College Trigonometry

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LSSU Math 131

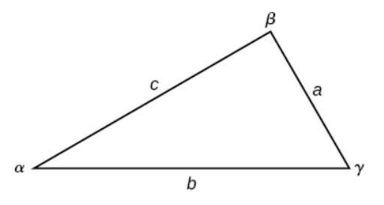
1. Introduction to Further Applications of Trigonometry
   - Non-right Triangles: Law of Sines
   - Non-right Triangles: Law of Cosines
   - Polar Coordinates
   - Polar Coordinates: Graphs
   - Polar Form of Complex Numbers
   - Parametric Equations
   - Parametric Equations: Graphs
   - Vectors

Subsection 1

Non-right Triangles: Law of Sines

The Law of Sines

- The **Law of Sines** for the triangle

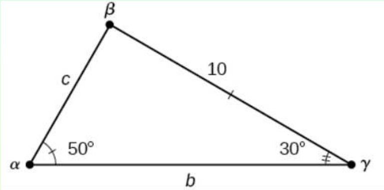


is based on proportions and is presented symbolically in two ways:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \quad \text{or} \quad \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Solving an AAS Triangle

- Solve the triangle shown to the nearest tenth.



We get

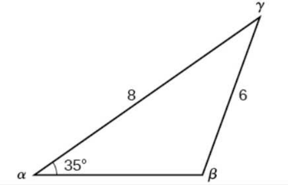
$$\beta = 180^\circ - 50^\circ - 30^\circ = 100^\circ,$$

$$\frac{c}{\sin(30^\circ)} = \frac{10}{\sin(50^\circ)} \Rightarrow c = \frac{10\sin(30^\circ)}{\sin(50^\circ)} \approx 6.53,$$

$$\frac{b}{\sin(100^\circ)} = \frac{10}{\sin(50^\circ)} \Rightarrow b = \frac{10\sin(100^\circ)}{\sin(50^\circ)} \approx 12.86.$$

Solving an Oblique SSA Triangle

- Solve the triangle for the missing side and find the missing angle measures to the nearest tenth.



We get

$$\frac{\sin\beta}{8} = \frac{\sin(35^\circ)}{6} \Rightarrow \sin\beta = \frac{8\sin(35^\circ)}{6} \approx 0.765$$

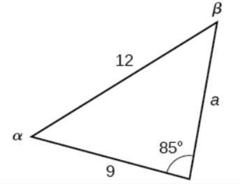
$$\Rightarrow \beta = 180^\circ - \sin^{-1}(0.765) \approx 130.1^\circ,$$

$$\gamma \approx 180^\circ - 35^\circ - 130.1^\circ = 14.9^\circ,$$

$$\frac{c}{\sin(14.9^\circ)} = \frac{6}{\sin(35^\circ)} \Rightarrow c = \frac{6\sin(14.9^\circ)}{\sin(35^\circ)} \approx 2.7.$$

Solving an SSA Triangle

- In the triangle shown, solve for the unknown side and angles. Round your answers to the nearest tenth.



We get

$$\frac{\sin\beta}{9} = \frac{\sin(85^\circ)}{12} \Rightarrow \sin\beta = \frac{9\sin(85^\circ)}{12} \approx 0.747$$

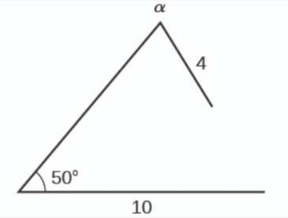
$$\Rightarrow \beta = \sin^{-1}(0.747) \approx 48.3^\circ,$$

$$\alpha \approx 180^\circ - 85^\circ - 48.3^\circ = 46.7^\circ,$$

$$\frac{a}{\sin(46.7^\circ)} = \frac{12}{\sin(85^\circ)} \Rightarrow a = \frac{12\sin(46.7^\circ)}{\sin(85^\circ)} \approx 8.8.$$

Finding the Triangles That Meet the Given Criteria

- Find all possible triangles if one side has length 4 opposite an angle of 50°, and a second side has length 10.



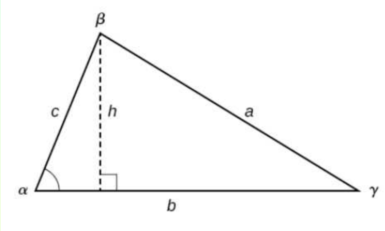
We get

$$\frac{\sin\alpha}{10} = \frac{\sin(50^\circ)}{4} \Rightarrow \sin\alpha = \frac{10\sin(50^\circ)}{4} \approx 1.9.$$

But this is impossible!

Area of an Oblique Triangle

- For an arbitrary triangle



the formula for its area is

$$\text{Area} = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ac\sin\beta = \frac{1}{2}ab\sin\gamma.$$

That is, the area equals one-half of the product of two sides times the sine of their included angle.

Finding the Area of an Oblique Triangle

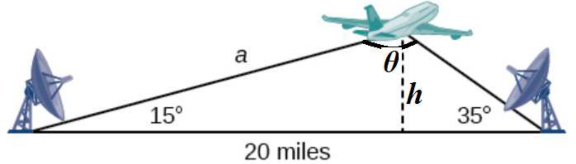
- Find the area of a triangle with sides $a = 90$, $b = 52$, and angle $\gamma = 102^\circ$. Round the area to the nearest integer.

  We get

$$\text{Area} = \frac{1}{2}ab\sin\gamma = \frac{1}{2}\cdot 90\cdot 52\cdot \sin(102^\circ) \approx 2289.$$

Finding an Altitude

- Find the altitude of the aircraft shown. Round the altitude to the nearest tenth of a mile.



We get

$$\theta = 180^\circ - 15^\circ - 35^\circ = 130^\circ,$$

$$\frac{a}{\sin(35^\circ)} = \frac{20}{\sin(130^\circ)} \Rightarrow a = \frac{20\sin(35^\circ)}{\sin(130^\circ)} \approx 14.975,$$

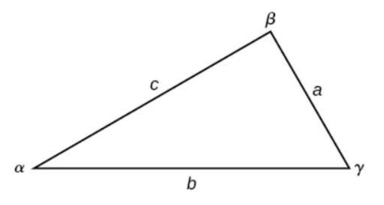
$$\sin(15^\circ) = \frac{h}{14.975} \Rightarrow h = 14.975\sin(15^\circ) \approx 3.9 \text{ miles}.$$

Subsection 2

Non-right Triangles: Law of Cosines

The Law of Cosines

- The **Law of Cosines** for the triangle

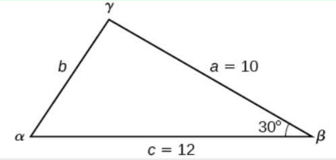


states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle.

$$a^2 = b^2 + c^2 - 2bc\cos\alpha,$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta,$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

Finding the Unknown Side and Angles of a SAS Triangle

- Find the unknown side and angles of the triangle in



We have

$$b^2 = a^2 + c^2 - 2ac\cos\beta = 10^2 + 12^2 - 2\cdot 10\cdot 12\cos(30^\circ)$$

$$\approx 36.15 \;\Rightarrow\; b \approx 6.01,$$

$$\frac{\sin\alpha}{10} = \frac{\sin(30^\circ)}{6.01} \;\Rightarrow\; \sin\alpha = \frac{10\sin(30^\circ)}{6.01} \approx 0.83$$

$$\Rightarrow\; \alpha \approx \sin^{-1}(0.83) \approx 56.3^\circ,$$

$$\gamma \approx 180^\circ - 30^\circ - 56.3^\circ = 93.7^\circ.$$

Solving for an Angle of a SSS Triangle

- Find the angle α for the given triangle if side $a = 20$, side $b = 25$ and side $c = 18$.

  We have

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc} = \frac{25^2+18^2-20^2}{2\cdot 25\cdot 18} = 0.61$$

$$\alpha = \cos^{-1}(0.61) \approx 52.41^\circ.$$

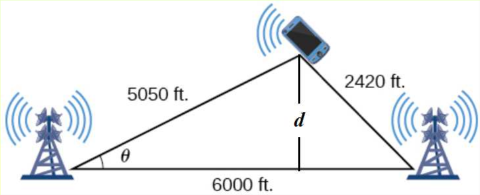
Law of Cosines for a Communication Problem

- Suppose there are two cell phone towers within range of a cell phone.

