EXAM 2: SOLUTIONS - MATH 111
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Problem 1 *Find the vertex, the opening direction, the* x*- and* y*-intercepts and sketch the graph of* $f(x) = -\frac{1}{2}x^2 + x + \frac{3}{2}$.

Solution:
The vertex has x-coordinate

$$x = -\frac{b}{2a} = -\frac{1}{2 \cdot (-\frac{1}{2})} = 1$$

and y-coordinate $f(1) = -\frac{1}{2} + 1 + \frac{3}{2} = 2$. Hence, it is the point $(1, 2)$.

The parabola opens down, since $a = -\frac{1}{2} < 0$.

The y-intercept is found by setting $x = 0$. We have then $y = \frac{3}{2}$. Thus $(0, \frac{3}{2})$ is the y-intercept. The x-intercepts are found by setting $y = 0$ and solving $-\frac{1}{2}x^2 + x + \frac{3}{2} = 0$. Use the quadratic

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm 2}{-1} = -1 \text{ or } 3.$$

Thus, the x-intercepts are the points $(-1, 0)$ and $(3, 0)$.

The graph follows

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Problem 2 *Find the equation of the parabola that has vertex* $V = (-2, 1)$ *and goes through the point* $(1, 3)$.

Solution:
Since the vertex is at $V = (-2, 1)$, we have equation

$$f(x) = a(x + 2)^2 + 1.$$

But the parabola goes through $(1, 3)$, whence

$$3 = a(1 + 2)^2 + 1, \quad \text{i.e.,} \quad 3 = 9a + 1,$$

which yields $a = \frac{2}{9}$. Thus, the equation of the parabola is

$$f(x) = \frac{2}{9}(x+2)^2 + 1.$$

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Problem 3 *The supply and the demand of a specific item are modelled by* $p = \frac{1}{2}q^2 + q + \frac{1}{2}$ *and* $p = -\frac{1}{2}q^2 - q + \frac{7}{2}$*, respectively, where* p *denotes price and* q *number of items. Find the equilibrium price and the equilibrium supply.*

Solution:

To find the equilibrium supply set the two price functions equal:

$$\frac{1}{2}q^2 + q + \frac{1}{2} = -\frac{1}{2}q^2 - q + \frac{7}{2}.$$

This gives $q^2 + 2q - 3 = 0$. This quadratic factors as $(q+3)(q-1) = 0$. Hence $q = -3$ or $q = 1$. Since supply has to be positive, we get $q = 1$. The equilibrium price is then $p = \frac{1}{2} + 1 + \frac{1}{2} = 2$. ■

Problem 4 *Find the vertical and the horizontal asymptotes and the* x*- and* y*-intercepts of the function* $f(x) = -\frac{2x-3}{5x+6}$ *and roughly sketch its graph.*

Solution:

The vertical asymptote is $x = -\frac{6}{5}$. The horizontal asymptote is given by $y = -\frac{2}{5}$. The y-intercept is obtained by setting $x = 0$. We get $y = \frac{1}{2}$. Thus, it is the point $(0, \frac{1}{2})$. The x-intercept is found by setting $y = 0$. This gives $x = \frac{3}{2}$. The graph follows

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Problem 5 *Solve the exponential equation* $4^{x^2+2} = 64^x$.

Solution:

We have by transforming both sides to base 2: $(2^2)^{x^2+2} = (2^6)^x$, i.e., $2^{2(x^2+2)} = 2^{6x}$. Hence $2(x^2+2) = 6x$, whence $x^2 + 2 = 3x$ and, therefore, $x^2 - 3x + 2 = 0$. This factors as $(x-2)(x-1) = 0$. Therefore, we obtain the two solutions $x = 1$ or $x = 2$. ■

Problem 6 *Solve the logarithmic equation* $\log_2 x - \log_2 (x-1) = 3$.

Solution:

We have $\log_2 x - \log_2 (x-1) = 3$ implies $\log_2 \frac{x}{x-1} = 3$, whence $\frac{x}{x-1} = 2^3$, i.e., $\frac{x}{x-1} = 8$. Multiply both sides by $x-1$ to get $x = 8(x-1)$. This gives $x = 8x - 8$, whence $7x = 8$, i.e., $x = \frac{8}{7}$. Check to see that this is an accepted solution. ■