College Trigonometry

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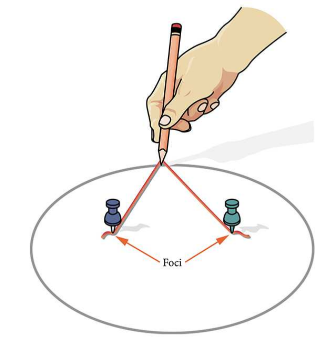
LSSU Math 131

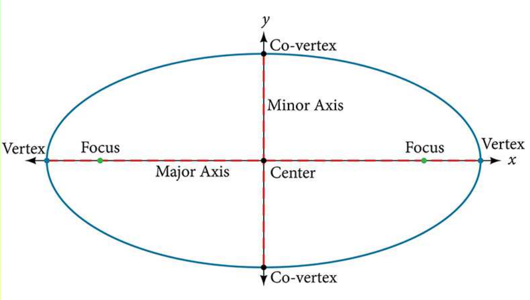
1. Analytic Geometry
   - The Ellipse
   - The Hyperbola
   - The Parabola
   - Conic Sections in Polar Coordinates

Subsection 1

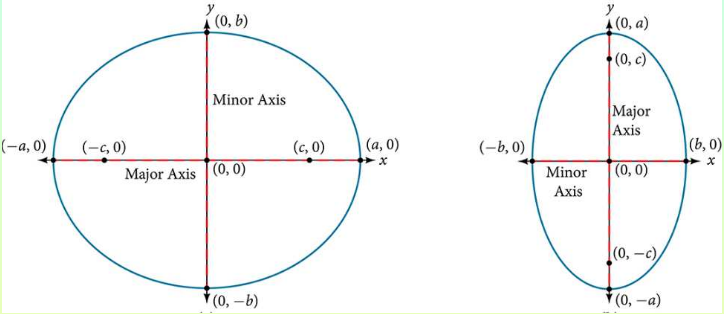
The Ellipse

Ellipse, Features and Terminology





Equation of an Ellipse at the Origin in Standard Form



- The equations are:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$$

- The key equation relating the parameters is $a^2 = b^2 + c^2$.

Equation of an Ellipse at the Origin in Standard Form

- What is the standard form equation of the ellipse that has vertices $(\pm 8, 0)$ and foci $(\pm 5, 0)$?

  We know that $c = 5$ and $a = 8$.

  We then calculate

  $$b^2 = a^2 - c^2 = 8^2 - 5^2 = 39.$$

  So the equation is

  $$\frac{x^2}{64} + \frac{y^2}{39} = 1.$$

An Ellipse Centered at a Point Other Than the Origin

- What is the standard form equation of the ellipse that has vertices $(-2,-8)$ and $(-2,2)$ and foci $(-2,-7)$ and $(-2,1)$?

  The ellipse has center $(-2,-3)$.

  It is vertically placed, with $a=5$ and $c=4$.

  So $b^2=a^2-c^2=5^2-4^2=9$.

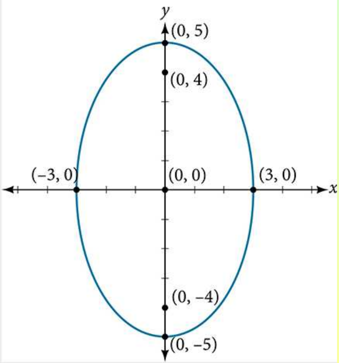
  If it was centered at the origin, the equation would be $\frac{y^2}{25}+\frac{x^2}{9}=1$, but, since its center is at $(h,k)=(-2,-3)$, we get the equation

$$\frac{(y+3)^2}{25}+\frac{(x+2)^2}{9}=1.$$

Graphing an Ellipse Centered at the Origin

- Graph the ellipse given by the equation, $\frac{x^2}{9} + \frac{y^2}{25} = 1$. Identify and label the center, vertices, co-vertices and foci.

  First identify all parameters: $a = 5$, $b = 3$ and $c^2 = a^2 - b^2 = 25 - 9 = 16$; so $c = 4$.



  So we have:
  - Center at the origin;
  - Vertices at $(0, \pm 5)$;
  - Co-vertices at $(\pm 3, 0)$;
  - Foci at $(0, \pm 4)$.

