EXAM 3: SOLUTIONS - MATH 111
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Problem 1 *Linda owes \$10,000 to a furniture store. She has agreed to pay the amount in 6 months at an interest rate of 10%. 3 months before the loan is due, the store needs \$12,000 to pay a wholesaler's bill. The bank agrees to discount the note at a rate of 12%. How much of the \$12,000 owed will be taken care of?*

Solution:

Linda's note will be worth $A = 10000(1 + 0.1 \cdot 0.5) = 10500$ in 6 months. When the store asks the bank for a discount loan, the amount of the loan will A at 12% for only 3 months. Hence the proceeds are

$$P = A(1 - 0.12 \cdot 0.25) = 10500(1 - 0.03) = 10500 \cdot 0.97.$$

This is the amount that will be taken care of. ■

Problem 2 *George deposits \$5,000 at the beginning of each semiannual period for 5 years in an account paying 10% compounded semiannually. After this period, he leaves the money alone with no further deposits for an additional 5 years. Find the final amount in the account at the end of the entire 10 year period.*

Solution:

In the first 5 years, the amount in the annuity will be

$$A = 5000[\frac{(1 + 0.05)^{11} - 1}{0.05} - 1].$$

Hence, the amount after another 5 years with semiannual compounding but without any more deposits will be

$$B = 5000[\frac{(1 + 0.05)^{11} - 1}{0.05} - 1](1 + 0.05)^{10}.$$

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Problem 3 *Solve the following system* **by substitution** $\left\{ \begin{array}{rcrcr} 2x & + & y & = & -1 \\ -5x & + & 2y & = & 16 \end{array} \right\}$

Solution:

The first equation gives $y = -2x - 1$. Substituting this value for y in the second equation, we obtain $-5x + 2(-2x - 1) = 16$, whence $-5x - 4x - 2 = 16$, i.e., $-9x = 18$ which gives $x = -2$. Back substituting into the equation for y, we get $y = 3$. Thus the solution of the given system is $(x, y) = (-2, 3)$. ■

Problem 4 *Solve the following system* **by the Gauss-Jordan method**

$$\left\{\begin{array}{rcrcrcr} x & + & y & + & z & = & -1 \\ -x & + & 3y & - & z & = & 1 \\ 2x & + & y & - & 2z & = & 6 \end{array}\right\}.$$

Solution:
Take the augmented matrix and perform row operations:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & -1 \\ -1 & 3 & -1 & 1 \\ 2 & 1 & -2 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 1 & 1 & -1 \\ 0 & 4 & 0 & 0 \\ 0 & -1 & -4 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -4 & 8 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

Hence $(x, y, z) = (1, 0, -2)$. ■

Problem 5 *Let* $X = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$. *Solve the matrix equation* $X^2 = 2X + \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$.

Solution:
We have $X^2 = 2X + \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ implies

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \cdot \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = 2 \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}.$$

After performing the matrix operations this becomes

$$\begin{bmatrix} x^2 & 0 \\ 0 & y^2 \end{bmatrix} = \begin{bmatrix} 2x-1 & 0 \\ 0 & 2y+3 \end{bmatrix}.$$

Hence, $x^2 = 2x - 1$ and $y^2 = 2y + 3$. The first gives $x^2 - 2x + 1 = 0$ and the second $y^2 - 2y - 3 = 0$. Hence $(x-1)^2 = 0$ and $(y+1)(y-3) = 0$. Therefore $x = 1$ and $y = -1$ or $y = 3$. Thus, there are two matrix solutions for X. The first is the matrix $X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and the second the matrix $X = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. ■

Problem 6 *Find the inverse of the matrix* $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$.

Solution:
We have

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -2 & -1 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right].$$

Thus

$$A^{-1} = \left[\begin{array}{ccc} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right].$$

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