\tan\alpha = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4};$$

$$\sin\beta = -\sqrt{1-\cos^2\beta} = -\sqrt{1-\frac{25}{169}} = -\sqrt{\frac{144}{169}} = -\frac{12}{13};$$

$$\tan\beta = \frac{-\frac{12}{13}}{-\frac{5}{13}} = \frac{12}{5}.$$

Finding Multiple Sums and Differences of Angles

- We had been given $\sin\alpha = \frac{3}{5}$, $\cos\beta = -\frac{5}{13}$, and we computed $\cos\alpha = \frac{4}{5}$, $\tan\alpha = \frac{3}{4}$, $\sin\beta = -\frac{12}{13}$ and $\tan\beta = \frac{12}{5}$.

  Now we are ready to calculate the numbers requested:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha = \frac{3}{5}\cdot\left(-\frac{5}{13}\right) + \left(-\frac{12}{13}\right)\cdot\frac{4}{5} = -\frac{15}{65} - \frac{48}{65} = -\frac{63}{65};$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{4}{5}\cdot\left(-\frac{5}{13}\right) - \frac{3}{5}\cdot\left(-\frac{12}{13}\right) = -\frac{20}{65} + \frac{36}{65} = \frac{16}{65};$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{\frac{3}{4}+\frac{12}{5}}{1-\frac{3}{4}\cdot\frac{12}{5}} = \frac{\frac{63}{20}}{-\frac{16}{20}} = -\frac{63}{16};$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} = \frac{\frac{3}{4}-\frac{12}{5}}{1+\frac{3}{4}\cdot\frac{12}{5}} = \frac{-\frac{33}{20}}{\frac{56}{20}} = -\frac{33}{56}.$$

Finding a Cofunction with the Same Value

- Write $\tan\frac{\pi}{9}$ in terms of its cofunction.

  We have

$$\tan\frac{\pi}{9} = \cot\left(\frac{\pi}{2} - \frac{\pi}{9}\right) = \cot\left(\frac{9\pi}{18} - \frac{2\pi}{18}\right) = \cot\left(\frac{7\pi}{18}\right).$$

Verifying an Identity Involving Sine

- Verify the identity $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$.

  We have

$$\begin{aligned}&\sin(\alpha+\beta)+\sin(\alpha-\beta)\\&= \sin\alpha\cos\beta+\sin\beta\cos\alpha+\sin\alpha\cos\beta-\sin\beta\cos\alpha\\&=2\sin\alpha\cos\beta.\end{aligned}$$

Verifying an Identity Involving Tangent

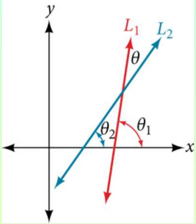
- Verify the following identity. $\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} = \tan\alpha - \tan\beta.$

  We have

$$\begin{aligned}
&\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} \\
&= \frac{\sin\alpha\cos\beta - \sin\beta\cos\alpha}{\cos\alpha\cos\beta} \\
&= \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta} \\
&= \frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta} \\
&= \tan\alpha - \tan\beta.
\end{aligned}$$

Using Sum and Difference Formulas in Applications

- Let L_1 and L_2 denote two non-vertical intersecting lines, and let θ be the acute angle between L_1, L_2. Show that $\tan\theta = \frac{m_2-m_1}{1+m_1m_2}$, where m_1 and m_2 are the slopes of L_1 and L_2 respectively.



By the triangle, $\theta_1 = \theta + \theta_2$. So $\theta = \theta_1 - \theta_2$.

Now we get

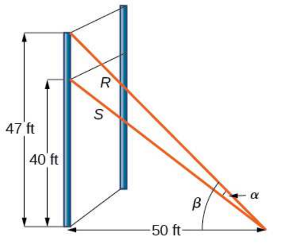
$$\tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

Investigating a Guy-wire Problem

- For a climbing wall, a guy-wire R is attached 47 feet high on a vertical pole.