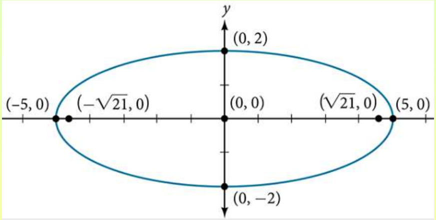
Graphing from an Equation Not in Standard Form

- Graph the ellipse given by the equation $4x^2 + 25y^2 = 100$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices and foci.

  Convert into standard form: $\frac{x^2}{25} + \frac{y^2}{4} = 1$.

  Identify all parameters: $a = 5$, $b = 2$ and $c^2 = a^2 - b^2 = 25 - 4 = 21$; so $c = \sqrt{21}$.

  So we have:
  - Center at the origin;
  - Vertices at $(\pm 5, 0)$;
  - Co-vertices at $(0, \pm 2)$;
  - Foci at $(\pm\sqrt{21}, 0)$.

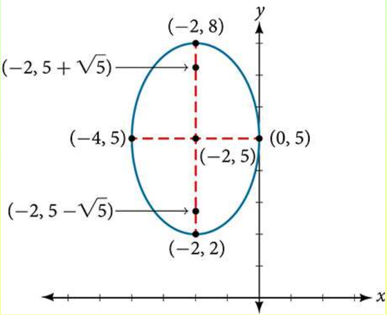


Graphing an Ellipse Centered at (h, k)

- Graph the ellipse given by the equation, $\frac{(x+2)^2}{4} + \frac{(y-5)^2}{9} = 1$. Identify and label the center, vertices, co-vertices and foci.

  First identify all parameters: $(h, k) = (-2, 5)$, $a = 3$, $b = 2$ and $c^2 = a^2 - b^2 = 9 - 4 = 5$; so $c = \sqrt{5}$.

  So we have:
  - Center at $(-2, 5)$;
  - Vertices at $(-2, 5 \pm 3)$;
  - Co-vertices at $(-2 \pm 2, 5)$;
  - Foci at $(-2, 5 \pm \sqrt{5})$.



An Ellipse Centered at (h, k) Not in Standard Form

- Graph the ellipse given by $4x^2 + 9y^2 - 40x + 36y + 100 = 0$. Identify and label the center, vertices, co-vertices and foci.

  Convert into standard form:

$$4x^2 + 9y^2 - 40x + 36y + 100 = 0$$

$$4x^2 - 40x + 9y^2 + 36y = -100$$

$$4(x^2 - 10x) + 9(y^2 + 4y) = -100$$

$$4(x^2 - 10x + 25) + 9(y^2 + 4y + 4) = -100 + 100 + 36$$

$$4(x-5)^2 + 9(y+2)^2 = 36$$

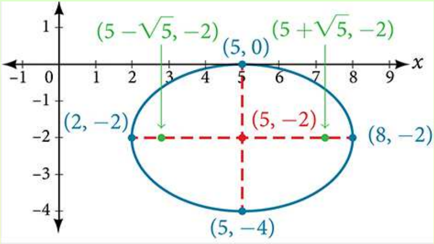
$$\frac{(x-5)^2}{9} + \frac{(y+2)^2}{4} = 1.$$

  Identify all parameters: $(h, k) = (5, -2)$, $a = 3$, $b = 2$ and $c^2 = a^2 - b^2 = 9 - 4 = 5$; so $c = \sqrt{5}$.

An Ellipse Centered at (h, k) Not in Standard Form

- We found $\frac{(x-5)^2}{9} + \frac{(y+2)^2}{4} = 1$
  and identified the parameters: $(h, k) = (5, -2)$, $a = 3$, $b = 2$ and $c^2 = a^2 - b^2 = 9 - 4 = 5$; so $c = \sqrt{5}$.