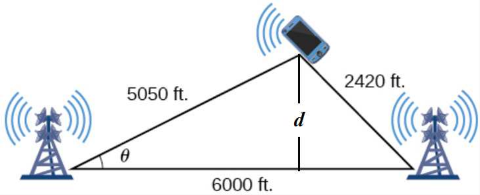
  The two towers are located 6000 feet apart along a straight highway, running east to west, and the cell phone is north of the highway.

  Based on the signal delay, it can be determined that the signal is 5,050 feet from the first tower and 2,420 feet from the second tower.

  Determine the position of the cell phone north and east of the first tower, and determine how far it is from the highway.



Communication Problem: Solution



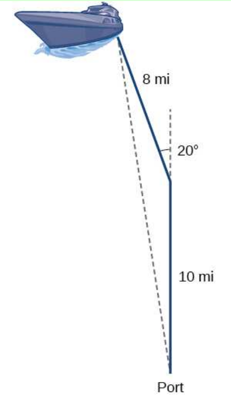
- Since $2420^2 = 5050^2 + 6000^2 - 2 \cdot 5050 \cdot 6000 \cos\theta$,

$$\cos\theta = \frac{5050^2+6000^2-2420^2}{2 \cdot 5050 \cdot 6000} \approx 0.918$$
$$\Rightarrow \theta \approx \cos^{-1}(0.918) \approx 23.33^\circ.$$

$$\sin\theta = \frac{d}{5050} \Rightarrow d \approx 5050\sin(23.33^\circ) \approx 2000 \text{ feet.}$$

Calculating Distance Traveled Using a SAS Triangle

- Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat?



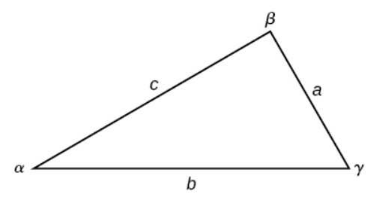
We have

$$d^2 = 10^2 + 8^2 - 2 \cdot 10 \cdot 8 \cos(160^\circ)$$

$$\Rightarrow \; d^2 \approx 314.35 \; \Rightarrow \; d \approx 17.73 \text{ miles.}$$

Heron's Formula for the Area of a Triangle

- **Herons Formula** finds the area of arbitrary triangles in which sides a, b and c are known.

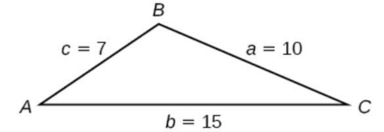


$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \dfrac{a+b+c}{2}$ is one half of the perimeter of the triangle, called the **semi-perimeter**.

Using Heron's Formula to Find the Area

- Find the area of the triangle using Heron's formula.



We have

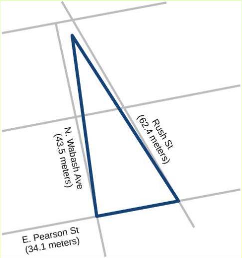
$$\begin{aligned} s &= \frac{a+b+c}{2} = \frac{10+15+7}{2} = 16, \\ A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{16(16-10)(16-15)(16-7)} \\ &\approx 29.4. \end{aligned}$$

Applying Heron's Formula to a Real-World Problem

- A Chicago city developer wants to construct a building on a triangular lot bordered by Rush Street, Wabash Avenue and Pearson Street.

  The frontage along Rush Street is approximately 62.4 meters, along Wabash Avenue it is approximately 43.5 meters, and along Pearson Street it is approximately 34.1 meters.

  How many square meters are available to the developer?



We have

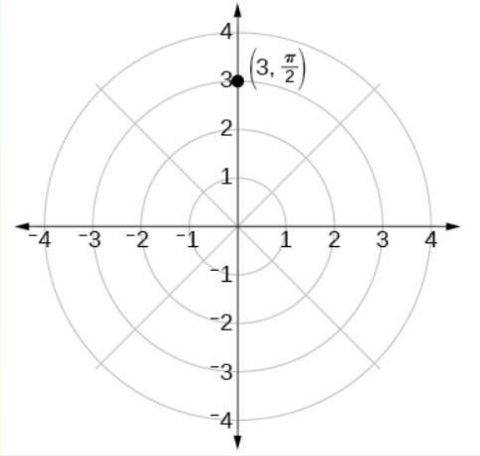
$$\begin{aligned} s &= \frac{62.4+43.5+34.1}{2} = 70, \\ A &= \sqrt{70(70-62.4)(70-43.5)(70-34.1)} \\ &\approx 711.4 \text{ meters}^2 \end{aligned}$$

Subsection 3

Polar Coordinates

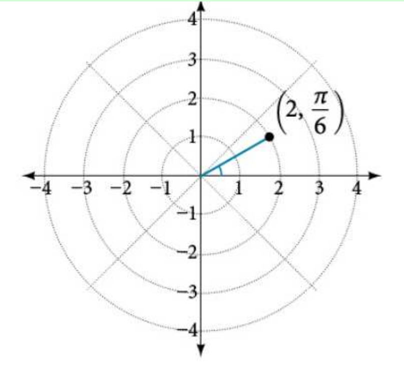
Plotting a Point on the Polar Grid

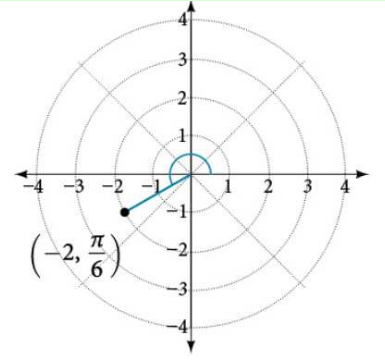
- Plot the point $(3, \frac{\pi}{2})$ on the polar grid.



Plotting a Point with a Negative Component

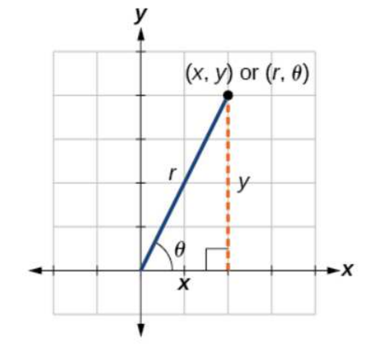
- Plot the point $(-2, \frac{\pi}{6})$ on the polar grid.





From Polar to Rectangular Coordinates

- Referring to the figure,



we obtain:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Writing Polar Coordinates as Rectangular Coordinates

- Write the polar coordinates $(3, \frac{\pi}{2})$ as rectangular coordinates.

  We get
  $$x = r\cos\theta = 3\cos\frac{\pi}{2} = 0$$
  and
  $$y = r\sin\theta = 3\sin\frac{\pi}{2} = 3 \cdot 1 = 3.$$

  So, in rectangular coordinates the point is $(0, 3)$.

Writing Polar Coordinates as Rectangular Coordinates

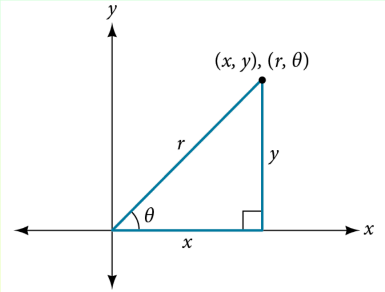
- Write the polar coordinates $(-2, 0)$ as rectangular coordinates.

  We get
  $$x = r\cos\theta = -2\cos 0 = -2 \cdot 1 = -2$$
  and
  $$y = r\sin\theta = -2\sin 0 = -2 \cdot 0 = 0.$$

  So, in rectangular coordinates the point is $(-2, 0)$.

From Rectangular to Polar Coordinates

- Referring to the figure,



we obtain:

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

Writing Rectangular Coordinates as Polar Coordinates

- Convert the rectangular coordinates $(3, 3)$ to polar coordinates.