  A second guy-wire S attached 40 feet above ground on the same pole.

  If the wires are attached to the ground 50 feet from the pole, find the angle α between the wires.



  We get

$$\begin{aligned} \tan\alpha &= \tan(\beta-\gamma) = \frac{\tan\beta - \tan\gamma}{1+\tan\beta\tan\gamma} \\ &= \frac{\frac{47}{50}-\frac{40}{50}}{1+\frac{47}{50}\cdot\frac{40}{50}} \approx 0.08. \end{aligned}$$

  We conclude $\alpha = \tan^{-1}(0.08) \approx 4.57^\circ$.

Subsection 3

Double-Angle, Half-Angle and Reduction Formulas

Double-Angle Formulas

- Double-Angle Formulas:

$$\begin{aligned} \sin(2\theta) &= 2\sin\theta\cos\theta, \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1, \\ \tan(2\theta) &= \tfrac{2\tan\theta}{1-\tan^2\theta}. \end{aligned}$$

- Reduction Formulas:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}, \quad \cos^2\theta = \frac{1+\cos(2\theta)}{2}, \quad \tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}.$$

- Half-Angle Formulas:

$$\begin{aligned} \sin\left(\tfrac{\alpha}{2}\right) &= \pm\sqrt{\tfrac{1-\cos\alpha}{2}}, \\ \cos\left(\tfrac{\alpha}{2}\right) &= \pm\sqrt{\tfrac{1+\cos\alpha}{2}}, \\ \tan\left(\tfrac{\alpha}{2}\right) &= \pm\sqrt{\tfrac{1-\cos\alpha}{1+\cos\alpha}} = \tfrac{\sin\alpha}{1+\cos\alpha} = \tfrac{1-\cos\alpha}{\sin\alpha}. \end{aligned}$$

Using a Double-Angle Formula to Find Exact Values

- Given that $\tan\theta = -\frac{3}{4}$ and θ is in quadrant II, find $\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$.

  Use the triangle method to determine $\sin\theta = \frac{3}{5}$ and $\cos\theta = -\frac{4}{5}$.

  Now we apply the Double-Angle Formulas:

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\frac{3}{5}\cdot\left(-\frac{4}{5}\right) = -\frac{24}{25},$$

$$\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(-\frac{4}{5}\right)^2 - 1 = \frac{7}{25},$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\cdot\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2} = \frac{-\frac{3}{2}}{\frac{7}{16}} = -\frac{24}{7}.$$

Using the Double-Angle Formula for Cosine

- Use the double-angle formula for cosine to write $\cos(6x)$ in terms of $\cos(3x)$.

  We have
  $$\begin{aligned} \cos(6x) &= \cos(2 \cdot (3x)) \\ &= 2\cos^2(3x) - 1. \end{aligned}$$

Using the Double-Angle Formulas to Verify an Identity

- Verify the following identity using double-angle formulas: $1+\sin(2\theta)=(\sin\theta+\cos\theta)^2$.

  We get

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\ &= 1+\sin(2\theta).\end{aligned}$$

Verifying a Double-Angle Identity for Tangent

- Verify the identity: $\tan(2\theta) = \frac{2}{\cot\theta - \tan\theta}$.

  We get

$$\begin{aligned}\frac{2}{\cot\theta - \tan\theta} &= \frac{2}{\frac{1}{\tan\theta} - \tan\theta} \\ &= \frac{2\tan\theta}{\tan\theta(\frac{1}{\tan\theta} - \tan\theta)} \\ &= \frac{2\tan\theta}{1 - \tan^2\theta} \\ &= \tan(2\theta).\end{aligned}$$

Equivalent Expression Not Containing Powers ≥ 1

- Write an equivalent expression for $\cos^4 x$ that does not involve any powers of sine or cosine greater than 1.