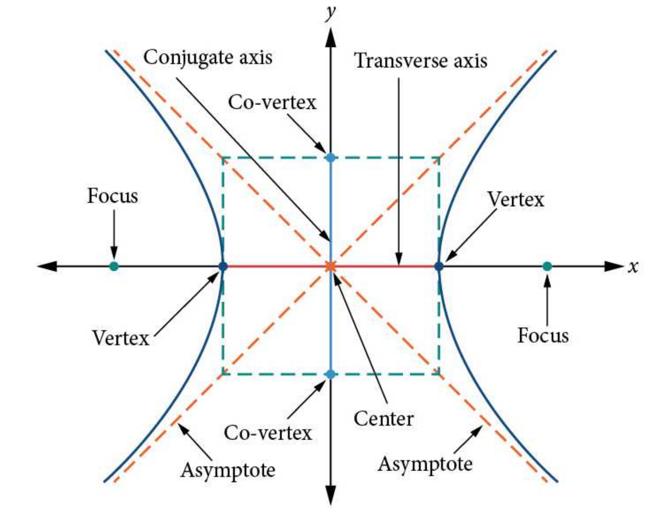
  So we have:
  - Center at $(5, -2)$;
  - Vertices at $(5 \pm 3, -2)$;
  - Co-vertices at $(5, -2 \pm 2)$;
  - Foci at $(5 \pm \sqrt{5}, -2)$.



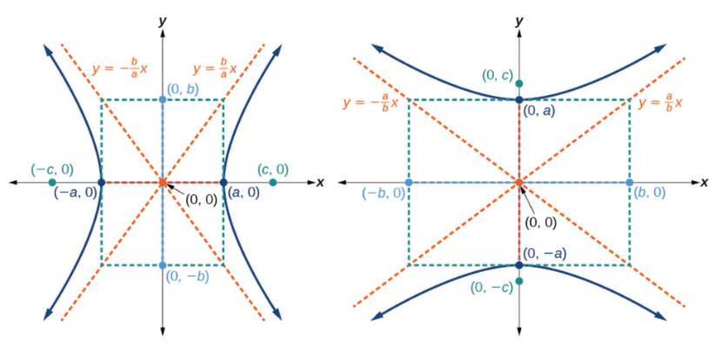
Subsection 2

The Hyperbola

Hyperbola, Features and Terminology



Equation of a Hyperbola at the Origin in Standard Form



- The equations are:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

- The key equation relating the parameters is $c^2 = a^2 + b^2$.

Locating a Hyperbola's Vertices and Foci

- Identify the vertices and foci of the hyperbola $\frac{y^2}{49} - \frac{x^2}{32} = 1$.
  It is a vertically opening hyperbola, with
  - $a = 7$,
  - $b = \sqrt{32} = 4\sqrt{2}$,
  - $c^2 = a^2 + b^2 = 49 + 32 = 81$; so $c = 9$.

  So it has vertices at $(0, \pm 7)$ and foci at $(0, \pm 9)$.

Finding a Hyperbola at $(0,0)$ Given its Foci and Vertices

- What is the standard form equation of the hyperbola that has vertices $(\pm 6, 0)$ and foci $(2\sqrt{10}, 0)$?

  We identify the parameters:

  - $a = 6$;
  - $c = 2\sqrt{10}$;
  - $b^2 = c^2 - a^2 = 40 - 36 = 4$; so $b = 2$.

  Now we get

$$\frac{x^2}{36} - \frac{y^2}{4} = 1.$$

Finding a Hyperbola at (h, k) Given its Foci and Vertices

- What is the standard form equation of the hyperbola that has vertices at $(0, -2)$ and $(6, -2)$ and foci at $(-2, -2)$ and $(8, -2)$?

  It is centered at $(3, -2)$.

  Identify the parameters:

  - $a = 3$;
  - $c = 5$;
  - $b^2 = c^2 - a^2 = 25 - 9 = 16$; so $b = 4$.

  The equation at the origin would be $\frac{x^2}{9} - \frac{y^2}{16} = 1$

  but since it is centered at $(h, k) = (3, -2)$, we get

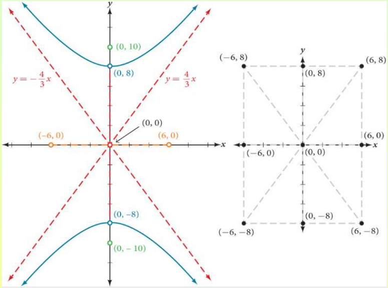
$$\frac{(x-3)^2}{9} - \frac{(y+2)^2}{16} = 1.$$

Graphing a Hyperbola at $(0,0)$ in Standard Form

- Graph the hyperbola given by the equation $\frac{y^2}{64} - \frac{x^2}{36} = 1$. Identify and label the vertices, co-vertices, foci and asymptotes.

