EXTRA HOMEWORK: SOLUTIONS - MATH 110
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Problem 1 *Solve the following system of linear equations using the Gauss-Jordan method:*

$$\left\{\begin{array}{rcrcrcr} x & + & y & + & z & = & 3 \\ -x & + & 2y & - & z & = & 3 \\ 2x & - & 3y & + & 2z & = & -4 \end{array}\right\}$$

Solution:

Form the augmented matrix and perform row operations:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ -1 & 2 & -1 & 3 \\ 2 & -3 & 2 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ 0 & 3 & 0 & 6 \\ 0 & -5 & 0 & -10 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & 0 & -10 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore, the system has infinitely many solutions

$$(x, y, z) = (1 - z, 2, z), \quad z \text{ arbitrary.}$$

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Problem 2 *Solve the following system by* **all three** *of the following methods: substitution, inverse matrix method and the Cramer's method.*

$$\left\{\begin{array}{rcrcr} -x & + & 2y & = & -18 \\ 2x & - & 3y & = & 29 \end{array}\right\}$$

Solution:

1. (Substitution) Solve the first equation for x. $x = 2y + 18$. Now substitute this value of x for x in the second equation:

$$2(2y + 18) - 3y = 29, \text{ i.e., } 4y + 36 - 3y = 29$$

which gives $y = -7$. Therefore, back-substituting in the first equation: $x = 2(-7)+18$, which gives $x = 4$. Thus, the given equation has the unique solution $(x, y) = (4, -7)$.

2. (Inverse Matrix Method) Compute the inverse matrix of the matrix of the coefficients $A = \left[\begin{array}{rr} -1 & 2 \\ 2 & -3 \end{array}\right]$. The determinant is $\det(A) = (-1)(-3) - 2 \cdot 2 = 3 - 4 = -1 \neq 0$. Hence, the inverse matrix is

$$A^{-1} = \frac{1}{\det(A)} \left[\begin{array}{rr} d & -b \\ -c & a \end{array}\right] = \frac{1}{-1} \left[\begin{array}{rr} -3 & -2 \\ -2 & -1 \end{array}\right] = \left[\begin{array}{rr} 3 & 2 \\ 2 & 1 \end{array}\right].$$

Hence the solutions are given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} -18 \\ 29 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -18 \\ 29 \end{bmatrix} = \begin{bmatrix} 3(-18)+2\cdot 29 \\ 2(-18)+29 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}.$$

3. (Cramer's Method)

$$x = \frac{\begin{vmatrix} -18 & 2 \\ 29 & -3 \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix}} = \frac{54-58}{-1} = 4 \quad \text{and} \quad y = \frac{\begin{vmatrix} -1 & -18 \\ 2 & 29 \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix}} = \frac{-29+36}{-1} = -7$$

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Problem 3 *Taking the long view of your education, you go to the Prestige Corporation and ask what you should do in college to be hired when you graduate. The Personnel Director replies that you will be hired* only if *you major in mathematics or computer science, get a B or better average, and take accounting. You do, in fact, become a math major, get a* B^+ *average and take accounting. You return to the Prestige Corporation, make a formal application and are turned down. Did the Personnel Director lie to you?*

Solution:
This problem has to do only with your understanding of how the phrase "only if" is used to connect sentences in formal logic. Recall that "P only if Q" means "If P then Q". Therefore, what the Personnel Director really said was "If you are hired, then you have majored in mathematics or computer science, gotten a B or better average, and took accounting". As we recall, this is not logically equivalent to "If you major in mathematics or computer science, get a B or better average, and take accounting, then you will be hired". Therefore, logically speaking, the Personnel Director did not lie to you. You were just not careful enough to untangle the rhetoric scheme that he used. ■

Problem 4 *"If compound X is boiling, then its temperature must be at least* 250^oF*". Assuming that this statement is true, which of the following must also be true?*

1. *If the temperature of compound X is at least* 250^oF*, then compound X is boiling.*
2. *If the temperature of compound X is less than* 250^oF*, then compound X is not boiling.*
3. *Compound X will boil only if its temperature is at least* 250^oF*.*
4. *If compound X is not boiling, then its temperature is less than* 250^oF*.*
5. *A necessary condition for compound X to boil is that its temperature be at least* 250^oF*.*
6. *A sufficient condition for compound X to boil is that its temperature be at least* 250^oF*.*

Solution:

Let $P =$ "compound X is boiling" and $Q =$ "its temperature must be at least $250^o F$". Then the given statement is $P \to Q$.

1. This statement is the statement $Q \to P$. Recall that $Q \to P \not\equiv P \to Q$, whence the fact that $P \to Q$ is true does not necessarily mean that $Q \to P$ is also true.

2. This statement is $\neg Q \to \neg P$. By the contrapositive rule, $\neg Q \to \neg P \equiv P \to Q$. Therefore, this statement must necessarily be true.

3. "P only if Q" is equivalent to "if P then Q", whence the given statement is also equivalent to the statement $P \to Q$ and, thus, it must also be true.

4. This statement is the statement $\neg P \to \neg Q$. Recall that $\neg P \to \neg Q \not\equiv P \to Q$. Therefore this statement is not necessarily true.