  We get

$$\begin{aligned}
\cos^4 x &= (\cos^2 x)^2 = (\tfrac{1+\cos 2x}{2})^2 \\
&= \tfrac{1+2\cos(2x)+\cos^2(2x)}{4} \\
&= \tfrac{1}{4} + \tfrac{1}{2}\cos(2x) + \tfrac{1}{4}\cos^2(2x) \\
&= \tfrac{1}{4} + \tfrac{1}{2}\cos(2x) + \tfrac{1}{4}\tfrac{1+\cos(4x)}{2} \\
&= \tfrac{1}{4} + \tfrac{1}{2}\cos(2x) + \tfrac{1}{8} + \tfrac{1}{8}\cos(4x) \\
&= \tfrac{3}{8} + \tfrac{1}{2}\cos(2x) + \tfrac{1}{8}\cos(4x).
\end{aligned}$$

Using the Power-Reducing Formulas to Prove an Identity

- Use the power-reducing formulas to prove $\sin^3(2x) = [\frac{1}{2}\sin(2x)][1-\cos(4x)]$.

  We have

$$\begin{aligned}\sin^3(2x) &= \sin(2x)\sin^2(2x) \\ &= \sin(2x)\frac{1-\cos(4x)}{2} \\ &= \frac{1}{2}\sin(2x)[1-\cos(4x)].\end{aligned}$$

Using a Half-Angle Formula to Find an Exact Value

- Find $\sin(15^\circ)$ using a half-angle formula.

  We use the half-angle formula:

$$\begin{aligned}\sin(15^\circ) &= \sin\left(\tfrac{30^\circ}{2}\right) = \sqrt{\tfrac{1-\cos 30^\circ}{2}} \\ &= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} \\ &= \frac{\sqrt{2-\sqrt{3}}}{2}.\end{aligned}$$

Finding Exact Values Using Half-Angle Identities

- Given that $\tan\alpha = \frac{8}{15}$ and α lies in quadrant III, find the exact values of $\sin\left(\frac{\alpha}{2}\right)$, $\cos\left(\frac{\alpha}{2}\right)$, $\tan\left(\frac{\alpha}{2}\right)$.

  We first use the triangle method to get $\cos\alpha = -\frac{15}{17}$.

  Now we get

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} = \pm\sqrt{\frac{1-(-\frac{15}{17})}{2}} = \pm\sqrt{\frac{16}{17}} = \frac{4}{\sqrt{17}},$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} = \pm\sqrt{\frac{1+(-\frac{15}{17})}{2}} = \pm\sqrt{\frac{1}{17}} = -\frac{1}{\sqrt{17}},$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-cos\alpha}{1+\cos\alpha}} = \pm\sqrt{\frac{1-(-\frac{15}{17})}{1+(-\frac{15}{17})}} = \pm\sqrt{\frac{32}{2}} = -4.$$

Finding the Measurement of a Half Angle

- A bicycle ramp is constructed for high-level competition with an angle of θ formed by the ramp and the ground.

  Another ramp is to be constructed half as steep for novices.

  If $\tan\theta = \frac{5}{3}$ for higher-level competition, what is the measurement of the angle for novice competition?

  Use the triangle method to find $\cos\theta = \frac{3}{\sqrt{34}}$.

  Then we get

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\frac{3}{\sqrt{34}}}{1+\frac{3}{\sqrt{34}}}} = \sqrt{\frac{\sqrt{34}-3}{\sqrt{34}+3}} \approx 0.57.$$

  Therefore, $\frac{\theta}{2} = \tan^{-1}(0.57) \approx 29.7^\circ$.