  First identify all parameters: $a = 8$, $b = 6$ and $c^2 = a^2 + b^2 = 64 + 36 = 100$; so $c = 10$.

  So we have:
  - Center at the origin;
  - Vertices at $(0, \pm 8)$;
  - Co-vertices at $(\pm 6, 0)$;
  - Foci at $(0, \pm 10)$;
  - Asymptotes $y = \pm \frac{a}{b}x$, i.e., $y = \pm \frac{4}{3}x$.



A Hyperbola at (h, k) in General Form

- Graph the hyperbola given by $9x^2 - 4y^2 - 36x - 40y - 388 = 0$. Identify and label the center, vertices, co-vertices, foci and asymptotes.

  Convert into standard form:

$$9x^2 - 4y^2 - 36x - 40y - 388 = 0$$

$$9x^2 - 36x - 4y^2 - 40y = 388$$

$$9(x^2 - 4x) - 4(y^2 + 10y) = 388$$

$$9(x^2 - 4x + 4) - 4(y^2 + 10y + 25) = 388 + 36 - 100$$

$$9(x - 2)^2 - 4(y + 5)^2 = 324$$

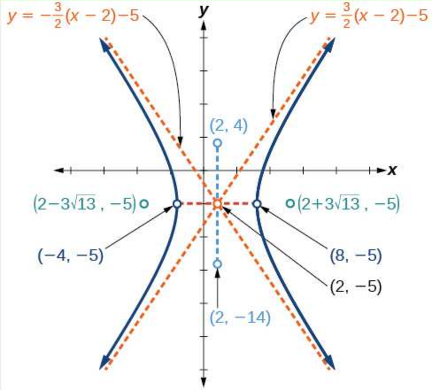
$$\frac{(x-2)^2}{36} + \frac{(y+5)^2}{81} = 1.$$

  Identify all parameters: $(h, k) = (2, -5)$, $a = 6$, $b = 9$ and $c^2 = a^2 + b^2 = 36 + 81 = 117$; so $c = 3\sqrt{13}$.

A Hyperbola at (h, k) in General Form

- We found $\frac{(x-2)^2}{36} + \frac{(y+5)^2}{81} = 1$
  and identified the parameters: $(h, k) = (2, -5)$, $a = 6$, $b = 9$ and $c^2 = a^2 + b^2 = 36 + 81 = 117$; so $c = 3\sqrt{13}$.

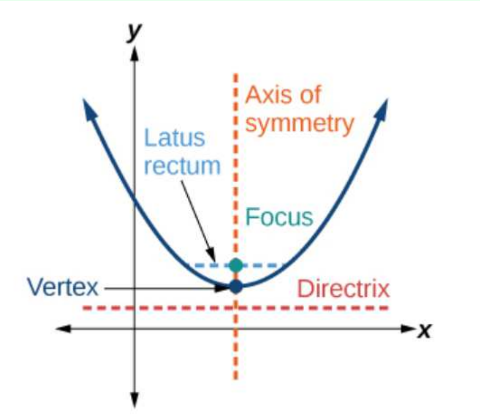
  So we have:
  - Center at $(2, -5)$;
  - Vertices at $(2 \pm 6, -5)$;
  - Co-vertices at $(2, -5 \pm 9)$;
  - Foci at $(2 \pm 3\sqrt{13}, -5)$;
  - Asymptotes $y + 5 = \pm\frac{3}{2}(x - 2)$.