  We get
  $$r^2 = 3^2 + 3^2 = 18 \;\Rightarrow\; r = 3\sqrt{2}$$
  and
  $$\tan\theta = \frac{y}{x} = \frac{3}{3} = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}.$$

  So, in polar coordinates the point is $(3\sqrt{2}, \frac{\pi}{4})$.

Writing a Cartesian Equation in Polar Form

- Write the Cartesian equation $x^2 + y^2 = 9$ in polar form.

  Recall $x = r\cos\theta$ and $y = r\sin\theta$.

  So

$$\begin{aligned} x^2 + y^2 &= 9 \\ (r\cos\theta)^2 + (r\sin\theta)^2 &= 9 \\ r^2\cos^2\theta + r^2\sin^2\theta &= 9 \\ r^2(\cos^2\theta + \sin^2\theta) &= 9 \\ r^2 &= 9 \\ r &= \pm 3. \end{aligned}$$

Rewriting a Cartesian Equation as a Polar Equation

- Rewrite the Cartesian equation $x^2 + y^2 = 6y$ as a polar equation.

  We have

$$\begin{aligned} x^2 + y^2 &= 6y \\ (r\cos\theta)^2 + (r\sin\theta)^2 &= 6r\sin\theta \\ r^2\cos^2\theta + r^2\sin^2\theta &= 6r\sin\theta \\ r^2(\cos^2\theta + \sin^2\theta) &= 6r\sin\theta \\ r^2 &= 6r\sin\theta \\ r &= 6\sin\theta. \end{aligned}$$

Rewriting a Cartesian Equation in Polar Form

- Rewrite the Cartesian equation $y = 3x + 2$ as a polar equation.

  We have

$$\begin{aligned} y &= 3x + 2 \\ r\sin\theta &= 3r\cos\theta + 2 \\ r\sin\theta - 3r\cos\theta &= 2 \\ r(\sin\theta - 3\cos\theta) &= 2 \\ r &= \frac{2}{\sin\theta - 3\cos\theta}. \end{aligned}$$

Polar Equation to a Rectangular Equation

- Covert the polar equation $r = 2\sec\theta$ to a rectangular equation, and draw its corresponding graph.

  We get

$$r = 2\sec\theta$$
$$r = \frac{2}{\cos\theta}$$
$$r\cos\theta = 2$$
$$x = 2.$$

Rewriting a Polar Equation in Cartesian Form

- Rewrite the polar equation $r = \dfrac{3}{1 - 2\cos\theta}$ as a Cartesian equation.

  We get

$$\begin{aligned} r &= \frac{3}{1 - 2\cos\theta} \\ r(1 - 2\cos\theta) &= 3 \\ r - 2r\cos\theta &= 3 \\ \pm\sqrt{x^2 + y^2} - 2x &= 3. \end{aligned}$$

Rewriting a Polar Equation in Cartesian Form

- Rewrite the polar equation $r = \sin(2\theta)$ in Cartesian form.

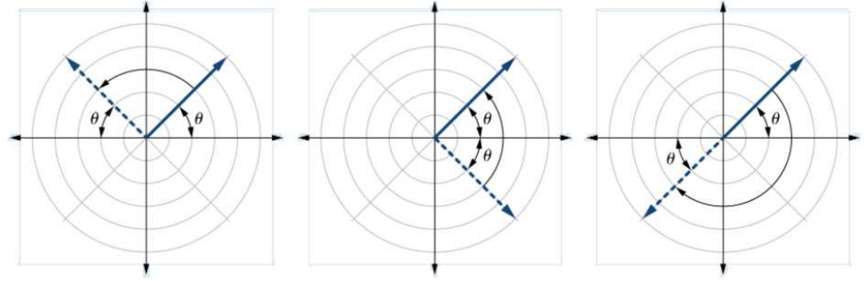
  We get

$$r = \sin(2\theta)$$
$$r = 2\sin\theta\cos\theta$$
$$r^3 = 2r^2\sin\theta\cos\theta$$
$$r^3 = 2r\sin\theta r\cos\theta$$
$$\pm\sqrt{(x^2+y^2)^3} = 2xy.$$

Subsection 4

Polar Coordinates: Graphs

Three Tests for Symmetry



- We substitute for (r,θ):
  - $(-r,-\theta)$ to test for $\theta=\frac{\pi}{2}$-symmetry;
  - $(r,-\theta)$ to test for polar axis symmetry;
  - $(-r,\theta)$ to test for pole symmetry.

  If the resulting equations are equivalent in one or more of the tests, the graph possesses the corresponding symmetry.

Testing a Polar Equation for Symmetry

- Test the equation $r = 2\sin\theta$ for symmetry.

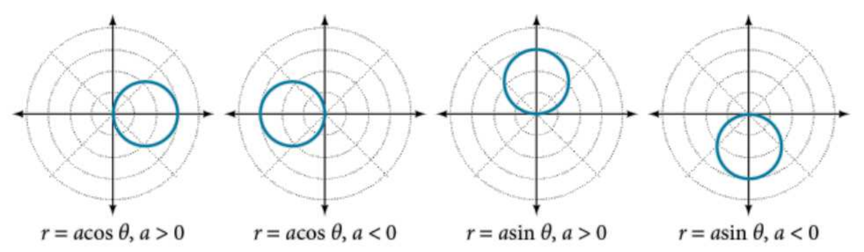
  We do the tests and then draw the appropriate conclusions:
  - Test for $\theta = \frac{\pi}{2}$-symmetry:
    $-r = 2\sin(-\theta) \Rightarrow -r = -2\sin\theta \Rightarrow r = 2\sin\theta$
    Test is positive!
  - Test for polar axis symmetry:
    $r = 2\sin(-\theta) \Rightarrow r = -2\sin\theta$
    Test is negative!
  - Test for pole symmetry:
    $-r = 2\sin\theta \Rightarrow r = -2\sin\theta$
    Test is negative!

  So $r = 2\sin\theta$ is symmetric with respect to $\theta = \frac{\pi}{2}$ but is not symmetric with respect to either the polar axis or the pole.

Finding Zeros and Maximum Values for a Polar Equation

- Find the zeros and maximum $|r|$ and, if necessary, the polar axis intercepts of $r = 2\sin\theta$.
- Work as follows:
  - To find the zeros, we set $r = 0$ and solve for θ.
  - To find the maximum $|r|$ use the fact that sine and cosine take values between -1 and 1.
  - To find the polar axis intercepts, find r for $\theta = \pi k$ for k any integer.

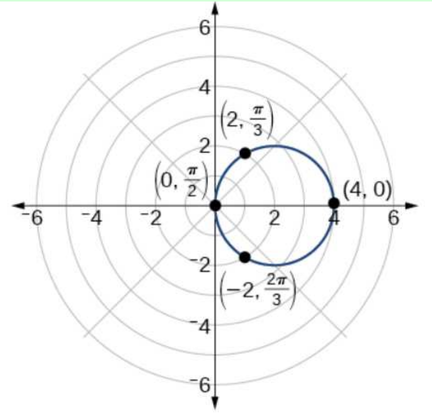
Polar Equation for a Circle



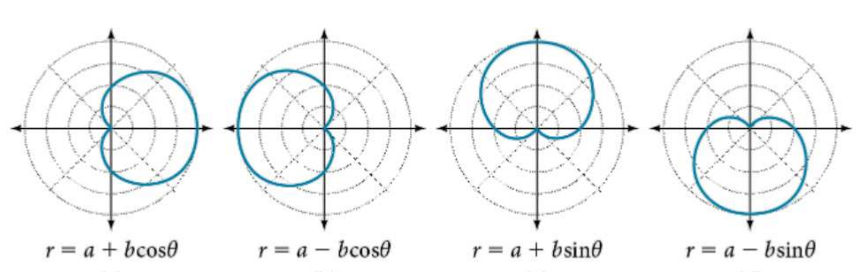
Sketching the Graph of a Polar Equation for a Circle

- Sketch the graph of $r = 4\cos\theta$.