Subsection 4

Sum-to-Product and Product-to-Sum Formulas

Product-to-Sum and Sum-to-Product Formulas

- Product-to-Sum Formulas:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)],$$
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)],$$
$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)].$$

- Sum-to-Product Formulas:

$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),$$
$$\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right),$$
$$\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right),$$
$$\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).$$

Writing a Product as a Sum

- Write the following product of cosines as a sum: $2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right)$.
  We have

$$\begin{aligned}2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right) &= \cos\left(\frac{7x}{2}-\frac{3x}{2}\right)+\cos\left(\frac{7x}{2}+\frac{3x}{2}\right)\\ &= \cos\left(\frac{4x}{2}\right)+\cos\left(\frac{10x}{2}\right)\\ &= \cos(2x)+\cos(5x).\end{aligned}$$

Writing the Product as a Sum with only Sine or Cosine

- Express the following product as a sum containing only sine or cosine and no products: $\sin(4\theta)\cos(2\theta)$.

  We have

$$\begin{aligned}\sin(4\theta)\cos(2\theta) &= \frac{1}{2}[\sin(4\theta+2\theta)+\sin(4\theta-2\theta)] \\ &= \frac{1}{2}[\sin(6\theta)+\sin(2\theta)].\end{aligned}$$

Express the Product as a Sum or Difference

- Write $\cos(3\theta)\cos(5\theta)$ as a sum or difference.

  We have

$$\begin{aligned}\cos(3\theta)\cos(5\theta) &= \frac{1}{2}[\cos(3\theta-5\theta)+\cos(3\theta+5\theta)] \\ &= \frac{1}{2}[\cos(-2\theta)+\cos(8\theta)] \\ &= \frac{1}{2}[\cos(2\theta)+\cos(8\theta)].\end{aligned}$$

Writing the Difference of Sines as a Product

- Write the following difference of sines expression as a product: $\sin(4\theta) - \sin(2\theta)$.

  We have

$$\begin{aligned} \sin(4\theta) - \sin(2\theta) &= 2\sin\left(\frac{4\theta-2\theta}{2}\right)\cos\left(\frac{4\theta+2\theta}{2}\right) \\ &= 2\sin\left(\frac{2\theta}{2}\right)\cos\left(\frac{6\theta}{2}\right) \\ &= 2\sin(\theta)\cos(3\theta). \end{aligned}$$

Evaluating Using the Sum-to-Product Formula

- Evaluate $\cos(15^\circ) - \cos(75^\circ)$.

  We have

$$\begin{aligned} \cos(15^\circ) - \cos(75^\circ) &= -2\sin\left(\frac{15^\circ+75^\circ}{2}\right)\sin\left(\frac{15^\circ-75^\circ}{2}\right) \\ &= -2\sin(45^\circ)\sin(-30^\circ) \\ &= -2\frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right) \\ &= -\frac{\sqrt{2}}{2}. \end{aligned}$$

Proving an Identity

- Prove the identity: $\dfrac{\cos(4t) - \cos(2t)}{\sin(4t) + \sin(2t)} = -\tan t.$

  We have

$$\begin{aligned}\frac{\cos(4t) - \cos(2t)}{\sin(4t) + \sin(2t)} &= \frac{-2\sin\left(\frac{4t+2t}{2}\right)\sin\left(\frac{4t-2t}{2}\right)}{2\sin\left(\frac{4t+2t}{2}\right)\cos\left(\frac{4t-2t}{2}\right)} \\ &= \frac{-2\sin(3t)\sin(t)}{2\sin(3t)\cos(t)} \\ &= -\tan t.\end{aligned}$$

Using Double-Angle and Reciprocal Identities

- Verify the identity $\csc^2\theta - 2 = \dfrac{\cos(2\theta)}{\sin^2\theta}$.

  We have

$$\begin{aligned}\frac{\cos(2\theta)}{\sin^2\theta} &= \frac{1-2\sin^2\theta}{\sin^2\theta} \\ &= \frac{1}{\sin^2\theta} - \frac{2\sin^2\theta}{\sin^2\theta} \\ &= \csc^2\theta - 2.\end{aligned}$$

Subsection 5

Solving Trigonometric Equations

Solving a Linear Equation Involving Cosine

- Find all possible exact solutions for the equation $\cos\theta = \frac{1}{2}$.