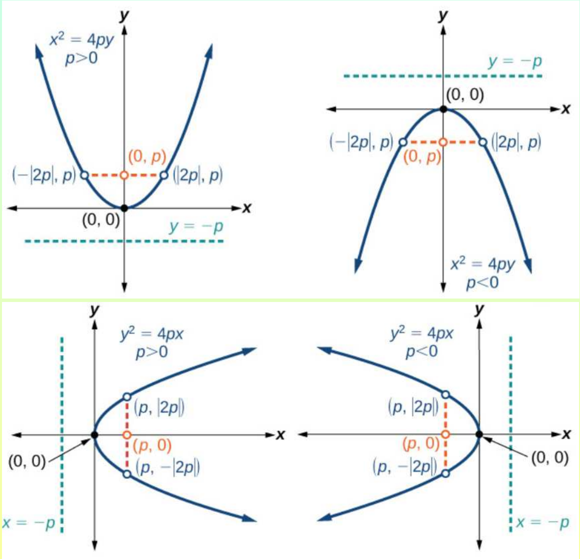
Subsection 3

The Parabola

Parabola, Features and Terminology



Equation of a Parabola at the Origin in Standard Form

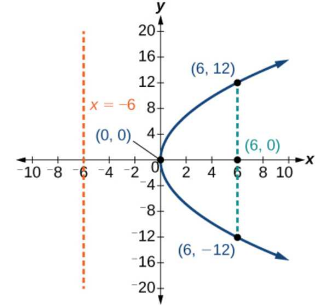


A Parabola at $(0, 0)$ With x-Symmetry

- Graph $y^2 = 24x$. Identify and label the focus, directrix and endpoints of the latus rectum.

  First identify p by comparing with $y^2 = 4px$: $p = 6$.

  So we have:
  - Vertex at the origin and opening right;
  - Focus at $(6, 0)$;
  - Directrix $x = -6$;
  - Endpoints of latus rectum $(6, \pm 12)$.

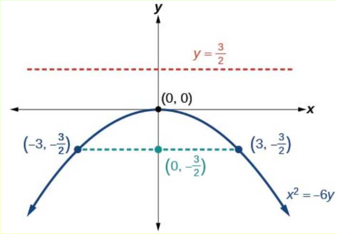


A Parabola at $(0, 0)$ With y-Symmetry

- Graph $x^2 = -6y$. Identify and label the focus, directrix and endpoints of the latus rectum.

  First identify p by comparing with $x^2 = 4py$: $p = -\frac{3}{2}$.

  So we have:
  - Vertex at the origin and opening down;
  - Focus at $(0, -\frac{3}{2})$;
  - Directrix $y = \frac{3}{2}$;
  - Endpoints of latus rectum $(\pm 3, -\frac{3}{2})$.



A Parabola Given its Focus and Directrix

- What is the equation for the parabola with focus $(-\frac{1}{2}, 0)$ and directrix $x = \frac{1}{2}$?

  First find the vertex, opening direction and p.

  Then it is easy to write an equation for the parabola.

  We have:

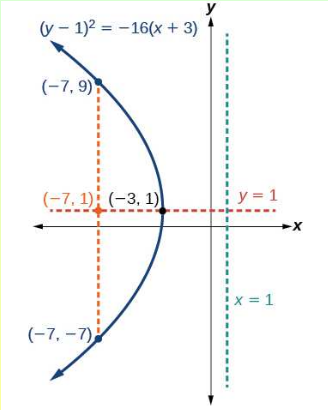
  - Vertex $(0, 0)$;
  - Opens left;
  - $p = -\frac{1}{2}$.

  So equation is $y^2 = 4(-\frac{1}{2})x$, i.e., $y^2 = -2x$.

Parabola at (h, k) With Horizontal Symmetry

- Graph $(y-1)^2 = -16(x+3)$. Identify and label the vertex, axis of symmetry, focus, directrix and endpoints of the latus rectum.

  Compare with $(y-k)^2 = 4p(x-h)$. So $p = -4$.



  So we have:
  - Vertex $(h, k) = (-3, 1)$;
  - Opens left;
  - Focus at $(-3-4, 1)$;
  - Directrix $x = -3+4$;
  - Endpoints of latus rectum $(-3-4, 1 \pm 8)$.

A Parabola from an Equation Given in General Form

- Graph $x^2 - 8x - 28y - 208 = 0$. Identify and label the vertex, axis of symmetry, focus, directrix and endpoints of the latus rectum.