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	2	0	2	-4



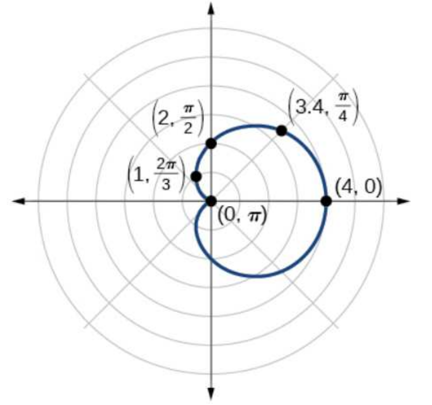
Polar Equation for a Cardioid



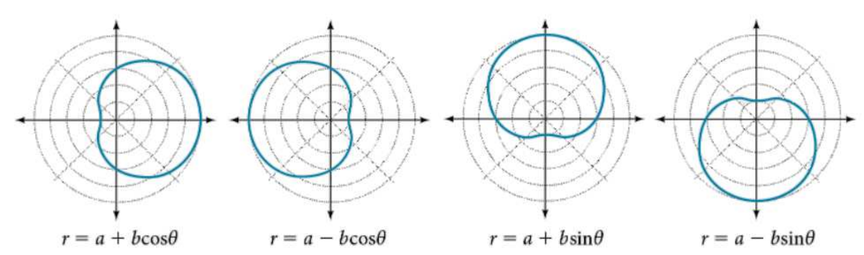
Sketching the Graph of a Cardioid

- Sketch the graph of $r = 2 + 2\cos\theta$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	3.4	2	1	0



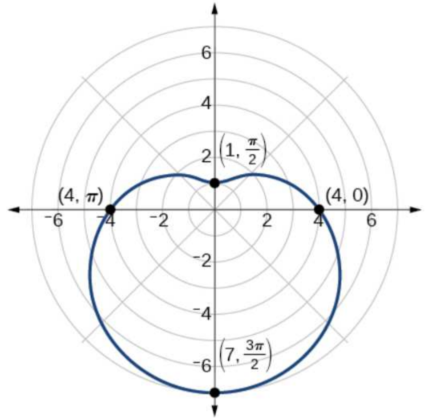
Polar Equation for a One-Loop Limaçon



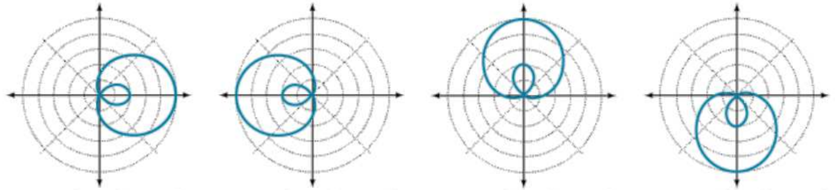
Sketching the Graph of a One-Loop Limaçon

- Graph the equation $r = 4 - 3\sin\theta$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
r	4	2.5	1.4	1	1.4	2.5
	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
	4	5.5	6.6	7	6.6	5.5



Polar Equation for an Inner-Loop Limaçon

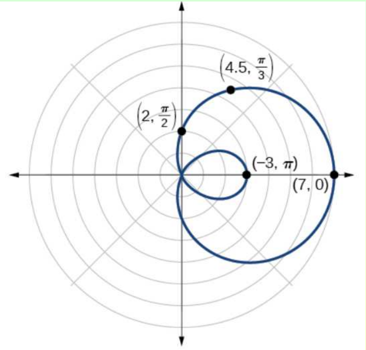


$r = a + b\cos\theta,\ a < b$ $\quad r = a - b\cos\theta,\ a < b$ $\quad r = a + b\sin\theta,\ a < b$ $\quad r = a - b\sin\theta,\ a < b$

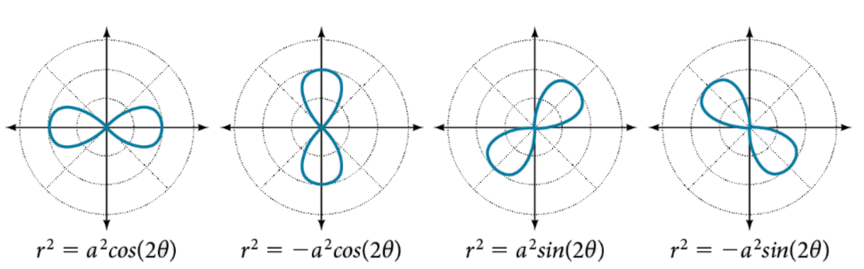
Sketching the Graph of an Inner-Loop Limaçon

- Sketch the graph of $r = 2 + 5\cos\theta$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	7	6.3	4.5	2
	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$
	-0.5	-2.3	-3	-2.3
	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
	-0.5	2	4.5	6.3



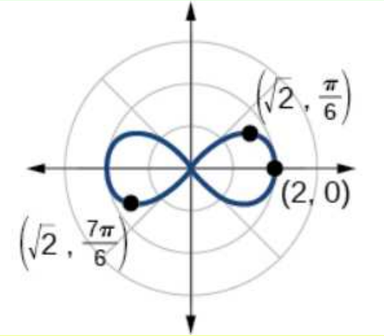
Polar Equation for a Lemniscate



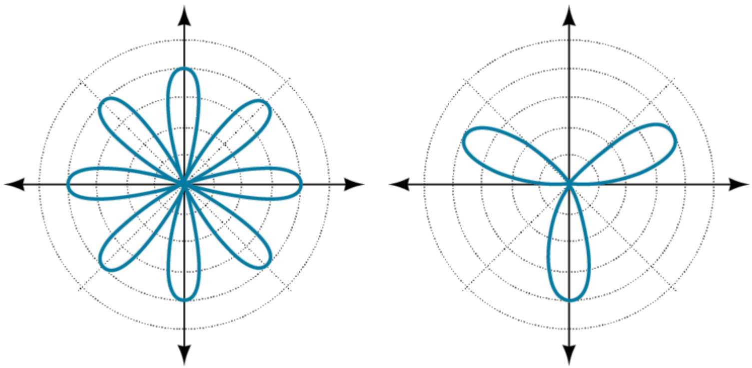
Sketching the Graph of a Lemniscate

- Sketch the graph of $r^2 = 4\cos(2\theta)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	± 2	$\pm\sqrt{2}$	0



Polar Equation for a Rose Curve



$r = a\cos(n\theta)$, n even

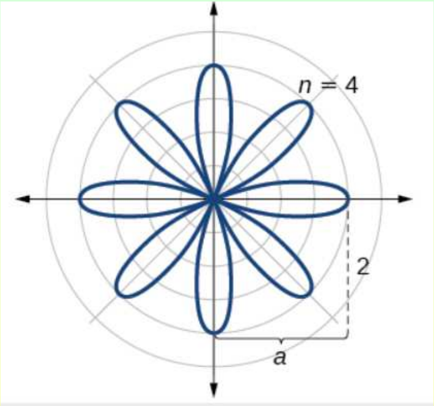
$r = a\sin(n\theta)$, n odd

Sketching the Graph of a Rose Curve (n Even)

- Sketch the graph of $r = 2\cos(4\theta)$.

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$
r	2	0	-2	0

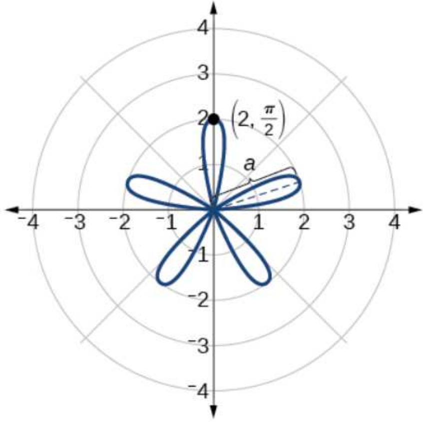
	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	$\frac{3\pi}{4}$	
	2	0	-2	



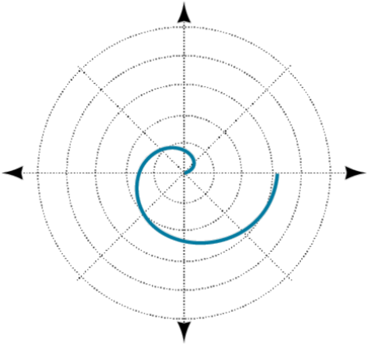
Sketching the Graph of a Rose Curve (n Odd)

- Sketch the graph of $r = 2\sin(5\theta)$.