  First, find the angles θ in $[0, 2\pi)$ such that $\cos\theta = \frac{1}{2}$.

  These are $\theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$.

  Then, the solutions are all coterminal angles:

$$\theta = \frac{\pi}{3} + 2\pi k, \ \theta = \frac{5\pi}{3} + 2\pi k, \quad k \text{ any integer.}$$

Solving a Linear Equation Involving the Sine Function

- Find all possible exact solutions for the equation $\sin t = \frac{1}{2}$.
  First, find the angles t in $[0, 2\pi)$ such that $\sin t = \frac{1}{2}$.
  These are $t = \frac{\pi}{6}$ or $t = \frac{5\pi}{6}$.
  Then, the solutions are all coterminal angles:

$$t = \frac{\pi}{6} + 2\pi k, \;\; t = \frac{5\pi}{6} + 2\pi k, \quad k \text{ any integer.}$$

Solve the Linear Trigonometric Equation

- Solve the equation exactly: $2\cos\theta - 3 = -5$, $0 \le \theta < 2\pi$.

  We have

$$2\cos\theta - 3 = -5$$

$$2\cos\theta = -2$$

$$\cos\theta = -1$$

$$\theta = \pi.$$

Solving a Problem Involving a Single Trig Function

- Solve the problem exactly: $2\sin^2\theta - 1 = 0$, $0 \leq \theta < 2\pi$.

  We have

$$2\sin^2\theta - 1 = 0$$

$$2\sin^2\theta = 1$$

$$\sin^2\theta = \frac{1}{2}$$

$$\sin\theta = -\frac{\sqrt{2}}{2} \text{ or } \sin\theta = \frac{\sqrt{2}}{2}$$

$$\theta = \frac{5\pi}{4} \text{ or } \theta = \frac{7\pi}{4} \quad \text{or} \quad \theta = \frac{\pi}{4} \text{ or } \theta = \frac{3\pi}{4}.$$

Solving a Trigonometric Equation Involving Cosecant

- Solve the following equation exactly: $\csc\theta = -2$, $0 \leq \theta < 4\pi$.

  We have

$$\csc\theta = -2$$

$$\frac{1}{\sin\theta} = -2$$

$$\sin\theta = -\frac{1}{2}$$

$$\theta = \frac{7\pi}{6} \text{ or } \theta = \frac{11\pi}{6} \text{ or } \theta = \frac{19\pi}{6} \text{ or } \theta = \frac{23\pi}{6}.$$

Solving an Equation Involving Tangent

- Solve the equation exactly: $\tan(\theta - \frac{\pi}{2}) = 1$, $0 \le \theta < 2\pi$.

  We have

$$\begin{aligned} \tan(\theta - \tfrac{\pi}{2}) &= 1 \\ -\tan(\tfrac{\pi}{2} - \theta) &= 1 \\ -\cot\theta &= 1 \\ \cot\theta &= -1 \\ \cos\theta &= -\sin\theta \\ \theta = \tfrac{3\pi}{4} &\text{ or } \theta = \tfrac{7\pi}{4}. \end{aligned}$$

Identify all Solutions to the Equation Involving Tangent

- Identify all exact solutions to the equation $2(\tan x+3)=5+\tan x$, $0\leq x<2\pi$.

  We have

$$2(\tan x+3)=5+\tan x$$

$$2\tan x+6=5+\tan x$$

$$\tan x=-1$$

$$\sin x=-\cos x$$

$$x=\frac{3\pi}{4} \text{ or } x=\frac{7\pi}{4}.$$

Using a Calculator to Solve a Sine Equation

- Use a calculator to solve the equation $\sin\theta = 0.8$, where θ is in radians.