  Convert into standard form:

$$x^2 - 8x - 28y - 208 = 0$$

$$x^2 - 8x = 28y + 208$$

$$x^2 - 8x + 16 = 28y + 224$$

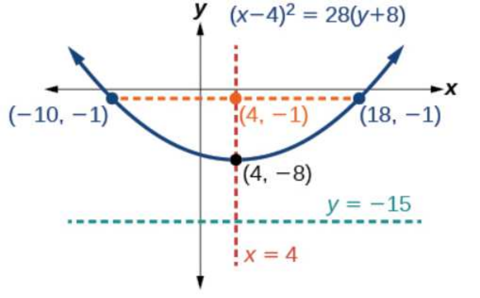
$$(x - 4)^2 = 28(y + 8).$$

  Identify all parameters: $(h, k) = (4, -8)$, opens up, $p = 7$.

A Parabola from an Equation Given in General Form

- We found equation $(x-4)^2 = 28(y+8)$.
  Moreover, we reasoned that
  - $(h,k) = (4,-8)$;
  - Opens up;
  - $p = 7$.

  So we have:
  - Vertex $(4,-8)$;
  - Axis of Symmetry $x = 4$;
  - Focus at $(4, -8+7)$;
  - Directrix $y = -8-7$;
  - Endpoints $(4 \pm 14, -8+7)$.

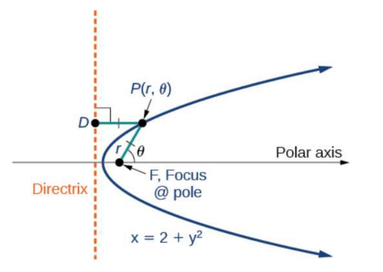


Subsection 4

Conic Sections in Polar Coordinates

Conic Sections in Polar: Features and Terminology

- F is the **focus** at the pole;
- D is the **directrix** at $x = \pm p$;
- $e > 0$ a fixed number, called the **eccentricity**;
- The set of all points P such that $e = \frac{PF}{PD}$ is a conic:
  - if $0 = e < 1$, the conic is an ellipse;
  - if $e = 1$, the conic is a parabola;
  - if $e > 1$, the conic is a hyperbola.



Equation of Conic Sections in Polar Coordinates

- For a conic with a focus at the origin, if the directrix is $x = \pm p$, where p is a positive real number, and the eccentricity is a positive real number e, the conic has a polar equation

$$r = \frac{ep}{1 \pm e\cos\theta}.$$

- For a conic with a focus at the origin, if the directrix is $y = \pm p$, where p is a positive real number, and the eccentricity is a positive real number e, the conic has a polar equation

$$r = \frac{ep}{1 \pm e\sin\theta}.$$

Identifying a Conic Given the Polar Form

- For each of the following equations, identify the conic with focus at the origin, the directrix and the eccentricity.
  (a) $r = \frac{6}{3+2\sin\theta}$
  (b) $r = \frac{12}{4+5\cos\theta}$
  (c) $r = \frac{7}{2-2\sin\theta}$

(a) $r = \frac{6}{3+2\sin\theta} \Rightarrow r = \frac{2}{1+\frac{2}{3}\sin\theta} \Rightarrow e = \frac{2}{3}$ and $ep = 2 \Rightarrow p = 3$.
So directrix is $y = 3$ and the conic is an ellipse ($e < 1$).

(b) $r = \frac{12}{4+5\cos\theta} \Rightarrow r = \frac{3}{1+\frac{5}{4}\cos\theta} \Rightarrow e = \frac{5}{4}$ and $ep = 3 \Rightarrow p = \frac{12}{5}$.
So directrix is $x = \frac{12}{5}$ and the conic is a hyperbola ($e > 1$).

(c) $r = \frac{7}{2-2\sin\theta} \Rightarrow r = \frac{\frac{7}{2}}{1-\sin\theta} \Rightarrow e = 1$ and $ep = \frac{7}{2} \Rightarrow p = \frac{7}{2}$.
So directrix is $y = -\frac{7}{2}$ and the conic is a parabola ($e = 1$).

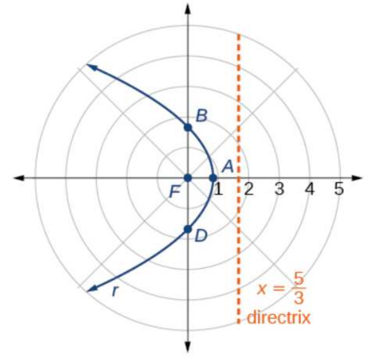
Graphing a Parabola in Polar Form

- Graph $r = \frac{5}{3+3\cos\theta}$.