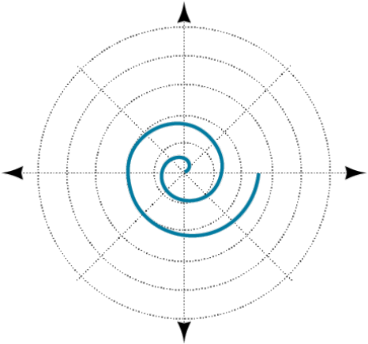
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	1	-1.73	2
	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	
	-1.73	1	0	



Polar Equation for an Archimedes' Spiral



$r = \theta,\ [0, 2\pi]$

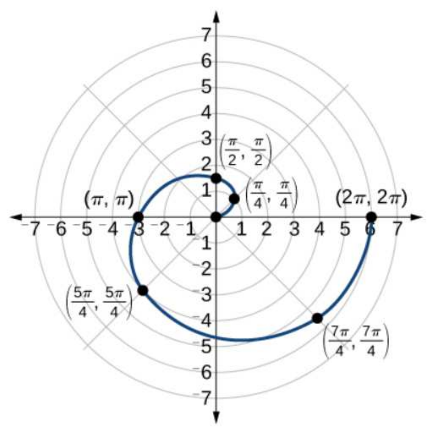


$r = \theta,\ [0, 4\pi]$

Sketching the Graph of an Archimedes' Spiral

- Sketch the graph of $r = \theta$ over $[0, 2\pi]$.

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0.785	1.57	3.14	4.71	5.50	6.28



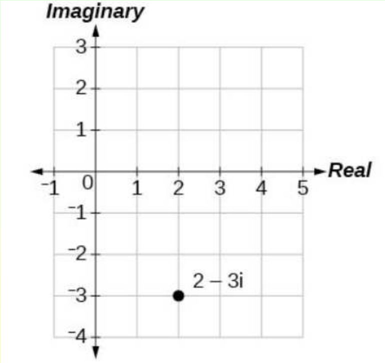
Subsection 5

Polar Form of Complex Numbers

Plotting a Complex Number in the Complex Plane

- Plot the complex number $2-3i$ in the complex plane.

  If $z=a+bi$, then z is represented by the point (a,b).



Finding the Absolute Value of a Complex Number

- Find the absolute value of $z = \sqrt{5} - i$.

  If $z = a + bi$, then
  $$|z| = \sqrt{a^2 + b^2}.$$
  Note that this is also the distance of the point (a, b), which represents the complex number z on the plane, from the origin.

  So, for $z = \sqrt{5} - i$, we get
  $$|z| = \sqrt{(\sqrt{5})^2 + (-1)^2} = \sqrt{6}.$$

Finding the Absolute Value of a Complex Number

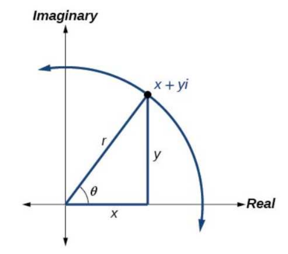
- Given $z = 3 - 4i$, find $|z|$.

  We get
  $$|z| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

Complex Number in Polar Coordinates

- If $z = x + yi$, we calculate r and θ, such that

$$x = r\cos\theta \text{ and } y = r\sin\theta,$$



then

$$z = x + yi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = r\mathrm{cis}(\theta).$$

$\mathrm{cis}(\theta)$ stands for $\cos\theta + i\sin\theta$.
The expression above is called the **polar form** of z, with r the **modulus** and θ the **argument**.

Expressing a Complex Number Using Polar Coordinates

- Express the complex number $4i$ using polar coordinates.

  Plot $4i$.

  We see that $r = 4$ and $\theta = \frac{\pi}{2}$.

  So
  $$4i = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$$

Finding the Polar Form of a Complex Number

- Find the polar form of $-4+4i$.

  Instead of geometrically, we may find r and θ algebraically:
  - $r = \sqrt{a^2+b^2} = \sqrt{(-4)^2+4^2} = \sqrt{32} = 4\sqrt{2}$.
  - $\tan\theta = \frac{b}{a} = \frac{4}{-4} = -1 \ \Rightarrow\ \theta = \frac{3\pi}{4}$.

  So we get
  $$-4+4i = 4\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right).$$

Converting from Polar to Rectangular Form

- Convert the polar form of the given complex number to rectangular form: $z = \frac{1}{2}(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right))$.

  Here, we just compute the sine and the cosine and then distribute:

$$\begin{aligned} z &= \frac{1}{2}(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)) \\ &= \frac{1}{2}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) \\ &= \frac{\sqrt{3}}{4} + \frac{1}{4}i. \end{aligned}$$

Finding the Rectangular Form of a Complex Number

- Find the rectangular form of the complex number given $r = 20$ and $\tan\theta = \frac{1}{3}$.

  Recall that, if $z = a + bi$, then $\tan\theta = \frac{b}{a}$ and $r = \sqrt{a^2 + b^2}$.

  So we work as follows:

  - $\tan\theta = \frac{b}{a} \Rightarrow \frac{b}{a} = \frac{1}{3} \Rightarrow a = 3b$.
  - $r = \sqrt{a^2 + b^2} \Rightarrow 20 = \sqrt{(3b)^2 + b^2} \Rightarrow 20 = \sqrt{10b^2} \Rightarrow 10b^2 = 400 \Rightarrow b^2 = 40 \Rightarrow b = 2\sqrt{10}$.
  - Finally, $a = 3b = 3 \cdot 2\sqrt{10} = 6\sqrt{10}$.

  Therefore,

  $$z = a + bi = 6\sqrt{10} + 2\sqrt{10}i.$$

Product and Quotient in Polar Form

- If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the product and quotient of these numbers is given as:

$$\begin{aligned} z_1 z_2 &= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]. \end{aligned}$$

- Notice that the product calls for multiplying the moduli and adding the angles
- In the quotient the moduli are divided, and the angles are subtracted.

Finding the Product of Complex Numbers in Polar Form

- Find the product of $z_1 z_2$, given $z_1 = 4(\cos(80^\circ) + i \sin(80^\circ))$ and $z_2 = 2(\cos(145^\circ) + i \sin(145^\circ))$.

  We get

$$\begin{aligned} z_1 z_2 &= 4 \cdot 2(\cos(80^\circ + 145^\circ) + i \sin(80^\circ + 145^\circ)) \\ &= 8(\cos(225^\circ) + i \sin(225^\circ)) \\ &= 8(-\tfrac{\sqrt{2}}{2} - i\tfrac{\sqrt{2}}{2}) \\ &= -4\sqrt{2} - 4\sqrt{2}i. \end{aligned}$$

Finding the Quotient of Two Complex Numbers

- Find the quotient of $z_1 = 2(\cos(213^\circ) + i\sin(213^\circ))$ and $z_2 = 4(\cos(33^\circ) + i\sin(33^\circ))$.

  We get

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{2}{4}(\cos(213^\circ - 33^\circ) + i\sin(213^circ - 33^\circ)) \\
&= \frac{1}{2}(\cos(180^\circ) + i\sin(180^\circ)) \\
&= \frac{1}{2}(-1 + i \cdot 0) \\
&= -\frac{1}{2}.
\end{aligned}$$

De Moivre's Theorem

- If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)].$$

Evaluating an Expression Using De Moivre's Theorem

- Evaluate the expression $(1+i)^5$ using De Moivre's Theorem.

  We get

$$\begin{aligned}(1+i)^5 &= [\sqrt{2}(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4})]^5 \\ &= (\sqrt{2})^5[\cos(5\tfrac{\pi}{4}) + i\sin(5\tfrac{\pi}{4})] \\ &= 4\sqrt{2}(-\tfrac{\sqrt{2}}{2} - i\tfrac{\sqrt{2}}{2}) \\ &= -4-4i.\end{aligned}$$

The n-th Root Theorem

- To find the n-th root of a complex number $z = r(\cos\theta + i\sin\theta)$ in polar form, use the formula given as
$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right],$$
where $k = 0, 1, 2, 3, \ldots, n-1$.
We add $\frac{2k\pi}{n}$ to $\frac{\theta}{n}$ in order to obtain the periodic roots.