  Using a calculator, we get

  $$\sin^{-1}(0.8) \approx 0.9273 \text{ radians.}$$

  However, there is the additional solution

  $$\pi - 0.9273 = 2.2143.$$

  Therefore, the solutions are

  $$\theta \approx 0.9273 + 2\pi k, \ \theta = 2.2143 + 2\pi k \quad k \text{ any integer.}$$

Using a Calculator to Solve a Secant Equation

- Use a calculator to solve the equation $\sec\theta = -4$, giving your answer in radians.

  We have

  $$\sec\theta = -4 \ \Rightarrow\ \frac{1}{\cos\theta} = -4 \ \Rightarrow\ \cos\theta = -\frac{1}{4}.$$

  Using a calculator, we get

  $$\cos^{-1}\left(-\frac{1}{4}\right) \approx 1.8235 \text{ radians.}$$

  However, there is the additional solution

  $$2\pi - 1.8235 = 4.4597.$$

  Therefore, the solutions are

  $$\theta \approx 1.8235 + 2\pi k,\ \theta = 4.4597 + 2\pi k \quad k \text{ any integer.}$$

Solving a Trigonometric Equation in Quadratic Form

- Solve the equation exactly: $2\sin^2\theta - 5\sin\theta + 3 = 0$, $0 \le \theta \le 2\pi$.

  We have

$$2\sin^2\theta - 5\sin\theta + 3 = 0$$

$$(2\sin\theta - 3)(\sin\theta - 1) = 0$$

$$2\sin\theta - 3 = 0 \quad \text{or} \quad \sin\theta - 1 = 0$$

$$\sin\theta = \frac{3}{2} \quad \text{or} \quad \sin\theta = 1$$

$$\theta = \frac{\pi}{2}.$$

Solving a Trigonometric Equation Using Algebra

- Solve exactly: $2\sin^2\theta+\sin\theta=0$; $0\leq\theta<2\pi$.

  We have

$$2\sin^2\theta+\sin\theta=0$$

$$\sin\theta(2\sin\theta+1)=0$$

$$\sin\theta=0 \quad \text{or} \quad 2\sin\theta+1=0$$

$$\sin\theta=0 \quad \text{or} \quad \sin\theta=-\frac{1}{2}$$

$$\theta=0 \text{ or } \theta=\pi \quad \text{or} \quad \theta=\frac{7\pi}{6} \text{ or } \theta=\frac{11\pi}{6}.$$

Solving a Trigonometric Equation Quadratic in Form

- Solve the equation $2\sin^2\theta - 3\sin\theta + 1 = 0$, $0 \leq \theta < 2\pi$.

  We have

$$2\sin^2\theta - 3\sin\theta + 1 = 0$$

$$(2\sin\theta - 1)(\sin\theta - 1) = 0$$

$$2\sin\theta - 1 = 0 \quad \text{or} \quad \sin\theta - 1 = 0$$

$$\sin\theta = \frac{1}{2} \quad \text{or} \quad \sin\theta = 1$$

$$\theta = \frac{\pi}{6} \text{ or } \theta = \frac{5\pi}{6} \quad \text{or} \quad \theta = \frac{\pi}{2}.$$

Use Identities to Solve an Equation

- Use identities to solve exactly the trigonometric equation $\cos x \cos(2x) + \sin x \sin(2x) = \frac{\sqrt{3}}{2}$ over the interval $0 \leq x < 2\pi$.

  We have

$$\cos x \cos(2x) + \sin x \sin(2x) = \frac{\sqrt{3}}{2}$$

$$\cos(x - 2x) = \frac{\sqrt{3}}{2}$$

$$\cos(-x) = \frac{\sqrt{3}}{2}$$

$$\cos x = \frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{6} \text{ or } x = \frac{11\pi}{6}.$$

Solving the Equation Using a Double-Angle Formula

- Solve the equation exactly using a double-angle formula: $\cos(2\theta) = \cos\theta$.