5. This statement is "a necessary condition for P is Q". This translates to $P \to Q$. Whence the given statement must also be true.

6. This statement is "a sufficient condition for P is Q". This translates to $Q \to P$. But $Q \to P \not\equiv P \to Q$, whence the given statement is not necessarily true.

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Problem 5 *Determine of whether it is true that, for all sets A, B, C,*

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

If yes, give a proof. If not, give a counterexample.

Solution:

This statement is true. Its proof is as follows: To show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$ we need to show first that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ and second that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

For the first inclusion, suppose that $x \in A \times (B \cup C)$. This means, by the definition of $\times$ that $x = (a, d)$, for $a \in A$ and $d \in B \cup C$. Thus, by the definition of $\cup$, $x = (a, d)$ for $a \in A$ and $d \in B$ or $d \in C$. Therefore $x = (a, d)$ for $a \in A, d \in B$ or $x = (a, d)$ for $a \in A, d \in C$. Thus, by the definition of $\times$, $x \in A \times B$ or $x \in A \times C$, which, by the definition of $\cup$, shows that $x \in (A \times B) \cup (A \times C)$. Hence, we have proved that $x \in A \times (B \cup C)$ implies $x \in (A \times B) \cup (A \times C)$, i,e., that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

For the second inclusion, suppose $x \in (A \times B) \cup (A \times C)$. By the definition of $\cup$, this means that $x \in A \times B$ or $x \in A \times C$. This means that $x = (a, b)$ for $a \in A, b \in B$ or $x = (a', c)$ for $a' \in A, c \in C$. In the first case, since $B \subseteq B \cup C$, we have that $x = (a, b)$ for $a \in A, b \in B \cup C$, i.e., by the definition of $\times$, $x \in A \times (B \cup C)$, and, in the second, since $C \subseteq B \cup C$, that $x = (a', c)$ for $a' \in A, c \in B \cup C$. Therefore, by the definition of $\times$, in this case as well $x \in A\times(B\cup C)$. In any case, $x \in A\times(B\cup C)$ follows. Therefore, we have shown that $x \in (A\times B)\cup(A\times C)$ implies $x \in A\times(B\cup C)$, i.e., that $(A\times B)\cup(A\times C) \subseteq A\times(B\cup C)$.

Having shown that both $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ and $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ are true, we may now conclude that $A \times (B \cup C) = (A \times B) \cup (A \times C)$. ■

Problem 6 *A student council consists of three freshmen, four sophomores, three juniors and five seniors. How many committees of eight members of the council contain at least one member from each class?*

Solution:

Denote by F, S, J, N the sets of committees of eight members that contain at least one freshman, at least one sophomore, at least one junior and at least one senior, respectively. Thus, F^c, S^c, J^c and N^c denote the sets of committees containing no freshman, no sophomore, no junior and no senior, respectively. The set of the eight member committees with at least one member of each class is $F \cap S \cap J \cap N$. This is the same as $((F \cap S \cap J \cap N)^c)^c$, which, by de Morgan Law, is equal to $(F^c \cup S^c \cup J^c \cup N^c)^c$. What we will do is compute the number of elements in $F^c \cup S^c \cup J^c \cup N^c$ by using Inclusion-Exclusion, and then subtract this from the total to find the number of elements in $(F^c \cup S^c \cup J^c \cup N^c)^c$.

We have

$$\begin{aligned}
n(F^c \cup S^c \cup J^c \cup N^c) &= n(F^c) + n(S^c) + n(J^c) + n(N^c) - n(F^c \cap S^c) - n(F^c \cap J^c) \\
&\quad -n(F^c \cap N^c) - n(S^c \cap J^c) - n(S^c \cap N^c) - n(J^c \cap N^c) \\
&\quad +n(F^c \cap S^c \cap J^c) + n(F^c \cap S^c \cap N^c) + n(F^c \cap J^c \cap N^c) \\
&\quad +n(S^c \cap J^c \cap N^c) - n(F^c \cap S^c \cap J^c \cap N^c) \\
&= \tbinom{12}{8} + \tbinom{11}{8} + \tbinom{12}{8} + \tbinom{10}{8} - \tbinom{8}{8} - \tbinom{9}{8} \\
&\quad -0 - \tbinom{8}{8} - 0 - 0 \\
&\quad +0 + 0 + 0 \\
&\quad +0 - 0 \\
&= \tbinom{12}{8} + \tbinom{11}{8} + \tbinom{12}{8} + \tbinom{10}{8} - \tbinom{8}{8} - \tbinom{9}{8} - \tbinom{8}{8}.
\end{aligned}$$

Therefore $n((F^c \cup S^c \cup J^c \cup N^c)^c) = \tbinom{15}{8} - [\tbinom{12}{8} + \tbinom{11}{8} + \tbinom{12}{8} + \tbinom{10}{8} - \tbinom{8}{8} - \tbinom{9}{8} - \tbinom{8}{8}]$, i.e.,

$$n(F \cap S \cap J \cap N) = \binom{15}{8} - [\binom{12}{8} + \binom{11}{8} + \binom{12}{8} + \binom{10}{8} - \binom{8}{8} - \binom{9}{8} - \binom{8}{8}].$$

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