Finding the nth Root of a Complex Number

- Evaluate the cube roots of $z = 8(\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}))$.
  We get for $n = 3$ and $k = 0, 1, 2$,

$$\begin{aligned} z^{1/3} &= 8^{1/3}(\cos(\tfrac{\frac{2\pi}{3}}{3}) + i\sin(\tfrac{\frac{2\pi}{3}}{3})) \\ &= 2(\cos(\tfrac{2\pi}{9}) + i\sin(\tfrac{2\pi}{9})); \end{aligned}$$

$$\begin{aligned} z^{1/3} &= 8^{1/3}(\cos(\tfrac{\frac{2\pi}{3}}{3} + \tfrac{2\pi}{3}) + i\sin(\tfrac{\frac{2\pi}{3}}{3} + \tfrac{2\pi}{3})) \\ &= 2(\cos(\tfrac{2\pi}{9} + \tfrac{6\pi}{9}) + i\sin(\tfrac{2\pi}{9} + \tfrac{6\pi}{9})) \\ &= 2(\cos(\tfrac{8\pi}{9}) + i\sin(\tfrac{8\pi}{9})); \end{aligned}$$

$$\begin{aligned} z^{1/3} &= 8^{1/3}(\cos(\tfrac{\frac{2\pi}{3}}{3} + \tfrac{4\pi}{3}) + i\sin(\tfrac{\frac{2\pi}{3}}{3} + \tfrac{4\pi}{3})) \\ &= 2(\cos(\tfrac{2\pi}{9} + \tfrac{12\pi}{9}) + i\sin(\tfrac{2\pi}{9} + \tfrac{12\pi}{9})) \\ &= 2(\cos(\tfrac{14\pi}{9}) + i\sin(\tfrac{14\pi}{9})). \end{aligned}$$

Subsection 6

Parametric Equations

Parameterizing a Curve

- Parameterize the curve $y = x^2 - 1$ letting $x(t) = t$.

  We set $x(t) = t$.

  Then we obtain $y(t) = x(t)^2 - 1 = t^2 - 1$.

  So the parametric system (with parameter t) is

$$\left\{ \begin{array}{lcl} x(t) & = & t \\ y(t) & = & t^2 - 1 \end{array} \right\}.$$

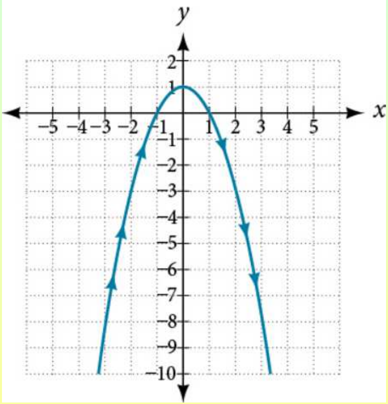
Finding a Pair of Parametric Equations

- Find a pair of parametric equations that models the graph of $y = 1 - x^2$, using the parameter $x(t) = t$.
  Plot some points and sketch the graph.
  We get

$$\left\{ \begin{array}{lcl} x(t) & = & t \\ y(t) & = & 1 - t^2 \end{array} \right\}$$

Here are a few points:

t	$x(t)$	$y(t)$
-2	-2	-3
-1	-1	0
0	0	1
1	1	0
2	2	-3



Finding Parametric Equations That Model Given Criteria

- An object travels at a steady rate along a straight path $(-5, 3)$ to $(3, -1)$ in the same plane in four seconds.

  The coordinates are measured in meters.

  Find parametric equations for the position of the object.

  The x-value of the object starts at -5 meters and goes to 3 meters.

  So x has changed by 8 m in 4 s, i.e. at a rate of $\frac{8}{4}$ m/s or 2 m/s.

  The x-coordinate is a linear function of time $x(t) = 2t - 5$.

  Similarly, for y we get $y = -t + 3$.

  Therefore,

$$\left\{ \begin{array}{lcl} x(t) & = & 2t - 5 \\ y(t) & = & -t + 3 \end{array} \right\}, \quad 0 \le t \le 4.$$

Eliminating the Parameter in Polynomials

- Given $x(t) = t^2 + 1$ and $y(t) = 2 + t$, eliminate the parameter, and write the parametric equations as a Cartesian equation.

  $y = 2 + t \;\Rightarrow\; t = y - 2.$

  So $x = (y-2)^2 + 1 \;\Rightarrow\; x = y^2 - 4y + 4 + 1 \;\Rightarrow\; x = y^2 - 4y + 5.$

Eliminating the Parameter in Exponential Equations

- Eliminate the parameter and write as a Cartesian equation:
  $x(t) = e^{-t}$ and $y(t) = 3e^t$, $t > 0$.
  $x = e^{-t} \Rightarrow x = \frac{1}{e^t} \Rightarrow e^t = \frac{1}{x}$.
  So we get $y = 3e^t \Rightarrow y = 3 \cdot \frac{1}{x} \Rightarrow y = \frac{3}{x}$.

Eliminating the Parameter in Logarithmic Equations

- Eliminate the parameter and write as a Cartesian equation: $x(t) = \sqrt{t} + 2$ and $y(t) = \log(t)$.
  $x = \sqrt{t} + 2 \Rightarrow \sqrt{t} = x - 2 \Rightarrow t = (x-2)^2$.
  So we get $y = \log((x-2)^2) \Rightarrow y = 2\log(x-2)$.

Eliminating the Parameter from a Trigonometric Pair

- Eliminate the parameter from the given pair of trigonometric equations where $0 \leq t \leq 2\pi$ and sketch the graph.
  $$\left\{\begin{array}{lcl} x(t) & = & 4\cos t \\ y(t) & = & 3\sin t \end{array}\right\}.$$
  We exploit the Pythagorean Identity $\sin^2 t + \cos^2 t = 1$.

  We get $\cos t = \frac{x}{4}$ and $\sin t = \frac{y}{3}$.

  And we square and add:
  $\cos^2 t + \sin^2 t = 1 \Rightarrow (\frac{x}{4})^2 + (\frac{y}{3})^2 = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1.$

Finding a Cartesian Equation

- Find the Cartesian equation equivalent to the given set of parametric equations. $\left\{\begin{array}{rcl} x(t) & = & 3t-2 \\ y(t) & = & t+1 \end{array}\right\}$.

  Solve the second for t: $y=t+1 \Rightarrow t=y-1$.

  Substitute into the first: $x=3t-2 \Rightarrow x=3(y-1)-2 \Rightarrow x=3y-3-2 \Rightarrow x=3y-5 \Rightarrow 3y=x+5 \Rightarrow y=\frac{x+5}{3}$.

Parametric Equations from Rectangular Equations

- Find a set of equivalent parametric equations for $y=(x+3)^2+1$.

  If we set $x=t$, then $y=(t+3)^2+1$.

  So
  $$\left\{\begin{array}{lcl} x(t) & = & t \\ y(t) & = & (t+3)^2+1 \end{array}\right\}$$

  But we can also set $t=x+3$. Then $y=t^2+1$.

  Therefore, in this case
  $$\left\{\begin{array}{lcl} x(t) & = & t-3 \\ y(t) & = & t^2+1 \end{array}\right\}$$

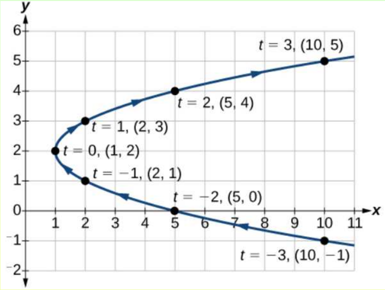
Subsection 7

Parametric Equations: Graphs

Sketching the Graph of Parametric Equations by Points

- Sketch the graph of the parametric equations $x(t) = t^2 + 1$, $y(t) = 2 + t$.