  $r = \frac{5}{3+3\cos\theta} \Rightarrow r = \frac{\frac{5}{3}}{1+\cos\theta} \Rightarrow e = 1, ep = \frac{5}{3} \Rightarrow p = \frac{5}{3}$.

  We have a parabola with directrix $x = \frac{5}{3}$.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{5}{3+3\cos\theta}$	$\frac{5}{6}$	$\frac{5}{3}$	$-$	$\frac{5}{3}$



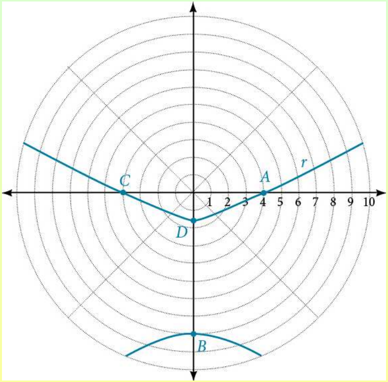
Graphing a Hyperbola in Polar Form

- Graph $r = \frac{8}{2-3\sin\theta}$.

  $r = \frac{8}{2-3\sin\theta} \Rightarrow r = \frac{4}{1-\frac{3}{2}\sin\theta} \Rightarrow e = \frac{3}{2}, ep = 4 \Rightarrow p = \frac{8}{3}$.

  We have a hyperbola with directrix $y = -\frac{8}{3}$.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{8}{2-3\sin\theta}$	4	-8	4	$\frac{8}{5}$



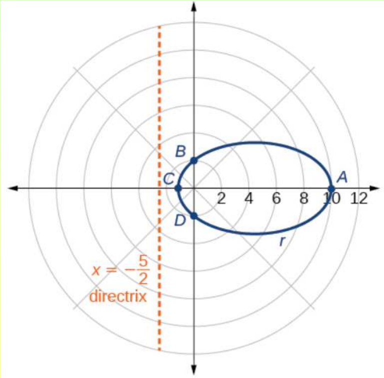
Graphing an Ellipse in Polar Form

- Graph $r = \frac{10}{5-4\cos\theta}$.

  $r = \frac{10}{5-4\cos\theta} \Rightarrow r = \frac{2}{1-\frac{4}{5}\cos\theta} \Rightarrow e = \frac{4}{5}, ep = 2 \Rightarrow p = \frac{5}{2}$.

  We have an ellipse with directrix $x = -\frac{5}{2}$.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{10}{5-4\cos\theta}$	10	-2	$\frac{10}{9}$	2



Finding the Polar Form of a Vertical Conic

- Find the polar form of the conic given a focus at the origin, $e = 3$ and directrix $y = -2$.

  Identify $p = 2$ and choose one of

  $$r = \frac{ep}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ep}{1 \pm e\sin\theta}.$$

  We have

  $$r = \frac{ep}{1 - e\sin\theta} \Rightarrow r = \frac{6}{1 - 3\sin\theta}.$$

Finding the Polar Form of a Horizontal Conic

- Find the polar form of a conic given a focus at the origin, $e = \frac{3}{5}$, and directrix $x = 4$.

  Identify $p = 4$ and choose one of

  $$r = \frac{ep}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ep}{1 \pm e\sin\theta}.$$

  We have

  $$r = \frac{ep}{1 + e\cos\theta} \Rightarrow r = \frac{\frac{12}{5}}{1 + \frac{3}{5}\cos\theta} \Rightarrow r = \frac{12}{5 + 3\cos\theta}.$$

Converting a Conic in Polar Form to Rectangular Form

- Convert the conic $r = \frac{1}{5-5\sin\theta}$ to rectangular form.

  We get

$$\begin{aligned} r &= \frac{1}{5-5\sin\theta} \\ r(5-5\sin\theta) &= 1 \\ 5r - 5r\sin\theta &= 1 \\ 5r &= 1 + 5r\sin\theta \\ 25r^2 &= (1+5r\sin\theta)^2 \\ 25(x^2+y^2) &= (1+5y)^2 \\ 25x^2 + 25y^2 &= 1 + 10y + 25y^2 \\ 25x^2 - 10y &= 1. \end{aligned}$$