  We have

$$\cos(2\theta) = \cos\theta$$

$$2\cos^2\theta - 1 = \cos\theta$$

$$2\cos^2\theta - \cos\theta - 1 = 0$$

$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$

$$2\cos\theta + 1 = 0 \quad \text{or} \quad \cos\theta - 1 = 0$$

$$\cos\theta = -\frac{1}{2} \quad \text{or} \quad \cos\theta = 1$$

$$\theta = \frac{2\pi}{3} \text{ or } \theta = \frac{4\pi}{3} \quad \text{or} \quad \theta = 0.$$

Solving an Equation Using an Identity

- Solve the equation $3\cos\theta + 3 = 2\sin^2\theta$, $0 \leq \theta < 2\pi$.
  We have
  $$3\cos\theta + 3 = 2\sin^2\theta$$
  $$3\cos\theta + 3 = 2(1 - \cos^2\theta)$$
  $$3\cos\theta + 3 = 2 - 2\cos^2\theta$$
  $$2\cos^2\theta + 3\cos\theta + 1 = 0$$
  $$(2\cos\theta + 1)(\cos\theta + 1) = 0$$
  $$2\cos\theta + 1 = 0 \quad \text{or} \quad \cos\theta + 1 = 0$$
  $$\cos\theta = -\tfrac{1}{2} \quad \text{or} \quad \cos\theta = -1$$
  $$\theta = \tfrac{2\pi}{3} \text{ or } \theta = \tfrac{4\pi}{3} \quad \text{or} \quad \theta = \pi.$$

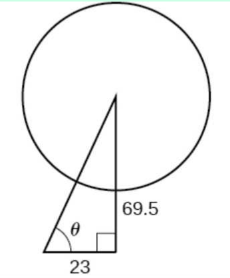
Solving a Multiple Angle Trigonometric Equation

- Solve exactly: $\cos(2x) = \frac{1}{2}$ on $[0, 2\pi)$.
  First note that
  $$0 \leq x < 2\pi \quad \text{implies} \quad 0 \leq 2x < 4\pi.$$
  Now, we have
  $$\cos(2x) = \frac{1}{2}$$
  $$2x = \frac{\pi}{3} \text{ or } 2x = \frac{5\pi}{3} \text{ or } 2x = \frac{7\pi}{3} \text{ or } 2x = \frac{11\pi}{3}$$
  $$x = \frac{\pi}{6} \text{ or } x = \frac{5\pi}{6} \text{ or } x = \frac{7\pi}{6} \text{ or } x = \frac{11\pi}{6}.$$

Using the Pythagorean Theorem to Model an Equation

- One of the cables that anchors the center of the London Eye Ferris wheel to the ground must be replaced.

  The center of the Ferris wheel is 69.5 meters above the ground, and the second anchor on the ground is 23 meters from the base of the Ferris wheel.



  Approximately how long is the cable, and what is the angle of elevation (from ground up to the center of the Ferris wheel)?

  We have;

$$\ell = \sqrt{69.5^2 + 23^2} = \sqrt{5359.25} \approx 73.2 \text{ meters.}$$

  Moreover,

$$\tan\theta = \frac{69.5}{23} \approx 3.022 \ \Rightarrow \ \theta = \tan^{-1}(3.022) \approx 71.69^\circ.$$

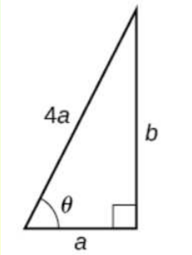
Using the Pythagorean Theorem for an Abstract Problem

- OSHA safety regulations require that the base of a ladder be placed 1 foot from the wall for every 4 feet of ladder length.

  Find the angle that a ladder of any length forms with the ground and the height at which the ladder touches the wall.

  We have

$$b^2 + a^2 = (4a)^2 \Rightarrow b^2 = 15a^2 \Rightarrow b = \sqrt{15}a.$$



  Moreover:

$$\cos\theta = \frac{a}{4a} = \frac{1}{4} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{4}\right) \approx 75.52^\circ.$$