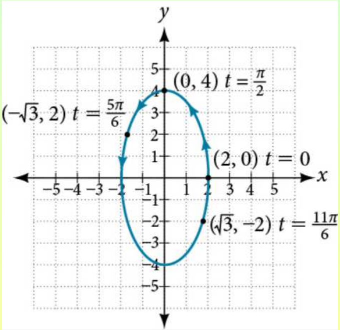
t	$x(t) = t^2 + 1$	$y(t) = 2 + t$
-3	10	-1
-2	5	0
-1	2	1
0	1	2
1	2	3
2	5	4
3	10	5



Sketching the Graph of Trigonometric Equations

- Construct a table of values for the given parametric equations and sketch the graph: $\left\{\begin{array}{lcl} x & = & 2\cos t \\ y & = & 4\sin t \end{array}\right\}$.

t	$x = 2\cos t$	$y = 4\sin t$
0	2	0
$\frac{\pi}{6}$	$\sqrt{3}$	2
$\frac{\pi}{3}$	1	$2\sqrt{3}$
$\frac{\pi}{2}$	0	4
π	-2	0
$\frac{3\pi}{2}$	0	-4
2π	2	0

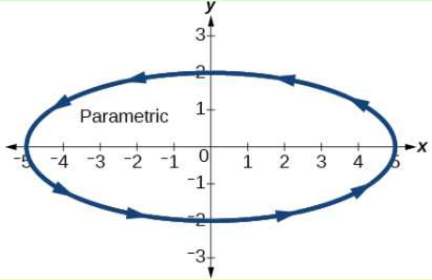


Graphing Parametric and Converting to Rectangular

(a) Graph the parametric equations $x = 5\cos t$ and $y = 2\sin t$.

(b) Convert into rectangular form.

t	$x = 5\cos t$	$y = 2\sin t$
0	5	0
$\frac{\pi}{4}$	$\frac{5\sqrt{2}}{2}$	$\sqrt{2}$
$\frac{\pi}{2}$	0	2
$\frac{3\pi}{2}$	$-\frac{5\sqrt{2}}{2}$	$\sqrt{2}$
π	-5	0
$\frac{3\pi}{2}$	0	-2
2π	5	0



For (b), we get

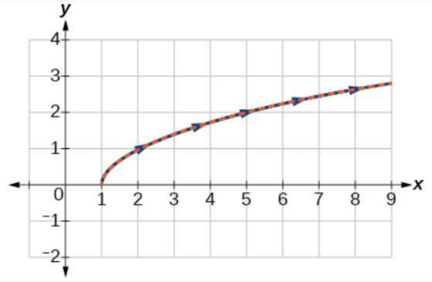
$$\cos^2 t + \sin^2 t = 1 \Rightarrow \left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \Rightarrow \frac{x^2}{25} + \frac{y^2}{4} = 1.$$

Graphing Parametric Equations

(a) Graph the parametric equations $x = t + 1$ and $y = \sqrt{t}$, $t \geq 0$.

(b) Convert into rectangular form.

t	$x = t + 1$	$y = \sqrt{t}$
0	1	0
1	2	1
4	5	2
9	10	3



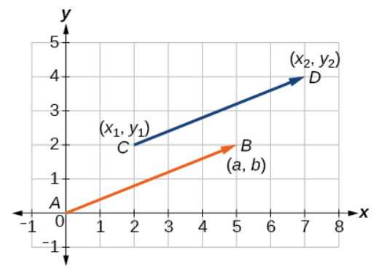
For (b), we get $y = \sqrt{x - 1}$.

Subsection 8

Vectors

Finding a Position Vector

- Given a vector $\overrightarrow{CD}$, with $C = (x_1, y_1)$ and $D = (x_2, y_2)$,



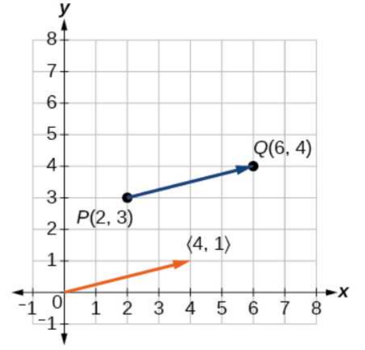
the **position vector** of $\overrightarrow{CD}$ is the vector $\overrightarrow{OB} = \langle a, b \rangle$, where

$$a = x_2 - x_1 \quad \text{and} \quad b = y_2 - y_1.$$

Find the Position Vector

- Consider the vector whose initial point is $P(2,3)$ and terminal point is $Q(6,4)$. Find the position vector.

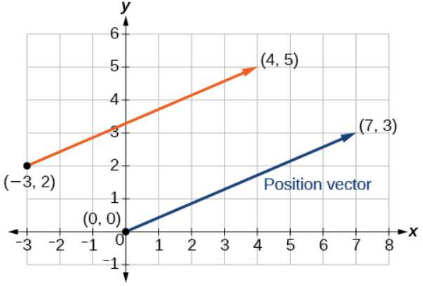
  We get
  $$\langle a,b\rangle = \langle 6-2, 4-3\rangle = \langle 4,1\rangle.$$



Drawing a Vector and Its Equivalent Position Vector

- Find the position vector given that vector **v** has an initial point at $(-3, 2)$ and a terminal point at $(4, 5)$, then graph both vectors in the same plane.

  We get
  $$\langle a, b\rangle = \langle 4 - (-3), 5 - 2\rangle = \langle 7, 3\rangle.$$



Finding the Magnitude and Direction of a Vector

- Find the magnitude and direction of the vector with initial point $P(-8,1)$ and terminal point $Q(-2,-5)$. Draw the vector.
- First we compute the corresponding position vector:

$$\mathbf{v} = \langle -2-(-8), -5-1 \rangle = \langle 6, -6 \rangle.$$

- The magnitude is the length

$$|\mathbf{v}| = \sqrt{6^2 + (-6)^2} = \sqrt{2 \cdot 6^2} = 6\sqrt{2}.$$

- The direction is the angle θ it forms with the x-axis:

$$\tan\theta = \frac{y}{x} = \frac{-6}{6} = -1 \ \Rightarrow\ \theta = -\frac{\pi}{4}.$$

Showing That Two Vectors Are Equal

(a) Show that vector **v** with initial point at $(5, -3)$ and terminal point at $(-1, 2)$ is equal to vector **u** with initial point at $(-1, -3)$ and terminal point at $(-7, 2)$.

(b) Find the magnitude and direction of each vector.

(a) We have

$$\begin{aligned} \mathbf{v} &= \langle -1-5, 2-(-3)\rangle = \langle -6, 5\rangle, \\ \mathbf{u} &= \langle -7-(-1), 2-(-3)\rangle = \langle -6, 5\rangle. \end{aligned}$$

(b) Now we get:

$$|\mathbf{v}| = \sqrt{(-6)^2+5^2} = \sqrt{36+25} = \sqrt{61}.$$

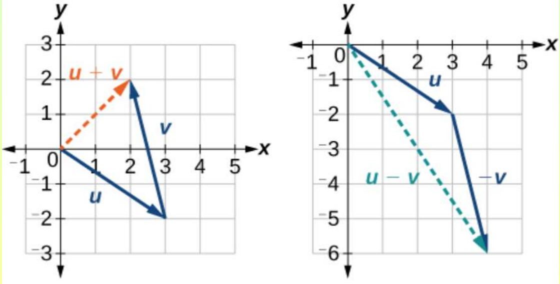
$$\tan\theta = \frac{5}{-6} \Rightarrow \theta = 180^\circ - \tan^{-1}\left(-\frac{5}{6}\right) \approx 180^\circ - 39.8^\circ = 140.2^\circ.$$

Adding and Subtracting Vectors

- Given $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -1, 4 \rangle$, find two new vectors $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

  We get

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \langle 3, -2 \rangle + \langle -1, 4 \rangle = \langle 2, 2 \rangle, \\ \mathbf{u} - \mathbf{v} &= \langle 3, -2 \rangle - \langle -1, 4 \rangle = \langle 4, -6 \rangle. \end{aligned}$$

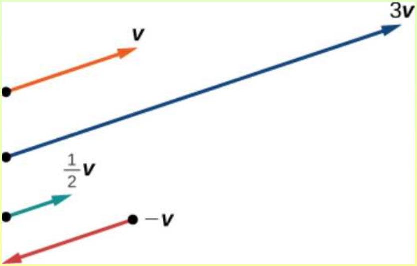


Performing Scalar Multiplication

- Given vector $\mathbf{v} = \langle 3, 1 \rangle$, find $3\mathbf{v}$, $\frac{1}{2}\mathbf{v}$ and $-\mathbf{v}$.

  We get

$$\begin{aligned} 3\mathbf{v} &= 3\langle 3, 1 \rangle = \langle 9, 3 \rangle, \\ \tfrac{1}{2}\mathbf{v} &= \tfrac{1}{2}\langle 3, 1 \rangle = \langle \tfrac{3}{2}, \tfrac{1}{2} \rangle, \\ -\mathbf{v} &= -\langle 3, 1 \rangle = \langle -3, -1 \rangle. \end{aligned}$$



Using Vector Addition and Scalar Multiplication

- Given $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -1, 4 \rangle$, find a new vector $\mathbf{w} = 3\mathbf{u} + 2\mathbf{v}$.

  First perform the scalar multiplication and then add/subtract:

$$\begin{aligned} \mathbf{w} &= 3\mathbf{u} + 2\mathbf{v} \\ &= 3\langle 3, -2 \rangle + 2\langle -1, 4 \rangle \\ &= \langle 9, -6 \rangle + \langle -2, 8 \rangle \\ &= \langle 7, 2 \rangle. \end{aligned}$$

Finding the Components of the Vector

- Find the components of the vector $\mathbf{v}$ with initial point $(3,2)$ and terminal point $(7,4)$.

  Write as a position vector:

$$\mathbf{v} = \langle 7-3, 4-2 \rangle = \langle 4, 2 \rangle.$$

  So $\mathbf{v} = \langle 4,0 \rangle + \langle 0,2 \rangle$, i.e., it has
  - horizontal component $\mathbf{v}_1 = \langle 4,0 \rangle$ and
  - vertical component $\mathbf{v}_2 = \langle 0,2 \rangle$.

Finding the Unit Vector in the Direction of $\mathbf{v}$

- Find a unit vector in the same direction as $\mathbf{v} = \langle -5, 12 \rangle$.
- The unit vector in the direction of $\mathbf{v}$ is the vector $\frac{1}{|\mathbf{v}|}\mathbf{v}$.
- So for our example, we get:

$$|\mathbf{v}| = \sqrt{(-5)^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13,$$

$$\frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{13}\langle -5, 12 \rangle = \langle -\frac{5}{13}, \frac{12}{13} \rangle.$$

Writing a Vector in Terms of **i** and **j**

- Given a vector $\mathbf{v}$ with initial point $P = (2, -6)$ and terminal point $Q = (-6, 6)$, write the vector in terms of $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$.

  Write as a position vector:

$$\begin{aligned}
\mathbf{v} &= \langle -6-2, 6-(-6)\rangle \\
&= \langle -8, 12\rangle \\
&= \langle -8, 0\rangle + \langle 0, 12\rangle \\
&= -8\langle 1, 0\rangle + 12\langle 0, 1\rangle \\
&= -8\mathbf{i} + 12\mathbf{j}.
\end{aligned}$$

Finding the Sum of the Vectors

- Find the sum of $\mathbf{v}_1 = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v}_2 = 4\mathbf{i} + 5\mathbf{j}$.

  We get

$$\begin{aligned} \mathbf{v}_1 + \mathbf{v}_2 &= 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{i} + 5\mathbf{j} \\ &= 6\mathbf{i} + 2\mathbf{j}. \end{aligned}$$

Writing a Vector in Terms of Magnitude and Direction

- Write a vector with length 7 at an angle of 135° to the positive x-axis in terms of magnitude and direction.

  We have

$$\begin{aligned}
\mathbf{v} &= x\mathbf{i} + y\mathbf{j} \\
&= |\mathbf{v}|\cos\theta\mathbf{i} + |\mathbf{v}|\sin\theta\mathbf{j} \\
&= 7\cos(135^\circ)\mathbf{i} + 7\sin(135^\circ)\mathbf{j} \\
&= 7(-\tfrac{\sqrt{2}}{2})\mathbf{i} + 7\tfrac{\sqrt{2}}{2}\mathbf{j} \\
&= -\tfrac{7\sqrt{2}}{2}\mathbf{i} + \tfrac{7\sqrt{2}}{2}\mathbf{j}.
\end{aligned}$$

The Dot Product of Two Vectors

- The **dot product** of two vectors $\mathbf{v} = \langle a, b \rangle$ and $\mathbf{u} = \langle c, d \rangle$ is the sum of the product of the horizontal components and the product of the vertical components.

$$\mathbf{v} \cdot \mathbf{u} = ac + bd.$$

- For vectors $\mathbf{v}$ and $\mathbf{u}$, we have

$$\mathbf{v} \cdot \mathbf{u} = |\mathbf{v}||\mathbf{u}| \cos \theta,$$

  where θ is the angle between them.

  So to find the angle θ between the two vectors, we use the formula

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}||\mathbf{u}|}.$$

Finding the Dot Product of Two Vectors

- Find the dot product of $\mathbf{v} = \langle 5, 12 \rangle$ and $\mathbf{u} = \langle -3, 4 \rangle$.

  We get

$$\begin{aligned} \mathbf{v} \cdot \mathbf{u} &= \langle 5, 12 \rangle \cdot \langle -3, 4 \rangle \\ &= 5 \cdot (-3) + 12 \cdot 4 \\ &= -15 + 48 \\ &= 33. \end{aligned}$$

Finding the Dot Product and the Angle

- Find the dot product of $\mathbf{v}_1 = 5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v}_2 = 3\mathbf{i} + 7\mathbf{j}$. Then, find the angle between the two vectors.

  For the dot product we have

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= (5\mathbf{i} + 2\mathbf{j}) \cdot (3\mathbf{i} + 7\mathbf{j}) \\ &= 5 \cdot 3 + 2 \cdot 7 \\ &= 29. \end{aligned}$$

  So, we compute the angle as follows:

$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1||\mathbf{v}_2|} = \frac{29}{\sqrt{29}\sqrt{58}} \approx 0.7071$$
$$\Rightarrow \ \theta = \cos^{-1}(0.7071) \approx 45^\circ.$$

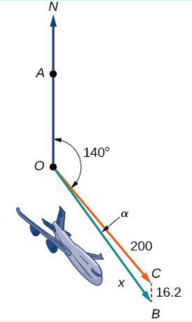
Finding the Angle between Two Vectors

- Find the angle between $\mathbf{u} = \langle -3, 4 \rangle$ and $\mathbf{v} = \langle 5, 12 \rangle$.

  We get

$$\begin{aligned} \cos\theta &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \\ &= \frac{-15+48}{\sqrt{9+16}\sqrt{25+144}} \\ &= \frac{33}{5\cdot 13} = \frac{33}{65}, \\ \theta &= \cos^{-1}\left(\frac{33}{65}\right) \approx 59.49^\circ. \end{aligned}$$

Finding Ground Speed and Bearing Using Vectors



- An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°.

  A north wind (from north to south) is blowing at 16.2 miles per hour.

  What are the ground speed and actual bearing of the plane?

  Find the vector **v** corresponding to the air velocity.

  $$\mathbf{v} = \langle 200\cos(-50^\circ), 200\sin(-50^\circ)\rangle = \langle 128.56, -153.21\rangle.$$

  The vector **u** giving the wind velocity is $\mathbf{u} = \langle 0, -16.2\rangle$.
  By adding them, we find the ground velocity of the plane.

  $$\mathbf{w} = \mathbf{v} + \mathbf{u} = \langle 128.56, -153.21\rangle + \langle 0, -16.2\rangle = \langle 128.56, -169.41\rangle.$$

  So $|\mathbf{w}| = 212.67$ and $\theta = 90^\circ - \tan^{-1}(\frac{-169.41}{128.56}) \approx 142.8^\circ$.